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S^3-BUNDLES AND EXOTIC ACTIONS

BY

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RÉSUMÉ. — Le but de ce travail est la construction explicite de représentants pour toutes fibrations principales avec fibres S^3 et $SO(4)$ sur S^4 et S^7. Comme conséquence on obtient les premières étapes d'une construction explicite des S^3-actions libres, sur chaque espace totale des S^3-fibrations principales sur S^7, ayant pour quotient des 7-sphères exotiques.

ABSTRACT. — We construct explicit representatives for all S^3 and $SO(4)$-principal bundles over S^4 and S^7. Moreover, the first steps are taken towards describing explicitly the free S^3-actions, on each of the total spaces of the S^3-principal bundles over S^7, with quotients exotic 7-spheres.

0. Introduction

In this note we construct explicit representatives for all principal bundles with group S^3 and $SO(4)$ over the spheres S^4 and S^7. As a consequence we get an insight into some of the exotic free S^3 actions on $S^7 \times S^3$. I. e., free actions with quotient a seven dimensional sphere with non-standard differentiable structure. Seven out of the fifteen exotic 7-spheres that are S^3 bundles over S^4 with group $SO(4)$ [E-K] appear as such quotients, each in an infinity of ways. It also turns out that there are such exotic free actions on $Sp(2)$ and on each of the other S^3-principal bindles over S^7. One could describe these actions in a way that will become clear in paragraph 4, generalizing the example of GROMOLL and MEYER [G-M]. In the present note we have not pursued these calculations.

Our motivations for seeking explicit descriptions for the S^3-principal bundles over S^4, and consequently over S^7, came from the following considerations:

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First, one expects some of the beauty of the S^1-principals over S^2, with total spaces the lense-spaces $S^3/\mathbb{Z}_n$, although the group S^3 is not commutative like S^1. *See* for example [S]. The trick we employed to bypass this non-commutativity was to keep increasing the size of the matrix.

Second, it appears from the work of ATIYAH, HITCHIN and SINGER [A-H-S] that bundles over S^4 and "natural" connections on them are of some interest to Theoretical Physicists. Building blocks for such bundles are the S^3-principals and some "natural" description of theirs could conceivably facilitate calculations and give some insight.

Finally, a problem in Differential Geometry suggested by the work of CHEEGER and GROMOLL [C-G] : Do all vector bundles over euclidean spheres admit complete riemannian metrics of non-negative sectional curvature? This problem begins to be non-trivial at exactly this point: the principal S^3-bundles over S^4.

See for example [R_2], [R_1], [We], [D-R]. Connection and curvature calculations will appear elsewhere.

Several routine homotopy arguments have not been written down explicitly for the purpose of avoiding excessive formality. We hope to have not made the note unclear by doing so.

We wish to thank Andrzej DERDZINSKI for many hepful discussions.

1. Preliminaries

Let $Sp(n)$ denote the group of quaternionic $n \times n$ matrices A such that $AA^* = A^*A = I$, where A^* denotes the conjugate transpose of A. If $A = (a_{ij})$ the above relations translate to:

(a) All rows have unit length: $R^i . \bar{R}^i = 1$ for all i;

(b) Rows are mutually orthogonal: $R^i . \bar{R}^j = 0$ for all $i \neq j$;

(a') Columns are of unit length: $\bar{C}_i . C_i = 1$ for all i;

(b') Columns are mutually orthogonal: $\bar{C}_i . C_j = 0$ for all $i \neq j$.

Where the product $R^i . \bar{R}^j$ is $\sum_{k=1}^n a_{ik} \bar{a}_{jk}$, etc., the conjugate of a quaternion $a = x_0 + x_1 i + x_2 j + x_3 k$ is $\bar{a} = x_0 - x_1 i - x_2 j - x_3 k$ and $\{(a'), (b')\}$ is equivalent to $\{(a), (b)\}$. Observe that the group of unit quaternions $Sp(1)$ is identified with S^3 in R^4 and if $\Delta : Sp(1) \to Sp(n)$ is the diagonal inclusion $\Delta(q) = \begin{pmatrix} q & & 0 \\ & \ddots & \\ 0 & & q \end{pmatrix}$ we denote the subgroup $\Delta(Sp(1))$ by

$\overset{n}{Sp(1)}$. Recall that $\pi_3 Sp(n) = \mathbf{Z}$ and that it is generated by any of the canonical inclusions of $Sp(1)$ in $Sp(n)$ as $q \to \begin{bmatrix} q & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & q & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \end{bmatrix}$.

Therefore the inclusion Δ induces the following map:

$$\Delta_* : \pi_3 Sp(1) \to \pi_3 Sp(n)$$

with $\Delta_*(1) = n$ which implies that $\pi_3(\overset{n}{Sp(1)} \backslash Sp(n)) = \mathbf{Z}_n$. Observe that the quotient is the one induced by *left* action of $\overset{n}{Sp(1)}$ on $Sp(n)$. Let now $n \geqslant 2$ and consider $Sp(n-1)$ acting from the right on the quotient and leaving the first column unaltered: A in $Sp(n-1)$ acts as $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ from the right.

CLAIM. — This is a free action with quotient QP^{n-1}.

Proof. — If B is in $Sp(n)$, q in $Sp(1)$ and A in $Sp(n-1)$ then $BA = (q)^n B$ implies that B and $(q)^n B$ have the same first column, so $q = 1$ and therefore $A = 1$ too. The quotient $\overset{n}{Sp(1)} \backslash Sp(n) / Sp(n-1)$ is also obtained as follows:

$$\begin{array}{ccc} Sp(1) \subsetneq Sp(n)/Sp(n-1) & & \cong S^{4n-1} \\ \Big\downarrow & & \Big\downarrow \text{Hopf} \\ \overset{n}{Sp(1)} \backslash Sp(n)/Sp(n-1) & = & QP^{n-1} \end{array}$$

Diagram 1

I. e., $B \mapsto$ (1st column of B) and q acts from the left on the first column as:

$$\begin{pmatrix} qb_1 \\ \vdots \\ qb_n \end{pmatrix} \text{ with quotient } QP^{n-1}.$$

In [G-M], $S^4 = QP^1$ is written as $2\, Sp(1) \backslash Sp(2) / Sp(1)$.

Now we have the principal bundles:

$$Sp(n-1) \overset{n}{\hookrightarrow} Sp(1) \backslash Sp(n) \overset{p_n}{\to} QP^{n-1}$$

and we will denote the elements of QP^{n-1} by $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ meaning the equivalence class of the corresponding element of S^{4n-1} under the action of $Sp(1)$, i. e., the quaternionic line defined by $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. As S^4 is QP^1 and the natural inclusion of QP_1 in QP^{n-1}: $\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} a \\ b \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ generates $\pi_4 QP^{n-1} = \mathbb{Z}$, we have:

CLAIM. — $X_n = p_n^{-1}(QP^1)$ is the total space of a principal $Sp(n-1)$ bundle over S^4 with $\pi_3(X_n) = \mathbb{Z}_n$.

