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ON THE EXTENSION IN THE HARDY CLASSES 
AND IN THE NEVANLINNA CLASS 

BY 
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1. Introduction

1.1. Recently Järvi ([9]; Theorem 1, p. 597) gave the following result.

Let $G$ be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in $G$ and polar. Let $f : G \setminus E \rightarrow \mathbb{C}$ be a holomorphic function such that for some $p > 0$, $|f|^p$ has a harmonic majorant in $G \setminus E$. Then $f$ has a unique holomorphic extension $f^* : G \rightarrow \mathbb{C}$ such that $|f^*|^p$ has a harmonic majorant in $G$.

In the case $n = 1$ Järvi’s result is contained in Parreau’s classical result ([13]; Théorème 20, p. 182). In the case $n \geq 1$ Järvi’s result generalized Cima’s and Graham’s result ([3]; Theorem A, p. 241) which stated that analytic subvarieties are removable singularities for certain subdomains of
Note that in [15]; Theorem 3.2, p. 285 a similar result was given to Järvi's result, however, only in the case $p \geq 2$.

Järvi's proof was based on a lemma of Parreau ([9]; Lemma, pp. 596-597) (see also [7]; Lemma 1, p. 18) concerning quasibounded harmonic functions. In section 2 below we give a perhaps more elementary proof to the above result of Järvi. Our proof applies also to the case of $n$-harmonic, i.e. separately harmonic functions. In this case our exceptional sets are $n$-small. For the definition of these sets see [16]; Definition 2.2 and 2.2 below.

In [15]; Theorem 3.9, p. 287, it was observed that the following result is a direct consequence of [10]; Theorem 2, p. 279 (see also [11]; Theorem 4, p. 35 and [5]; Theorem 1.2 (b), p. 704) and of [1]; Corollary 2.10, p. 425.

Let $G$ be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in $G$ and polar. Let $f : G \setminus E \to \mathbb{C}$ be a holomorphic function such that $\log^+ |f|$ has a pluriharmonic majorant in $G \setminus E$. Then $f$ has a unique meromorphic extension to $G$.

In the case $n=1$ this result is contained in the result of Parreau ([13]; Théorème 20, p. 182) (see also [1]; Corollary 2.10, p. 425). In the case $n \geq 2$ the above result generalized Cima's and Graham's result ([3]; Theorem C, p. 241) which stated that in this situation analytic subvarieties are removable singularities for certain subdomains of $\mathbb{C}^n$.

In section 3 below we show that in the above result it is sufficient to suppose that $\log^+ |f|$ has a harmonic majorant in $G \setminus E$. Our result gives thus a positive answer to a question posed by Cima and Graham ([3]; Remarks 7.4, p. 255).

The results for subharmonic functions are due to the first author, the results for $n$-hypoharmonic, i.e. separately hypoharmonic functions (except Remark 2.8) and for functions in the Nevanlinna class are due to the second author.

1.2. We use mainly the same notation as in [8]. See also [16]. However, we recall the following.

If $a \in \mathbb{R}^k$, $k \geq 1$, and $r > 0$, we write

$$B^k(a, r) = \{ x \in \mathbb{R}^k \mid |x-a| < r \}, \quad U = B^2(0, 1).$$

The complex space $\mathbb{C}^n$, $n \geq 1$, will be identified with the real space $\mathbb{R}^{2n}$. If $z_0 \in \mathbb{C}$ and $r > 0$, we write $S^1(z_0, r) = \partial B^2(z_0, r)$. If
\( z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \ n \geq 1, \) we set for each \( j, \ 1 \leq j \leq n, \)

\[ Z_j = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbb{C}^{n-1} \quad \text{and} \quad (z_j, Z_j) = z. \]

If \( G \subset \mathbb{C}^n \) and \( z_0 = (z_0^1, z_0^2) \) we write

\[
G(z_0^1) = \{ Z_j \in \mathbb{C}^{n-1} \mid (z_j^0, Z_j) \in G \},
\]

\[
G(Z_0^1) = \{ z_j \in \mathbb{C} \mid (z_j, Z_0^1) \in G \}.
\]