Proof. — Immediate from the homotopy sequence of the pull back diagram.

CLAIM. — The bundle $Sp(n-1) \hookrightarrow X_n \to S^4$ reduces to a principal $Sp(1) \hookrightarrow P_n \to S^4$ with $\pi_3(P_n) = \mathbb{Z}_n$.

Proof. — Such a reduction exists if and only if there is a section σ of the associated bundle:

$$Sp(n-1)/Sp(1) \hookrightarrow X_n/Sp(1) \to S^4,$$

and then $P_n=\mu^{-1}(\sigma(S^4))$, where $\mu : X_n \to X_n/_{Sp(1)}$ is the projection (*see* [K-N]). In our case, such a section always exists because the fibre $Sp(n-1)/Sp(1)$ is at least 3-connected. That $\pi_3 P_n = \mathbb{Z}_n$ follows then immediately from the commutative

$$\begin{array}{ccc} Sp(1) & \subsetneq & Sp(n-1) \\ \downarrow & & \downarrow \\ P_n & \subsetneq & X_n \\ \downarrow & & \downarrow \\ S^4 & = & S^4 \end{array}$$

Diagram 2

Instead of seeking sections σ_n we only retain the following information:

(i) Each P_n lives in $Sp(1)\backslash Sp(n)$, its first column looks like $\begin{bmatrix} a \\ b \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and the $Sp(1)$ free action is from the *right* on, say, the *last* column.

Now we pull-back the P_n's by the Hopf-fibration $S^7 \xrightarrow{h} S^4$ as in the diagram:

$$\begin{array}{ccccc} & & Sp(1) & & Sp(1) \\ & & \downarrow & & \downarrow \\ Sp(1) & \subsetneq & \tilde{P}_n & \xrightarrow{H} & P_n \\ & & \downarrow{\scriptstyle \tilde{p}_n} & & \downarrow{\scriptstyle p_n} \\ S^3 & \subsetneq & S^7 & \xrightarrow[h]{} & S^4 \end{array}$$

Diagram 3

Recall that if $S^7 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{H}^2 \,|\, a\bar{a} + b\bar{b} = 1 \right\}$ then:

$$h\begin{pmatrix} a \\ b \end{pmatrix} = (2\,\bar{a}b,\ a\bar{a} - b\bar{b}) \equiv \begin{bmatrix} a \\ b \end{bmatrix}$$

and we write S^7 as $\begin{pmatrix} a \\ b \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with n quaternionic coordinates.

From information (i) and the above diagram we have that $\tilde{P}_n$ is a 10 dimensional submanifold of $Sp(n)$ with first column of the form $\begin{pmatrix} a \\ b \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, an $Sp(1)$ *right* action on the last column by quaternionic multiplication, producing S^7 as quotient and an $Sp(1)$ *left* free action with quotient P_n.

Therefore $\tilde{P}_n$ also comes about as a pull back of the following type:

$$\begin{array}{ccc} Sp(1) & & Sp(1) \\ \downarrow & & \downarrow \\ \tilde{P}_n & \xrightarrow{i_n} & Sp(n) \\ \downarrow & & \downarrow \\ S^7 & \xrightarrow{i_n} & Sp(n)/Sp(1) \end{array}$$

Diagram 4

where the i_n's are inclusions.

From the homotopy ladder of this diagram follows that $\pi_3 \tilde{P}_n = \mathbb{Z}$ and that $\tilde{i}_n : \pi_3 \tilde{P}_n \to \pi_3 Sp(n)$ is an isomorphism. This implies that the inclusion of $Sp(1)$ induces the following map of π_3's $: \mathbb{Z} \to \mathbb{Z}$ with $1 \to n$. Therefore the quotient $P_n = Sp(1) \setminus \tilde{P}_n$ has $\pi_3(P_n) = \mathbb{Z}_n$, and the bundle P_n over S^4 is classified by its size. In the next section we construct an infinite sequence of the P_n's, but before we do so we classify them.

2. S^3-bundles over S^7

The S^3-principal bundles over S^7 are classified by $\pi_6 S^3 = \mathbb{Z}_{12}$ and generated by $Sp(1) \ldots Sp(2) \to S^7$ [Hu].

We denote the total spaces of these bundles by E_i, with $E_1 = Sp(2)$, $E_2 =$ (twice $Sp(2)$), $\ldots$, $E_0 \equiv E_{12} =$ (twelve times $Sp(2)$) and diffeomorphic to $S^7 \times S^3$. Here, (twice $Sp(2)$), etc., means the pull back of $(Sp(2) \xrightarrow{Sp(1)} S^7)$ over S^7, by a map of degree two $f_2 : S^7 \to S^7$.

For the classification of $\tilde{P}_n$ as an S^3-principal bundle over S^7 we increase Diagram 3, page 6, as follows:

$$\begin{array}{ccccccccc}
 & & S^3 & & S^3 & & S^3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & Sp(1) \hookrightarrow & \tilde{P}_n & \dashrightarrow & P_n & \longrightarrow & S^7 & & \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h} & & \\
S^3 \hookrightarrow & & S^7 & \xrightarrow[h]{} & S^4 & \xrightarrow[f_n]{} & S^4 & \xrightarrow[j]{} & BS^3
\end{array}$$

Diagram 5

where f_n is a map of degree n and j is the inclusion of $S^4 \cong QP^1$ in $BS^3 = \underline{\lim}_n QP^n$. The classifying map for $\tilde{P}_n$ is $j \circ f_n \circ h$. We shall confuse maps and their homotopy classes when this causes no aparent disaster.

First we calculate $f_n \circ h$ in $\pi_7 S^4 \cong \mathbb{Z} + \mathbb{Z}_{12}$, using the following theorem of Hilton [H].

THEOREM. — *If g is in $\pi_m(S^n)$, $m \leqslant 3n-3$ and F_1, F_2 are in $\pi_n(X)$ then:*

$$(F_1 + F_2) \circ g = F_1 \circ g + F_2 \circ g + [F_1, F_2] H(g),$$

where $[F_1, F_2]$ denotes the Whitehead product of F_1 and F_2 and $H(g)$ the Hopf invariant of g.

In our case $g \equiv h$ in $\pi_7 S^4$ with $H(g) = 1$.

For calculating the Whitehead product we use the formula ([Hu], p. 330).

$$[\iota, \iota] = 2h - \varepsilon \Sigma(\xi),$$

where ι is the identity element of $\pi_4 S^4 = \mathbb{Z}$, ε is $+1$ or -1 depending on orientation conventions and $\Sigma(\xi)$ is the suspension of $\xi : S^6 \to S^3$ that generates $\pi_6 S^3 = \mathbb{Z}_{12}$.