If \( r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n, \) we write \( R_1 = (r_2, \ldots, r_n) \) and

\[
D^*(z_0, r) = B^2(z_0^1, r_1) \times D^{n-1}(Z_0^1, R_1),
\]

where

\[
D^{n-1}(Z_1^0, R_1) = B^2(z_1^0, r_2) \times \ldots \times B^2(z_n^0, r_n).
\]

If \( G \subset \mathbb{C}^n \) is open and \( f : G \rightarrow \mathbb{C} \) (resp. \([-\infty, \infty]) \) we write for each \( Z_1 \in \mathbb{C}^{n-1} f_{z_1} : G(Z_1) \rightarrow \mathbb{C} \) (resp. \([-\infty, \infty]),

\[
f_{z_1}(z_1) = f(z_1, Z_1).
\]

For the Laplace of \( f \) (in the distribution sense) we write

\[
\Delta f = \sum_{j=1}^n \Delta_j f,
\]

where

\[
\Delta_j f = 4 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}.
\]

For the definition of \( n \)-hyperharmonic, i.e. separately hyperharmonic functions see [8]. A function \( u : G \rightarrow [-\infty, \infty) \) is \( n \)-hypoharmonic, if \(-u\) is \( n \)-hyperharmonic. Note that a function \( u : G \rightarrow (-\infty, \infty] \) (resp. \([-\infty, \infty)]) \) is superharmonic (resp. subharmonic) if \( u \) is hyperharmonic and \( \neq -\infty \) (resp. hypoharmonic and \( \neq -\infty \)) on each component of \( G \).

The \( k \)-dimensional Hausdorff measure is denoted by \( H_k \) (note the difference between the Hardy class \( H^p \)), the \( k \)-dimensional Lebesgue measure by \( m_k \).

For the theory of holomorphic functions, Hardy classes and Nevanlinna class see [18] and [21]. For potential theory see [8] and [6].

2. On the extension in the Hardy classes

2.1. Let \( G \) be an open set in \( \mathbb{R}^n, \ n \geq 2. \) Let \( p > 0. \) Set

\[ h^p(G) = \{ u : G \rightarrow \mathbb{R}_+ \mid u \text{ is subharmonic and } u^p \text{ has a harmonic majorant in } G \}. \]
If $G$ is an open set in $\mathbb{C}^n$, $n \geq 1$, set

$$h^p_\ast(G) = \{ u : G \to \mathbb{R}_+ \mid u \text{ is } n\text{-hypoharmonic and } u^p \text{ has an } n\text{-hyperharmonic majorant in } G \text{ which is } \neq \infty \text{ on each component of } G \};$$

$$H^p(G) = \{ f : G \to \mathbb{C} \mid f \text{ is holomorphic and } |f|^p \text{ has a harmonic majorant in } G \};$$

$$H^p_\ast(G) = \{ f : G \to \mathbb{C} \mid f \text{ is holomorphic and } |f|^p \text{ has an } n\text{-hyperharmonic majorant in } G \text{ which is } \neq \infty \text{ on each component of } G \}.$$

In Theorem 2.5 below we give extension results for the classes $h^p$ and $h^p_\ast$. In the case of the class $h^p$ the exceptional set is polar and the proof is based on the well-known result which states that polar sets are removable singularities for subharmonic functions which are locally bounded above (see [8]; Theorem 2, p. 25). In the case of the class $h^p_\ast$ the exceptional set is $n$-small (see [16]; Definition 2.2 and 2.2 below) and the proof is based on a corresponding result according to which $n$-hypoharmonic functions which are locally bounded above can be extended across $n$-small sets (see [16]; Theorem 4.1). We recall here, however, the definition of $n$-small sets and give a property of these sets.

2.2. For each set $E \subset \mathbb{C}$ we define $\mathcal{C}^1(E) = \text{cap}^* E$, where $\text{cap}^*$ denotes the outer logarithmic capacity in $\mathbb{C}$. If $n \geq 2$, $1 \leq j \leq n$ and $\mathbb{C}^{n-1}$ is defined for subsets of $\mathbb{C}^{n-1}$, we define for $E \subset \mathbb{C}^n$

$$\mathcal{C}_j^m(E) = H_2 \{ z_j \in \mathbb{C} \mid \mathcal{C}_j^{m-1} \{ Z_j \in \mathbb{C}^{n-1} \mid (z, Z_j) \in E \} > 0 \}.$$

Finally, set

$$\mathcal{C}(E) = \max_{1 \leq j \leq n} \mathcal{C}_j^m(E).$$

We say that $E \subset \mathbb{C}^n$ is $n$-small, if $\mathcal{C}(E) = 0$.

2.3. Proposition. — Let $E \subset \mathbb{C}^n$, $n \geq 2$. Then $E$ is $n$-small, if for each $k$, $1 \leq k \leq n$, $H_{2n-2}(E_k) = 0$, where

$$E_k = \{ Z_k \in \mathbb{C}^{n-1} \mid \text{cap}^* \{ z_k \in \mathbb{C} \mid (z, Z_k) \in E \} > 0 \}.$$

Conversely, if $E$ is $n$-small and an $\mathcal{F}$-set, then $H_{2n-2}(E_k) = 0$ for each $k$, $1 \leq k \leq n$.

Proof. — The first part of the Proposition is proved in [16]; Proposition 2.4. Note that there are (at least Lebesgue nonmeasurable) $n$-small sets $E$ for which $H_{2n-2}(E_k) > 0$. See [16]; Remark 2.8. We give an induction proof for the second part. If $n = 2$ then the assertion clearly
holds. Suppose then that $n \geq 3$ and take $k$, $1 \leq k \leq n$, arbitrarily. Since the outer logarithmic capacity and the Hausdorff outer measure are subad- ditive, we may suppose that $E$ is compact. From [17]; Lemma 2.2.1, p. 87 it follows that $E_k$ is an $\mathcal{F}_r$-set and thus Lebesgue measurable.

Take $j \neq k$, $1 \leq j \leq n$, arbitrarily. Since $E$ is $n$-small, there is $B_j \subset \mathbb{C}$ such that $H_2 (B_j) = 0$ and that for each $z_j \notin B_j$ the set

$$E(z_j) = \{ Z_j \in C^{n-1} | (z_j, Z_j) \in E \}$$

is $(n-1)$-small. It follows from the induction hypothesis that $H_{2n-4} (E_k (z_j)) = 0$ for each $z_j \notin B_j$ where

$$E_k (z_j) = \{ Z_{kj} \in C^{n-2} | \text{cap}^{n} \{ z_k \in \mathbb{C} | (z_{kj}, Z_{kj}) \in E (z_j) \} > 0 \}.$$ 

If $\chi_{E_k}$ is the characteristic function of $E_k$, we get by Fubini’s theorem

$$m_{2n-4} (E_k) = \int_{C \setminus B_j} \left( \int_{C^{n-2} \setminus E_k (z_j)} \chi_{E_k} (z, Z_{kj}) \, dm_{2n-4} (Z_{kj}) \right) \, dm_2 (z_j) = 0,$$

concluding the proof.

2.4. Remark. — From [19]; Lemma 6, p. 115 (see also [12]; Corollary 3.3) and Proposition 2.2 it follows that polar sets are $n$-small. Note that Lebesgue measurable $n$-small sets are of Lebesgue measure zero ([16]; Remark 2.3).

2.5. Theorem. — Let $G$ be an open set in $\mathbb{R}^n$, $n \geq 2$ (resp. in $\mathbb{C}^n$, $n \geq 1$). Let $E \subset G$ be closed in $G$ and polar (resp. $n$-small). Let $p > 1$. If $u \in h^p (G \setminus E)$ [resp. $h^p_n (G \setminus E)$], then $u$ has a unique extension $u^* \in h^p (G)$ [resp. $h^p_n (G)$].

Proof. — Let $h$ be a harmonic majorant (resp. an $n$-hyperharmonic majorant which is $\neq \infty$ on each component of $G \setminus E$) of $u$ in $G \setminus E$. By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) $h$ has a unique superharmonic (resp. $n$-hyperharmonic which by [8]; Theorem, p. 31 is superharmonic) extension $h^*: G \to (-\infty, \infty)$. Thus the greatest harmonic minorant $v$ of $h^*$ in $G$ exists by [8]; Corollary 1, p. 10.