It follows from [Hu], p. 330 that $\Sigma(\xi)$ generates the torsion part of $\pi_7 S^4$ and h generates the free part. We shall simplify these notations to $h \equiv (1, 0)$ and $\Sigma(\xi) \equiv (0, 1)$ in $\mathbb{Z} \oplus \mathbb{Z}_{12}$. For the time being we shall leave $\varepsilon \equiv \pm 1$. The spheres are suspensions and therefore co-H-spaces, so the Whitehead product is bilinear when $X = S^4$.

From all the above it follows:

$$f_1 \circ h = (1, \pm 1),$$
$$f_2 \circ h = (\iota + \iota) \circ h = 2\iota h + [\iota, \iota] = (2, 0) + \{(2, 0) \pm (0, 1)\} = (4, \pm 1),$$
$$f_3 \circ h = (\iota + 2\iota) \circ h = \iota h + 2\iota h + [\iota, 2\iota] = (3, 0) + 2(2, \pm 1) = (7, \pm 2), \text{ etc.,}$$
$$f_n \circ h = (3n - 2, \pm(n-1)) \text{ in } \mathbb{Z} + \mathbb{Z}_{12}, \quad \text{for all } n.$$

Before determining $j \circ f_n \circ h$ observe that j is essentially the boundary map ∂ of $S^3 \subsetneq S^7 \to S^4$ and that $\partial : \mathbb{Z} + \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ maps (a, b) to b, i. e., j is the projection of $\pi_7 S^4$ to its torsion component.

We may now decide the exact value of ε by testing on $Sp(2)$: In our notation, $Sp(2) \cong E_1$, $j \circ f_2 \circ h$ is the generator 1 in $\mathbb{Z}_{12}$. Therefore $\varepsilon = 1$ and we have:

COROLLARY. — *$\tilde{P}_n \cong E_{(n-1) \bmod 12}$ for $n \geqslant 3$.*

In particular $\tilde{P}_{13}$, $\tilde{P}_{25}$, $\tilde{P}_{12k+1}$ are isomorphic to the trivial bundle $S^7 \times S^3$.

Now we are ready to give a concrete description for each $\tilde{P}_n$ and consequently each P_n, $n = 3, 4, \ldots$

First we construct $\tilde{P}_3$ a 10-dimensional submanifold of $Sp(3)$, invariant under $3\,Sp(1)$ acting from the left:

$$\tilde{P}_3 = \left\{ \begin{pmatrix} a & -b|b|^2 & x \\ b & b\bar{a}b & y \\ 0 & a\sqrt{1+|b|^2} & z \end{pmatrix} \text{ in } Sp(3) \right\}.$$

I. e., $\tilde{P}_3$ is the bundle of quaternionic 2-frames over S^7 with first vector the 2nd column.

The invariance with respect to the $3\,Sp(1)$-action works because each element of the 2nd column is a product of the form $b\bar{a}b$ or a or b multiplied by a real number, always starting with a or b (not $\bar{a}$ or $\bar{b}$) and having an *odd* number of a's and b's.

This $\tilde{P}_3$ is a principal S^3-bundle over S^7, by projecting to its first column, i. e., S^3 (or $Sp(1)$) acts by quaternionic multiplication from the *right* on the last column.

Now we construct $\tilde{P}_4$ as a 10-dimensional submanifold of $Sp(4)$ invariant under S^3 action on the last column from the right and therefore a principal S^3 bundle over S^7:

$$\tilde{P}_4=\left\{\begin{pmatrix} a & -b|b|^2L^{-1} & 0 & x \\ b & b\bar{a}bL^{-1} & 0 & y \\ 0 & a|a|^2L^{-1} & -b & z \\ 0 & a\bar{b}aL^{-1} & a & w \end{pmatrix} \text{ in } Sp(4)\right\},$$

where $L=\sqrt{|a|^4+|b|^4}$. Observe that the conditions for the first three columns to be mutually orthonormal are satisfied and that all entries are smooth in a and b.

Next we give an inductive process for constructing $\tilde{P}_{n+1}$ from $\tilde{P}_n$, and illustrate each step by performing gradually the construction of $\tilde{P}_5$ from $\tilde{P}_4$.

STEP 1

Forget all divisions by the lengths of the columns and also forget the last column of x_i's:

$$\begin{pmatrix} a & -b|b|^2 & 0 \\ b & b\bar{a}b & 0 \\ 0 & a|a|^2 & -b \\ 0 & a\bar{b}a & a \end{pmatrix},$$

STEP 2

Cut off the first two rows and the first column:

$$\begin{pmatrix} a|a|^2 & -b \\ a\bar{b}a & a \end{pmatrix},$$

STEP 3

Multiply each element of the first column by $a\bar{b}$ from the left and put the result as a new first column:

$$\begin{pmatrix} a\bar{b}a|a|^2 & a|a|^2 & -b \\ (a\bar{b})^2 a & a\bar{b}a & a \end{pmatrix}.$$

STEP 4

Put $-b$ over the second column and af_k over the first column where f_k is a function of $|a|^2$ and $|b|^2$ that makes the product of these two columns equal to zero. I. e.,

$$\overline{(\text{Col})_\alpha}\cdot(\text{Col})_\beta=0,$$

and complete the first row with zeroes:

$$\begin{pmatrix} af_k & -b & 0 \\ a\bar{b}a|a|^2 & a|a|^2 & -b \\ (a\bar{b})^2 a & a\bar{b}a & a \end{pmatrix}.$$

Here, $f_k \equiv f_0 = |a|^4$.

STEP 5

Put back the piece that we took out at step 2 completing with zeroes down the first column and the first two rows. Put back the last column of the x_i's:

$$\begin{pmatrix} a & -b|b|^2 & 0 & 0 & x_1 \\ b & b\bar{a}b & 0 & 0 & x_2 \\ 0 & a|a|^4 & -b & 0 & x_3 \\ 0 & a\bar{b}a|a|^2 & a|a|^2 & -b & x_4 \\ 0 & (a\bar{b})^2 a & a\bar{b}a & a & x_5 \end{pmatrix}$$

STEP 6

Divide each column by its length to become unitary.

Observe now the following: The last columns are essentially constant except for the zeroes that one adds on the top places. Denote by L_1 the length of the 2nd before the last column of $\tilde{P}_n$, by L_2 the length of the third before the last, etc., by L_{n-4} the length of the $(n-3)$-before the last column, which is the third column from the left. Let L denote the length of the second column.