Take $q$, $1 < q < p$, and $\varepsilon > 0$ arbitrarily. Then the function

$$u_\varepsilon : G \setminus E \to [-\infty, \infty),$$

$$u_\varepsilon (z) = u(z)^q - \varepsilon \, h(z),$$

is subharmonic (resp. $n$-hypoharmonic). Since

$$u_\varepsilon (z) \leq u(z)^q - \varepsilon \, u(z)^p$$

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for each \( z \in G \setminus E \) and \( q < p \), \( u_\varepsilon \) is bounded above in \( G \). By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) \( u_\varepsilon \) has a unique subharmonic (resp. \( n \)-hypoharmonic which by [8]; Theorem, p. 31 is subharmonic) extension \( u_\varepsilon^*: G \to [-\infty, \infty) \).

For each \( z \in G \setminus E \) we have
\[
h^*_\varepsilon (z) - u_\varepsilon^* (z) = h(z) - u(z)^p + \varepsilon h(z) \geq u(z)^p - u(z)^p + \varepsilon h(z) \geq -1.
\]

Since \( E \) is of Lebesgue measure zero, it follows that
\[
u \varepsilon^* (z) \leq h^*_\varepsilon (z) + 1
\]
for each \( z \in G \). Thus by [8]; Corollary 1, p. 10
\[
u \varepsilon^* (z) \leq v(z) + 1
\]
for each \( z \in G \). But then
\[
u (z)^p - \varepsilon h(z) \leq v(z) + 1
\]
for each \( z \in G \setminus E \). Since \( \varepsilon > 0 \) was arbitrary, we get
\[
u (z)^p \leq v(z) + 1
\]
for each \( z \in G \setminus E \). Therefore \( u \) is locally bounded above in \( G \) and thus by [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) has a unique subharmonic (resp. \( n \)-hypoharmonic) extension \( u^*: G \to [0, \infty) \). Since (A) holds for all \( q < p \) and \( E \) is of Lebesgue measure zero,
\[
u^* (z)^p \leq v(z) + 1
\]
for all \( z \in G \). Thus \( u^* \in h^p(G) \). [Resp. it follows directly that \( u^* (z)^p \leq h^*_\varepsilon (z) \) for each \( z \in G \). Thus \( u^* \in h^p(G) \).]

2.6. Corollary ([9]; Theorem 1, p. 597 and [16]; Theorem 5.1). — Let \( G \) be an open set in \( \mathbb{C}^n \), \( n \geq 1 \). Let \( E \subset G \) be closed in \( G \) and polar (resp. \( n \)-small). Let \( p > 0 \). If \( f \in H^p(G \setminus E) \) [resp. \( H^p_n(G \setminus E) \)], then \( f \) has a unique extension \( f^* \in H^p(G) \) [resp. \( H^p_n(G) \)].

Proof. — Choose \( u = |f|^p \) and observe that \( u \in h^2(G \setminus E) \) [resp. \( h^2_n(G \setminus E) \)]. By Theorem 2.5 \( u \) has a unique extension \( u^* \in h^2(G) \) [resp. \( h^2_n(G) \)]. Using then [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) to the harmonic functions \( \text{Re} \ f \) and \( \text{Im} \ f \) locally bounded in \( G \) we see that \( f \) has a unique extension \( f^* \in H^p(G) \) [resp. \( H^p_n(G) \)].
2.7. Corollary ([16]; Theorem 5.2). — Let \( E \subset U^n \), \( n \geq 1 \), be closed in \( U^n \) and \( n \)-small. Let \( f: U^n \setminus E \to \mathbb{C} \) be a holomorphic function such that for some \( p > 0 \), \( |f|^p \) has an \( n \)-harmonic majorant in \( U^n \setminus E \). Then \( f \) has a unique holomorphic extension \( f^*: U^n \to \mathbb{C} \) such that \(|f^*|^p\) has an \( n \)-harmonic majorant in \( U^n \).

**Proof.** — To see that \(|f^*|^p\) has an \( n \)-harmonic majorant in \( U^n \) just proceed as in [16]; proof of Theorem 5.2 (see also [15]; p. 287).