Then we have the following:

PROPOSITION:

$$L_1^2 = |a|^4 + |b|^2 = f_0 + |b|^2,$$
$$L_2^2 = |a|^{10} + |a|^6|b|^2 + |b|^2 = f_1 + |b|^2,$$
$$L_3^2 = f_2 + |b|^2$$
$$\vdots$$
$$L_{n-4}^2 = f_{n-5} + |b|^2,$$
$$L^2 = f_{n-4} + |b|^4,$$

where:

$$f_1=|a|^6(|a|^4+|b|^2)=|a|^6 L_1^2,$$
$$f_2=|a|^8 L_1^2 L_2^2,$$
$$f_3=|a|^{10} L_1^2 L_2^2 L_3^2,$$
$$\vdots$$
$$f_{n-4}=|a|^{2n-4} L_1^2 \dots L_n^2.$$

Also, for $n \geqslant 2$ *we have:*

$$f_{n+1}=|a|^2 f_n^2+|a|^4|b|^2 f_{n-1}^2+|a|^6|b|^4 f_{n-2}^2+\dots$$
$$+|a|^{2n}|b|^{2n-2} f_1^2+|a|^{2n+10}|b|^{2n}+|a|^{2n+6}|b|^{2n+2}.$$

The proof of this proposition is elementary, though quite tedious, and is omitted.

Observe that all L_i's and f_i's are smooth and that $f_i\begin{pmatrix} a \\ 0 \end{pmatrix}=1$ for all i, so L_i^2 is always positive and therefore bounded away from zero, with $L_i^2\begin{pmatrix} 0 \\ b \end{pmatrix}=1$.

From the clasification of $\tilde{P}_n$ we have that the first trivial one is $\tilde{P}_{13}$ which is illustrated below in matrix form:

a	$-b\|b\|^2 L^{-1}$	0	0	$\dots$	0	z_1
b	$b\bar{a}\bar{b} L^{-1}$	0	0	$\dots$	0	z_2
0	$af_8 L^{-1}$	$-b L_9^{-1}$	0	$\dots$	0	z_3
0	$(a\bar{b}) af_7 L^{-1}$	$af_7 L_9^{-1}$	$-b L_8^{-1}$	$\dots$	0	z_4
0	$(a\bar{b})^2 af_6 L^{-1}$	$(a\bar{b}) af_6 L_9^{-1}$	$af_6 L_8^{-1}$	$\dots$	0	z_5
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	$(a\bar{b})^9 a\|a\|^2 L^{-1}$	$(a\bar{b})^8 a\|a\|^2 L_9^{-1}$	$(a\bar{b})^7 a\|a\|^2 L_8^{-1}$	$\dots$	$-b$	z_{12}
0	$(a\bar{b})^{10} a L^{-1}$	$(a\bar{b})^9 a L_9^{-1}$	$(a\bar{b})^8 a L_8^{-1}$	$\dots$	a	z_{13}

An element of $\tilde{P}_{13}$.

In paragraph 4 we attempt the construction of a global section for $\tilde{P}_{13}$, i.e., an explicit diffeomorphism with $S^7 \times S^3$. The formulas, however, depend on a homotopy which we have not been able to write down explicity.

3. $SO(4)$-bundles and exotic actions

It follows from the construction of $\tilde{P}_n$ that there is a free $SO(4)$-action on $\tilde{P}_n/\mathbb{Z}_2$ with quotient S^4 (*see* also [R₂]). First we look at $\tilde{P}_3$.

Write $SO(4)$ as the semi-direct product $S^3 \times SO(3)$ with the following linear action on $\mathbb{R}^4$:

$$(p, \theta) \mapsto (\xi \mapsto p\,\theta\xi\bar{\theta})$$

and the following multiplication:

$$(p, \theta)(q, \eta) = (p\,\theta\,q\,\bar{\theta}, \theta\eta),$$

where all products are products of quaternions.

Let $\mathbb{Z}_2 = \{1, -1\}$ act on the last column of $\tilde{P}_3$ and take the quotient $P'_3 \equiv \tilde{P}_3/\mathbb{Z}_2$. The *right* free $SO(4)$*—action* on P'_3 is then as follows:

$$\begin{bmatrix} a & -b|b|^2 & x_1 \\ b & b\bar{a}b & x_2 \\ 0 & a\sqrt{1+|b|^2} & x_3 \end{bmatrix} * (p, \theta) := \begin{bmatrix} \bar{\theta}ap\theta & \bar{\theta}(-b|b|^2)\,p\theta & \bar{\theta}x_1 \\ \bar{\theta}bp\theta & \bar{\theta}(b\bar{a}b)\,p\theta & \bar{\theta}x_2 \\ 0 & \bar{\theta}(a\sqrt{1+|b|^2})\,p\theta & \bar{\theta}x_3 \end{bmatrix}.$$

In other words we multiply each element of the first two columns by the real 4×4 matrix (p, θ) from the right and each element of the last column by $\bar{\theta}$ from the left. Although θ is not well defined as a quaternion (it is the class $\{\theta, -\theta\}$ that is well defined), having divided by $\mathbb{Z}_2$ removes this ambiguity and the $SO(4)$ action is well defined. To check its freeness just look at the first and last column: at least one of a, b and one of x_1, x_2, x_3 is different from zero.

The quotient four-dimensional manifold is a homology four sphere as follows from the homotopy sequence of the fibration:

$$SO(4) \hookrightarrow P'_3 \to M^4.$$

A map from M^4 to S^4 may be constructed as follows:

$$\text{Orbit of } \begin{bmatrix} a & -b|b|^2 & x_1 \\ b & b\bar{a}b & x_2 \\ 0 & a\sqrt{1+b^2} & x_3 \end{bmatrix}$$

$$\mapsto (6^3\,5^{-5/2}\,\bar{x}_3\,a\bar{b}x_3,\ \pm(1-6^6\,5^{-5}\,|a|^2\,|b|^{10})^{1/2}),$$

with the "+" sign if $|a| \leqslant 1/\sqrt{6}$ and the "−" sign if $|a| \geqslant 1/\sqrt{6}$.

The coefficients and powers of $|a|$ and $|b|$ are a consequence of $|x_3| = |b|^2$ and the maximum value of $|\bar{x}_3 a\bar{b} x_3|$.

Now we see that M^4 is the union of two 4-discs glued along their boundaries by the identity. It follows that M^4 is diffeomorphic to S^4.

The projections of the $SO(4)$-bundles, to be considered further on, onto S^4 are completely analogous.

The classification of P'_3, and in general P'_n, as an $SO(4)$ bundle over S^4 may be carried out as follows:

From the pull back Diagram 4 one has that $\tilde{i}_* : \pi_3(\tilde{P}_n) \to \pi_3 Sp(n) = \mathbb{Z}$ is an isomorphism and that the inclusion of S^3 in $\tilde{P}_n$ as any of the diagonal elements induces an isomorphism on π_3's. Let now $\pi_3 SO(4) = \mathbb{Z} \oplus \mathbb{Z}$ be generated by $(1, 0)$ and $(0, 1)$ where $(1, 0)$ comes from the S^3-part, i. e., the p-component and $(0, 1)$ comes from the $SO(3)$-part, i. e., the θ-component.