2.8. Remark. — Note that in Corollary 2.7 it is not possible to replace the polydisc \( U^n \) by an arbitrary open set \( G \).

For example, let
\[
G = \{(z_1, z_2) \in B^4(0, 1) \mid 1/|1 - z_1| + 1/|1 - z_2| < \log(1/|z_1|) + \log(1/|z_2|)\},
\]
where conventionally \( \log \infty = \infty \). The function \( f: G \to \mathbb{C} \),
\[
f(z) = 1/(1 - z_1) + 1/(1 - z_2),
\]
betransits to the class \( H^2_2(G) \). Moreover, \(|f|\) has a 2-harmonic majorant outside the 2-small set
\[
E = \{z \in G \mid z_1 = 0 \text{ or } z_2 = 0\}.
\]

Since \( G \cap (\mathbb{C} \times \{0\}) = U \times \{0\} \) and the function
\[
U \ni z \mapsto 1/(1 - z) \in \mathbb{C}
\]
does not belong to \( H^1(U) \), it follows that \(|f|\) has no 2-harmonic majorant in \( G \).

2.9. Corollary. — Let \( G \) be an open set in \( \mathbb{C}^n \), \( n \geq 1 \). Let \( E \subset G \) be closed in \( G \) and polar (resp. \( n \)-small). Let \( f: G \setminus E \to \mathbb{C} \) be a holomorphic function such that for some \( p > 1 \), \((\log^+ |f|)^p\) has a harmonic majorant in \( G \setminus E \) (resp. \( n \)-hyperharmonic majorant which is \( \neq \infty \) on each component of \( G \setminus E \)). Then \( f \) has a unique holomorphic extension \( f^*: G \to \mathbb{C} \).

**Proof.** — Observe that the subharmonic (resp. \( n \)-hypoharmonic) function
\[
u(z) = \log^+ |f(z)|,
\]
has by Theorem 2.5 a unique extension \( u^* \in h^p(G) \) [resp. \( h^n_*(G) \)].
Therefore $|f|$ is locally bounded in $G$. Proceeding then as in the proof of Corollary 2.6 we see that $f$ has a unique holomorphic extension $f^* : G \to \mathbb{C}$.

3. On the extension in the Nevanlinna class

3.1. Let $G$ be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $f$ be meromorphic in $G$. It is easy to see that for each point $z_0 \in G$ there is a neighborhood $U_{z_0}$ in $G$ and an analytic subvariety $E_{z_0}$ in $U_{z_0}$ such that $f$ is holomorphic in $U_{z_0} \setminus E_{z_0}$ and $\log^+ |f|$ has a pluriharmonic majorant in $U_{z_0} \setminus E_{z_0}$.

In Theorem 3.4 below we consider the converse situation. To be more precise, we show that if $E \subset G$ is closed in $G$ and polar and if $f : G \setminus E \to \mathbb{C}$ is holomorphic such that $\log^+ |f|$ has a harmonic majorant in $G \setminus E$, then $f$ has a unique meromorphic extension to $G$. For the proof of this result we give two definitions and one Lemma.

3.2. Let $G$ be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in $G$. Let $\varphi : G \setminus E \to [-\infty, \infty)$ be subharmonic. By [8]; Theorem 1, p. 11 $\Delta \varphi$ is a measure in $G \setminus E$. We say that $\Delta \varphi$ has locally finite mass near $E$, if $\Delta \varphi(K \setminus E)$ is finite for each compact set $K \subset G$. Moreover, we say that $\varphi$ has locally a harmonic majorant near $E$, if for each $z_0 \in E$ there is $R > 0$ such that $B^{2n}(z_0, R) \subset G$ and a harmonic function $h : B^{2n}(z_0, R) \setminus E \to \mathbb{R}$ such that $\varphi(z) \leq h(z)$ for each $z \in B^{2n}(z_0, R) \setminus E$.

3.3. LEMMA (cf. [2]; p. 283). Let $G$ be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $\varphi : G \setminus E \to [-\infty, \infty)$ be subharmonic. If $\Delta \varphi$ has locally finite mass near $E$, then $\varphi$ has locally a harmonic majorant near $E$.