Therefore, the inclusion of a $SO(4)$ orbit in P'_3 induces the following map on π_3's:

$$(1, 0) \mapsto 2,$$
$$(0, 1) \mapsto -1,$$

i. e., $(a, b) \mapsto 2a - b$ in $\mathbb{Z}$.

The relevant part of the homotopy sequence of $SO(4) \ldots P'_3 \to S^4$ is:

$$\pi_4 S^4 \xrightarrow{\partial} \pi_3 SO(4) \xrightarrow{i} \pi_3 P'_3 \to 0 = \pi_3 S^4$$

and Image $\partial = \operatorname{Ker} i_*$. It follows the that Ker i_* is generated by $(1, 2)$ or by $(-1, -2)$, and therefore $\partial(1) = (1, 2)$ or $(-1, -2)$. As ∂ is essentially the classifying map at homotopy level for the bundles $SO(4) \hookrightarrow P_{m, n} \to S^4$ we have that $P'_3 \equiv P_{1, 2}$. Notations and conventions are the same as in [R_2].

This same reasoning implies that P'_n with the analogous $SO(4)$-action ($\xi * (p, \theta) := \bar{\theta}\xi p \theta$ for every entry of each column except the last and $(\pm x) * (p, \theta) = \pm \bar{\theta} x$ for the entries of the last column) is the principal $SO(4)$ bundle $P_{1, n-1}$ over S^4.

The following theorem was proved by Eells and Kuiper in [E-K] where different conventions were used.

THEOREM. — *The associated 3-sphere bundle to $P_{m, n}$ has total space homeomorphic to S^7 if and only if $m = 1$. This seven sphere will have an* *exotic differentiable structure if and only if $n(n+1)$ is not a multiple of 56. In fact $n(n+1)$ mod 56 provides a complete classification of the 7-spheres that appear as S^3-bundles over S^4 with structure group $SO(4)$. These are exactly sixteen out of the twentyeight 7-spheres.*

In [G-M] GROMOLL and MEYER constructed an exotic 7-sphere as the free quotient of $Sp(2)$ by an S^3 action. In [R_2], this action was seen from the angle of principal $SO(4)$-bundles over S^4. We want to generalize this point of view to include all sixteen of these homotopy 7-spheres.

LEMMA 1. — *If S^3 acts on $\tilde{P}_{k+1}$ by $q * \xi = \bar{q}\xi q$ for each element of any column except the last and by $q * x = \bar{q}x$ for each element of the last column then $\tilde{P}_{k+1}/S^3$ is diffeomorphic to $P'_{k+1} \times_{SO(4)} S^3$, where $SO(4)$ acts on S^3 by $(p, \theta) * q := \bar{\theta}\bar{p}q\theta$.*

Proof. — Let (A, X) represent an element of $\tilde{P}_{k+1}$ where A stands for any column except the last and X is the last column. If q is in S^3 we denote by Aq the column with entries $a_i q$ where a_i are the entries of A. Similarly with $\bar{q}Aq$, $\bar{q}X$, etc.

The following maps are smooth and inverse to each other:

$$\Phi : \quad P'_{k+1} \times_{SO(4)} S^3 \to \tilde{P}_{k+1}/S^3,$$

with:

$$\Phi\{(A, X), q\} = [(Aq, X)]$$

and:

$$\psi : \quad \tilde{P}_{k+1}/S^3 \to P'_{k+1} \times_{SO(4)} S^3,$$

by:

$$\psi[(A, X)] = \{(A, X), 1\},$$

where [ ] and { } denote the class in $\tilde{P}_{k+1}/S^3$ and the class in $P'_{k+1} \times_{SO(4)} S^3$ respectively.

The map Φ is well defined because:

$$\{(A, X), q\} = \{(\bar{\theta}A\,p\theta, \bar{\theta}X), \bar{\theta}\,\bar{p}q\theta\}$$

which is mapped by Φ to:

$$[(\bar{\theta}Aq\theta, \bar{\theta}X)] = [(Aq, X)] \equiv \Phi\{(A, X), q\}.$$

Similarly $[(A, X)] = [(\bar{\theta}A\theta, \bar{\theta}X)]$ which is mapped by ψ to:

$$\{(\bar{\theta}A\theta, \bar{\theta}X), 1\} = \{(A, X), 1\} \equiv \psi[(A, X)].$$

Finally,

$$\psi\circ\Phi\{(A, X), q\}=\psi[(Aq, X)]=\{(Aq, X), 1\}=\{(A, X), q\}$$

and

$$\Phi\circ\psi[(A, X)]=\Phi\{(A, X), 1\}=[(A, X)] \text{Q.E.D.}$$

This together with the Eeels-Kuiper theorem and the classification of the $\tilde{P}_n$'s implies that some of the principal S^3-bundles over each of the Eeels-Kuiper Σ^7's have the standard differentiable structure, i. e., their total space is diffeomorphic to the space of the corresponding S^3-principal bundle over S^7. One may say this in a different way: There exist nonstandard free actions of S^3 on $E_0 \cong S^7\times S^3$, $E_1\cong Sp(2), \ldots, E_{11}$ with quotients exotic seven spheres.

Before we straighten out the book-keeping we comment that the above statement is neither very remarkable nor peculiar to our kind of argument. For example, the vanishing of the group $L_{11}(0)=L_3(0)$ (*see* [W]) implies that for *all* exotic Σ^7's one has that $\Sigma^7\times S^3$ is diffeomorphic to $S^7\times S^3$.

We owe this observation R. Schultz. However, in our case, the actions of S^3 are explicit and the diffeomorphims between certain $\Sigma_i^7\times S^3$ and $S^7\times S^3$ should not be too complicated.

So, what seems interesting is that some of these actions have a good chance to be written down explicitly.

Now back to our book-keeping.

(*a*) The manifold $E_0\cong S^7\times S^3$ is diffeomorphic to $\tilde{P}_{12k+1}$ for all k and from the lemma we have $\tilde{P}_{12k+1}/S^3\cong\Sigma^7_{[12k(12k+1)]}$ where $[12k(12k+1)]$ is the Eeels-Kuiper index of the Σ^7's. The possible indices are, 0, 2, 6, 12, 14, 16, 20, 26, 28, 30, 34, 40, 42, 44, 48, 54.

The number $12k\,(12k+1)$ is divisible by 4 and therefore so must be its residue mod 56. In other words the only possible candidates are the spheres with indices 0, 12, 16, 20, 28, 40, 44, 48.

In fact all possibilities occur, each for infinitely many values of k. For example:

Value of k		Index of sphere obtained	Value of k		Index of sphere obtained
1	$\mapsto$	44	7	$\mapsto$	28
2	$\mapsto$	40	8	$\mapsto$	16
4	$\mapsto$	0	9	$\mapsto$	12
5	$\mapsto$	20	20	$\mapsto$	48

COROLLARY. — *There exist free actions of* S^3 *on* $S^7 \times S^3$ *with quotient each of the exotic seven-spheres* $\Sigma^7_{[s]}$ *for* $s=12$, 16, 20, 28, 40, 44, 48.

(*b*) The manifold $E_1 \cong Sp(2)$ is diffeomorphic to $\tilde{P}_{12k+2}$ for all k. With the same reasoning as above there exist exotic S^3 actions on $Sp(2)$ with quotient $\Sigma^7_{[(12k+1)(12k+2)]}$. The product 1.2 is divisible by 2, but not by 4, so the only possible indices are 0, 2, 6, 14, 26, 30, 34, 42, 54.

A quick checking implies again that all possibilities occur, each for infinitely many values of k.

Value of k		Index of sphere obtained	Value of k		Index of sphere obtained
0	$\mapsto$	2	4	$\mapsto$	42
1	$\mapsto$	14	5	$\mapsto$	30
2	$\mapsto$	34	6	$\mapsto$	26
3	$\mapsto$	6	13	$\mapsto$	54

COROLLARY. — *There exist free actions of* S^3 *on* $Sp(2)$ *with quotient the exotic seven sphere* $\Sigma^7_{[r]}$ *for* $r=2$, 6, 14, 26, 30, 34, 42, 54.

The reasoning for each of the remaining E_i's is the same. Each E_i, $i=2, \ldots, 11$ is diffeomorphic to $\tilde{P}_{12k+i+1}$ for all $k=1, 2, \ldots$, and we get as quotients of the S^3 actions on E_i one of the above two sets of exotic seven spheres. The set indexed by 12, 16, 20, 40, 44 and 48 if $i(i+1)$ is divisible by 4 or the set indexed by 2, 6, 14, 26, 30, 34, 42 and 54 if $i(i+1)$ is not divisible by 4.

COROLLARY. — *There exist free* S^3 *actions on each of* E_2, E_5, E_6, E_9, E_{10} *with quotient each of the seven spheres* $\Sigma^7_{[r]}$, $r=2$, 6, 14, 26, 30, 34, 42, 54. *And there exist free actions of* S^3 *on each of* E_3, E_4, E_7, E_8, E_{11} *with quotient each of the* $\sum^7_{[s]}$, $s=12$, 16, 20, 40, 44, 48.

To complete this section we remark that one can describe explicitly all principal $SO(4)$ bundles over S^4 using the $\tilde{P}_i$'s (*see* also [S], § 26.6, [J-W] and [T]).

For example, the free $SO(4)$ action on $\tilde{P}_{n+1}/\mathbb{Z}_2$ by $(p, \theta) * (A, X) := (\theta A, \theta X \bar{\theta} \bar{p})$ with the notation of Lemma 1, has quotient S^4 and projection: $(2\bar{a}b, a\bar{a}-b\bar{b})$. The $\mathbb{Z}_2$ action changes the sign of the A-part.

The inclusion i of an $SO(4)$-orbit induces the following map on $\pi'_3 S$:

$$i_* : \quad \mathbb{Z}+\mathbb{Z} \to \mathbb{Z}$$

with:

$$i_*(1, 0) = -1$$

and:

$$i_*(0, 1) = n-1.$$

Therefore the image of the classifying map:

$$\partial : \quad \pi_4 S^4 \to \pi_3 SO(4)$$

is generated by $(n-1, 1)$, so in our notation, $\tilde{P}_n/\mathbb{Z}_2$ with the above described action is the principal $SO(4)$-bundle $P_{n-1,\,1}$ over S^4.

The bundles $P_{n,\,0}$ have total spaces $P_n \times_{S^3} SO(4)$ with the obvious $SO(4)$ action from the right and $P_{0,\,n}$ have total spaces $(P_n/\mathbb{Z}_2) \times_{SO(3)} SO(4)$ where the $\mathbb{Z}_2$ action on P_n changes the sign of the last column.

The bundles $P_{1,\,-n}$ are obtained from $\tilde{P}_n/\mathbb{Z}_2$ in a similar way: $p(\theta) * (A, X) = ((p\theta) A \bar{\theta},\ (p\theta) X)$.

The bundles $P_{m,\,n}$ for m and n other than 0, 1, -1 can be obtained as quotients of $P'_{m,\,n}$ by $\mathbb{Z}_2$, where $P'_{m,\,n}$ are the $\mathrm{Spin}(4) \cong S^3 \times S^3$ principal bundles over S^4.

The homotopy ladder of the pull-back diagram:

$$\begin{array}{ccc} S^3 \times S^3 & & S^3 \times S^3 \\ \downarrow & & \downarrow \\ P'_{m,\,n} & \xrightarrow{\Delta'} & P_m \times P_n \\ \downarrow & & \downarrow \\ S^4 & \xrightarrow{\Delta} & S^4 \times S^4 \end{array}$$

implies immediately that $P'_{m,\,n}$ is indeed the pull-back by the diagonal Δ of the Cartesian product $P_m \times P_n$.

The same construction applies to the $\mathrm{Spin}(4)$ and $SO(4)$ bundles over S^7: just replace S^4 by S^7 and P_m, P_n by $\tilde{P}_m$, $\tilde{P}_n$.

In order to write down the action of $SO(4)$ we find it more convenient to use the description:

$$SO(4) \cong S^3 \times_{\mathbb{Z}_2} S^3,$$

with $(-1)(p, q)=(-p, -q)$, denoting the elements of $SO(4)$ by $\{p, q\}$ now, rather than the semidirect product $S^3 \times SO(3)$ we used up to now.

In the direct product case the linear action on $\mathbb{R}^4$ is:

$$\{p, q\}(\xi)=p\xi\bar{q}.$$

As an illustration we have:

$$P'_{2,3}=\left\{\begin{bmatrix} a & x & -b|b|^2 & x_1 \\ b & y; & b\bar{a}b & y_1 \\ 0 & 0 & a\sqrt{1+|b|^2} & z_1 \end{bmatrix}\right\},$$

where the first two columns are an element of $Sp(2)$, the first, third and fourth column are an element of $\tilde{P}_3$ and there is no constraint between the second and fourth columns. The bracket denotes the quotient by S^3 from the left acting on all columns. Therefore, $P_{2,3}$ is $P'_{2,3}/\mathbb{Z}_2$, with $\mathbb{Z}_2$ multiplying by -1 the second and fourth columns and where the element $\{p, q\}$ of $SO(4)$ acts by:

$$\begin{bmatrix} a & x\bar{p} & -b|b|^2 & x_1\bar{q} \\ b & y\bar{p}; & b\bar{a}b & y_1\bar{q} \\ 0 & 0 & a\sqrt{1+|b|^2} & z_1\bar{q} \end{bmatrix}.$$

4. A trivialization of $\tilde{P}_{13}$

Let $U=\left\{\begin{pmatrix} a \\ b\end{pmatrix} \text{ in } S^7 \,|\, a\neq 0\right\}$ and $V=\left\{\begin{pmatrix} a \\ b\end{pmatrix} \text{ in } S^7 \,|\, b\neq 0\right\}$. Then $U\cup V=S^7$ and $U\cap V$ is diffeomorphic to $S^3\times S^3\times(0, \pi/2)$. In fact, if $\alpha : U\cap V\to S^3\times S^3\times(0, \pi/2)$ is $\begin{pmatrix} a \\ b\end{pmatrix}\mapsto(a|a|^{-1}, b|b|^{-1}, \cos^{-1}|a|)$ and if $\beta : S^3\times S^3\times(0, \pi/2)\to U\cap V$ is:

$$((A, B), \theta)\mapsto\begin{pmatrix} \cos\theta A \\ \sin\theta B\end{pmatrix}$$

then $\alpha\circ\beta$ and $\beta\circ\alpha$ are the identity because $\sin(\cos^{-1}|a|)=|b|$.