Proof. — Take $z_0 \in E$ arbitrarily. Choose $R$ and $R_1$ such that $0 < R < R_1$ and $B^{2n}(z_0, R_1) \subset G$. Set $\mu = (1/c_2n) \Delta \varphi|(B^{2n}(z_0, R) \setminus E)$, where $c_{2n}$ is the Poisson constant (see [8]; p. 4). Proceeding as Cegrell in [2]; proof of Proposition, (ii) $\Rightarrow$ (i), p. 283 one gets the desired harmonic majorant as follows. Define $\psi$,

$$
\psi(z) = \varphi(z) + G_\mu(z)
$$

where $G_\mu$ is the Green potential of $\mu$ in $B^{2n}(z_0, R_1)$. By [8]; p. 4 one sees that $\Delta \psi = 0$ in $B^{2n}(z_0, R) \setminus E$ in the distribution sense. Using then Weyl's lemma ([8]; Corollary, p. 3) one finds a harmonic function $h : B^{2n}(z_0, R) \setminus E \to \mathbb{R}$ such that $h = \psi$ Lebesgue almost everywhere in $B^{2n}(z_0, R) \setminus E$. Since $G_\mu$ is positive, $h$ gives a harmonic majorant to $\varphi$ in $B^{2n}(z_0, R) \setminus E$. 

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3.4. THEOREM. Let $G$ be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in $G$ and polar. Let $f: G \setminus E \to \mathbb{C}$ be a holomorphic function such that $\log^+ |f|$ has a harmonic majorant $u$ in $G \setminus E$. Then $f$ has a unique meromorphic extension $f^*$ to $G$.

Proof. Since $E$ is polar, $\text{int } E = \emptyset$. Therefore it is sufficient to show that each point $z_0 \in E$ has a neighborhood $U_{z_0}$ in $G$ such that $f|_{U_{z_0} \setminus E}$ has a meromorphic extension to $U_{z_0}$.

Since $E$ is polar, $H_{2n-1}(E) = 0$ by [6]. Theorem 5.13, p. 225. Thus we find by [4] (see also [20]; Lemma 2, p. 114) a complex line $P$ through the point $z_0 = (z_1^0, Z_1^0)$ such that $H_1(E \cap P) = 0$. By [10]; Proposition 2, p. 266 (see also [11]; Theorem 2, p. 35) and by [6]; p. 55 we may rotate the coordinate system and thus assume that $P = \mathbb{C} \times \{Z_1^0\}$.

Using the facts that $H_1(E \cap (\mathbb{C} \times \{Z_1^0\})) = 0$ and $E$ is closed in $G$, we find

$$r_1, r'_1 \in \mathbb{R}_+, \quad 0 < r'_1 < r_1 \quad \text{and} \quad R_1 = (r_2, \ldots, r_n) \in \mathbb{R}_+^{n-1}$$

such that

$$\bar{B}^2(z_1^0, r_1) \times \bar{D}^{s-1}(Z_1^0, R_1) \subset G$$

and

$$(\bar{B}^2(z_1^0, r_1) \setminus \bar{B}^2(z_1^0, r'_1)) \times \bar{D}^{s-1}(Z_1^0, R_1) \subset G \setminus E.$$

Therefore

$$f|_{(\bar{B}^2(z_1^0, r_1) \setminus \bar{B}^2(z_1^0, r'_1)) \times \bar{D}^{s-1}(Z_1^0, R_1)}$$

is holomorphic.

Now we argue as in [2]; proofs of Proposition and Theorem, pp. 283-285. Since $E$ is polar, we see using [8]; Theorem 2, p. 25 that the subharmonic functions $\log^+ |f| - u$ and $-u$ in $G \setminus E$ have subharmonic extensions $\varphi_1: G \to [-\infty, \infty)$ and $\varphi_2: G \to [-\infty, \infty)$, respectively. But then

$$\log^+ |f(z)| = \varphi_1(z) - \varphi_2(z)$$

for each $z \in G \setminus E$. Since $\Delta \varphi_1$ and $\Delta \varphi_2$ are measures in $G$, $\Delta \log^+ |f|$ has locally finite mass near $E$.