We first construct a section over U by solving the linear system that consists of the 12 equations:

$$\overline{(\mathrm{Col}_i)}\,(\mathrm{Col}_{13})=0, \qquad i=1, \ldots, 12$$

plus a 13th:

$$|\mathrm{Row}_\alpha|=1$$

for one convenient α between 1 and 13:

$$(1) \qquad \bar{a}z_1+\bar{b}z_2=0 \;\Rightarrow\; z_1=-a\bar{b}|a|^{-2}z_2,$$

$$(2) \qquad -\bar{b}z_{12}+\bar{a}z_{13}=0 \;\Rightarrow\; z_{13}=a\bar{b}|a|^{-2}z_{12},$$

$\vdots$

$$(11) \quad -\bar{b}z_3+\bar{a}f_7z_4+\overline{(a\bar{b})}af_6z_5+\ldots+\overline{(a\bar{b})^6}af_1z_{10}+\overline{(a\bar{b})^7a}|a|^4z_{11} + \overline{(a\bar{b})^8a}|a|^2z_{12}+\overline{(a\bar{b})^9}az_{13}=0,$$

$$(12) \quad -\bar{b}|b|^2z_1+\bar{b}a\bar{b}z_2+\bar{a}f_8z_3+\overline{(a\bar{b})}af_7z_4+\ldots +\overline{(a\bar{b})^7}af_1z_{10}+\overline{(a\bar{b})^8a}|a|^4z_{11} +\overline{(a\bar{b})^9a}|a|^2z_{12}+\overline{(a\bar{b})^{10}}az_{13}=0,$$

$$(13) \qquad |b|^2+|b|^4|a|^2L^{-2}+|z_2|^2=1.$$

Recall from Step 6, in paragraph 2, that L and L_i are lengths of columns.

We first solve (2) and substitute in (3), then solve (3) and substitute in (4), etc., till at the end we have $z_4, z_5, \ldots, z_{13}$ as functions of z_3:

$$z_4=a\bar{b}|a|^{-2}L_8^{-2}z_3,$$
$$z_5=(a\bar{b})^2|a|^{-4}L_8^{-2}z_3,$$
$$z_6=(a\bar{b})^3|a|^{-6}(L_6L_7L_8)^{-2}z_3,$$
$$\vdots$$
$$z_{12}=(a\bar{b})^9|a|^{-18}(L_1\ldots L_8)^{-2}z_3,$$
$$z_{13}=(a\bar{b})^{10}|a|^{-20}(L_1\ldots L_8)^{-2}z_3.$$

Put these together with $z_1=-a\bar{b}|a|^{-2}z_2$ in equation (12) and obtain:

$$z_3=-(a\bar{b})^2|a|^{-4}L_9^{-2}z_2,$$
$$z_4=-(a\bar{b})^3|a|^{-6}(L_8 L_9)^{-2}z_2,$$
$$\vdots$$
$$z_{11}=-(a\bar{b})^{10}|a|^{-20}(L_1\ldots L_9)^{-2}z_2,$$
$$z_{12}=-(a\bar{b})^{11}|a|^{-22}(L_1\ldots L_9)^{-2}z_2,$$
$$z_{13}=-(a\bar{b})^{12}|a|^{-24}(L_1\ldots L_9)^{-2}z_2.$$

From (13) we have $|z_2|^2=|a|^2(1-|b|^4L^{-2})$ and by the proposition in paragraph 2, $|z_2|^2=|a|^2 f_9 L^{-2}$. As $f_9=|a|^{22}(L_1\ldots L_9)^2$,

$$|z_2|=|a|^{12}(L_1\ldots L_9)L^{-1}.$$

Let now $z_2:=-\bar{a}^{12}L_1\ldots L_9L^{-1}$.

We chose this value so that the transition function, to be determined later, can be factored through to S^6.

Putting this value of z_2 in the above equations we obtain a section:

$$X:\quad U\times S^3\to\tilde{P}_{13},$$

with:

$$\left(\begin{pmatrix}a\\b\end{pmatrix},g\right)\mapsto X\left(\begin{pmatrix}a\\b\end{pmatrix},g\right),$$

whose coordinates x_i, $i=1,\ldots,13$ of the last column are:

$$x_1=(a\bar{b})\bar{a}^{12}|a|^{-2}L_1\ldots L_9L^{-1}g,$$
$$x_2=-\bar{a}^{12}L_1\ldots L_9L^{-1}g,$$
$$x_3=(a\bar{b})^2\bar{a}^{12}|a|^{-4}L_1\ldots L_8(L_9L)^{-1}g,$$
$$x_4=(a\bar{b})^3\bar{a}^{12}|a|^{-6}L_1\ldots L_7(L_8L_9L)^{-1}g,$$
$$\vdots$$
$$x_{11}=(a\bar{b})^{10}\bar{a}^{12}|a|^{-20}(L_1\ldots L_9L)^{-1}g,$$
$$x_{12}=(a\bar{b})^{11}\bar{a}^{12}|a|^{-22}(L_1\ldots L_9L)^{-1}g,$$
$$x_{13}=(a\bar{b})^{12}\bar{a}^{12}|a|^{-24}(L_1\ldots L_9L)^{-1}g.$$

To obtain a section Y on $V=\{b\neq 0\}$ we solve the same set of equations (1)-(12), being allowed to divide by b now:

$$(1) \Rightarrow z_2=-b\bar{a}|b|^{-2}z_1,$$
$$(2) \Rightarrow z_{12}=b\bar{a}|b|^{-2}z_{13}.$$

Substitute in (3), etc.,

$$z_{11}=(b\bar{a})^2|b|^{-4}z_{13},$$
$$z_{10}=(b\bar{a})^3|b|^{-6}L_1^2 z_{13},$$
$$z_9=(b\bar{a})^4|b|^{-8}(L_1 L_2)^2 z_{13},$$
$$\vdots$$
$$z_3=(b\bar{a})^{10}|b|^{-20}(L_1 \dots L_8)^2 z_{13}.$$