Set $r = (r_1, R_1)$ and take an increasing sequence of test functions

$$\chi_j \in D_+(D^s(z_0, r) \setminus E), \quad j = 1, 2, \ldots,$$
such that $\chi_j(z) \to 1$ as $j \to \infty$ for each $z \in D^n(z_0, r) \setminus E$. Since $\Delta \log^+ | f |$ has locally finite mass near $E$, there is $M \in \mathbb{R}_+$ such that

$$\int \log^+ | f(z) | \Delta \chi_j(z) \, dm_2(z) \leq M$$

for each $j=1, 2, \ldots$ Since $\log^+ | f |$ is $n$-hypoharmonic, we see by [8]; Proposition 1, p. 33 that

$$\int \log^+ | f(z) | \Delta_1 \chi_j(z) \, dm_2(z) \leq \int \log^+ | f(z) | \Delta \chi_j(z) \, dm_2(z) \leq M$$

for each $j=1, 2, \ldots$ From Fubini's theorem it follows that

$$(B) \int \left( \int \log^+ | f_{z_1}(z_1) | \Delta \chi_{jz_1}(z_1) \, dm_2(z_1) \right) \, dm_2 \leq M$$

for each $j=1, 2, \ldots$ Since the functions $\log^+ | f_{z_1} |, \, Z_1 \in D^{n-1}(Z_1^0, R_1)$, are subharmonic, we see that the sequence

$$\int \log^+ | f_{z_1}(z_1) | \Delta \chi_{jz_1}(z_1) \, dm_2(z_1), \quad j=1, 2, \ldots,$$

is increasing for each $Z_1 \in D^{n-1}(Z_1^0, R_1)$. Using then Monotone convergence in (B), we find a set $B_1 \subset D^{n-1}(Z_1^0, R_1)$ such that $\Delta_2 \chi_{jz_1}(z_1) \, dm_2(z_1)$ and that

$$(C) \lim_{j \to \infty} \int \log^+ | f_{z_1}(z_1) | \Delta \chi_{jz_1}(z_1) \, dm_2(z_1) < \infty$$

for each $Z_1 \in B_1$.

And now we continue with our previous techniques (see [14]; proof of Theorem 3.1, pp. 147-148). Since $E$ is polar, there is by [19]; Lemma 6, p. 115 (see also [12]; Corollary 3.3) a set $B_2 \subset D^{n-1}(Z_1^0, R_1)$ such that $H_{2n-2}(B_2)=0$ and that for each $Z_1 \in B_2$ the set $E(Z_1)$ is polar in $C$.

Set $B = B_1 \cup B_2$. It follows from (C) that for each $Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B$ $\Delta \log^+ | f_{z_1} |$ has locally finite mass near $E(Z_1) \cap B^2(z_1^0, r_1)$. From Lemma 3.2 it follows that for each

$$Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B, \quad \log^+ | f_{z_1} | \cap B^2(z_1^0, r_1) \setminus E(Z_1)$$

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has locally a harmonic majorant near $E(Z_1) \cap B^2(z_1^0, r_1)$. Since then $E(Z_1)$ is polar in $C$, it follows from [13]; Théorème 20, p. 182 (see also [1]; Corollary 2.10, p. 425) that $f_{Z_1}$ has a unique meromorphic extension $f_{Z_1}'$ to $D^+(z_0, r)(Z_1) = B^+(z_1^0, r_1)$. Since $H_{2n-2}(B) = 0$, it follows from Levi's extension theorem ([5]; Theorem 2.1 (b), p. 710) (see also [14]; Lemma 2.4, p. 147) that $f | D^+(z_0, r) \setminus E$ has a unique meromorphic extension to $D^+(z_0, r)$.

3.5. Remark. — Using the fact that the Hardy classes are contained in the Nevanlinna class, Theorem 3.4 together with Cima's and Graham's rather difficult argument ([3]; pp. 251-252) we get another proof to the first part of Corollary 2.6 above.

REFERENCES


*Added in proof.* — In the meantime D. Singman has proved extension results for Hardy classes in his article *Removable singularities for n-harmonic functions and Hardy classes in polydiscs*, Proc. Amer. Math. Soc., Vol. 90, 1984, pp. 299-302.