Now put everything in (12):

$$z_1=(b\bar{a})^{11}|b|^{-22}(L_1 \dots L_9)^2 z_{13}$$

and:

$$z_2=-(b\bar{a})^{12}|b|^{-24}(L_1 \dots L_9)^2 z_{13}.$$

It is easier to get the length of z_{13} from the last coordinate of X than directly from the matrix $\tilde{P}_{13}$:

$$|z_{13}|=|b|^{12}(L_1 \dots L_9 L)^{-1}.$$

For the same reason as in choosing z_2 we set now:

$$z_{13}:=\bar{b}^{12}(L_1 \dots L_9 L)^{-1}.$$

The coordinates y_i, $i=1, \dots, 13$ of the last column of the section

$Y: V\times S^3 \to \tilde{P}_{13}$ with $\left(\begin{pmatrix} a \\ b \end{pmatrix}, q\right) \to Y\left(\begin{pmatrix} a \\ b \end{pmatrix}, q\right)$ are:

$$y_1=(b\bar{a})^{11}\bar{b}^{12}|b|^{-22}(L_1 \dots L_9)L^{-1}q,$$
$$y_2=-(b\bar{a})^{12}\bar{b}^{12}|b|^{-24}(L_1 \dots L_9)L^{-1}q,$$
$$y_3=(b\bar{a})^{10}\bar{b}^{12}|b|^{-20}(L_1 \dots L_8)(L_9 L)^{-1}q,$$
$$y_{11}=(b\bar{a})^2\bar{b}^{12}|b|^{-4}(L_1 \dots L_9 L)^{-1}q,$$
$$y_{12}=(b\bar{a})\bar{b}^{12}|b|^{-2}(L_1 \dots L_9 L)^{-1}q,$$
$$y_{13}=\bar{b}^{12}(L_1 \dots L_9 L)^{-1}q.$$

The transition function $\lambda_{UV} : U \cap V \to S^3$ is therefore:

$$\lambda_{UV}\begin{pmatrix} a \\ b \end{pmatrix} = a^{12}(b\bar{a})^{12}\bar{b}^{12}\,|a|^{-24}\,|b|^{-24}.$$

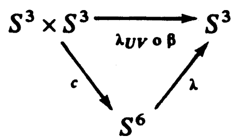
We use the map β defined at the beginning of this section to pass to $S^3 \times S^3$:

$$\lambda_{UV} \circ \beta : S^3 \times S^3 \times (0, \pi/2) \to S^3,$$
$$((A, B), \theta) \to A^{12}(B\bar{A})^{12}\bar{B}^{12},$$

i. e., it is independent of θ.

Recall that there is a continuous projection from $S^3 \times S^3$ to S^6, the equator of S^7, defined by collapsing $1 \times S^3$ and $S^3 \times 1$ to the same point, the base-point of S^6.

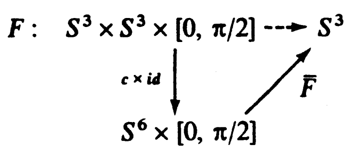
Call this map c and observe that $\lambda_{UV} \circ \beta$ factors through c to a map $\lambda : S^6 \to S^3$ making the following diagram commutative.



As $\tilde{P}_{13}$ is trivial, there is a homotopy $\bar{F} : S^6 \times [0, \pi/2] \to S^3$ with

$$\bar{F}_0\left(c\begin{pmatrix} a \\ b \end{pmatrix}\right) = 1 \text{ and } \bar{F}_{\pi/2}\left(c\begin{pmatrix} a \\ b \end{pmatrix}\right) = \lambda\left(c\begin{pmatrix} a \\ b \end{pmatrix}\right).$$

This homotopy is lifted to



the obvious way. We may take F to be smooth with $F_\theta\begin{pmatrix} a \\ b \end{pmatrix} = 1$ for all

θ in $[0, \pi/6]$, $F_\theta\begin{pmatrix} a \\ b \end{pmatrix} = a^{12}(b\bar{a})^{12}\bar{b}^{12}\,|a|^{-24}\,|b|^{-24}$ for all θ in $[\pi/3, \pi/2]$.

If $\theta=\cos^{-1}|a|$ in $[0, \pi/2]$ we have a global section of $\tilde{P}_{13}$ whose last column is:

$$w_1 =(a\bar{b})\bar{a}^{12}|a|^{-2}(L_1 \ldots L_9)L^{-1}F_\theta\begin{pmatrix}a\\b\end{pmatrix},$$

$$w_2 =-\bar{a}^{12}(L_1 \ldots L_9)L^{-1}F_\theta\begin{pmatrix}a\\b\end{pmatrix},$$

$$\vdots$$

$$w_{13}=(a\bar{b})^{12}\bar{a}^{12}|a|^{-24}(L_1 \ldots L_9 L)^{-1}F_\theta\begin{pmatrix}a\\b\end{pmatrix}.$$

A diffeomorphism $\Phi : S^7 \times S^3 \to \tilde{P}_{13}$ will then be:

$$\Phi\left(\begin{pmatrix}a\\b\end{pmatrix}, h\right)=\begin{bmatrix} a & -b|b|^2L^{-1} & 0 & \ldots & 0 & w_1 h\\ b & b\bar{a}b L^{-1} & 0 & \ldots & 0 & w_2 h\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & (a\bar{b})^{10}aL^{-1} & (a\bar{b})^9 a L_9^{-1} & \ldots & a & w_{13}h\end{bmatrix}.$$

From paragraph 3 we have that the free action of S^3 on $\tilde{P}_{13}$ with quotient $\Sigma^7_{[44]}$ is conjugation by $q\,(\xi \mapsto \bar{q}\xi q)$ on the entries of each column, except the last, and multiplication $(\omega \mapsto \bar{q}\omega)$ on the entries of the last column.

Therefore, a free action of S^3 on $S^7 \times S^3$, with quotient $\Sigma^7_{[44]}$ is:

$$q*\left(\begin{pmatrix}a\\b\end{pmatrix}, h\right)\equiv\Phi^{-1}\left(q*\Phi\left(\begin{pmatrix}a\\b\end{pmatrix}, h\right)\right)$$

$$=\left(\begin{pmatrix}\bar{q}aq\\ \bar{q}bq\end{pmatrix}, \overline{F_\theta\begin{pmatrix}\bar{q}aq\\ \bar{q}bq\end{pmatrix}}\,\bar{q}F_\theta\begin{pmatrix}a\\b\end{pmatrix}h\right),$$

where $\theta=\cos^{-1}|a|$ in $[0, \pi/2]$.

To obtain the other Σ^7's that are obtainable this way, according to paragraph 3, we have to consider the homotopy F between 1 and $a^{12k}(b\bar{a})^{12k}\bar{b}^{12k}$ for the appropriate values of k.

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