BULLETIN DE LA S. M. F.

JAAKKO HYVÖNEN
JUHANI RIIHENTAUS

On the extension in the Hardy classes and the Nevanlinna class

Bulletin de la S. M. F., tome 112 (1984), p. 469-480
<http://www.numdam.org/item?id=BSMF_1984__112__469_0>

Bull. Soc. math. France,
112, 1984, p. 469-480.

ON THE EXTENSION IN THE HARDY CLASSES AND IN THE NEVANLINNA CLASS

BY

JAAKKO HYVÖNEN and JUHANI RIIHENTAUS (*)

RÉSUMÉ. — En utilisant des méthodes de la théorie du potentiel on a établi des théorèmes d'extension pour les fonctions de quelques classes de Hardy et de la classe de Nevanlinna dans $\mathbb{C}^n$. Les ensembles exceptionnels sont polaires ou un peu plus grands ensembles n-petits selon que la fonction majorante sera harmonique ou séparément hyperharmonique.

ABSTRACT. — Using potential theoretic methods we give extension results for functions in various Hardy classes and in the Nevanlinna class in $\mathbb{C}^n$. Our exceptional sets are polar or slightly larger n-small sets depending whether the majorant is a harmonic or separately hyperharmonic function.

1. Introduction

1.1. Recently Järvi ([9]; Theorem 1, p. 597) gave the following result.

Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in G and polar. Let $f: G \setminus E \to \mathbb{C}$ be a holomorphic function such that for some $p>0$, $|f|^p$ has a harmonic majorant in $G \setminus E$. Then f has a unique holomorphic extension $f^*: G \to \mathbb{C}$ such that $|f^*|^p$ has a harmonic majorant in G.

In the case $n=1$ Järvi's result is contained in Parreau's classical result ([13]; Théorème 20, p. 182). In the case $n \geq 1$ Järvi's result generalized Cima's and Graham's result ([3]; Theorem A, p. 241) which stated that analytic subvarieties are removable singularities for certain subdomains of

(*) Texte reçu le 31 janvier 1984, révisé le 14 juin 1984.

J. HYVÖNEN, department of mathematics, University of Joensuu SF-80100 Joensuu 10, Finland.

J. RIIHENTAUS, department of mathematics, University of Joensuu SF-80100 Joensuu 10, Finland, and department of mathematics, University of Oulu, SF-90570 Oulu 57, Finland.

$\mathbb{C}^n$. Note that in [15]; Theorem 3.2, p. 285 a similar result was given to Järvi's result, however, only in the case $p \geqslant 2$.

Järvi's proof was based on a lemma of Parreau ([9]; Lemma, pp. 596-597) (*see* also [7]; Lemma 1, p. 18) concerning quasibounded harmonic functions. In section 2 below we give a perhaps more elementary proof to the above result of Järvi. Our proof applies also to the case of n-harmonic, i. e. separately harmonic functions. In this case our exceptional sets are n-small. For the definition of these sets *see* [16]; Definition 2.2 and 2.2 below.

In [15]; Theorem 3.9, p. 287, it was observed that the following result is a direct consequence of [10]; Theorem 2, p. 279 (*see* also [11]; Theorem 4, p. 35 and [5]; Theorem 1.2 (*b*), p. 704) and of [1]; Corollary 2.10, p. 425.

Let G be an open set in $\mathbb{C}^n$, $n \geqslant 1$. Let $E \subset G$ be closed in G and polar. Let $f : G \setminus E \to \mathbb{C}$ be a holomorphic function such that $\log^+ | f |$ *has a pluriharmonic majorant in $G \setminus E$. Then f has a unique meromorphic extension to G.*

In the case $n=1$ this result is contained in the result of Parreau ([13]; Théorème 20, p. 182) (*see* also [1]; Corollary 2.10, p. 425). In the case $n \geqslant 2$ the above result generalized Cima's and Graham's result ([3]; Theorem C, p. 241) which stated that in this situation analytic subvarieties are removable singularities for certain subdomains of $\mathbb{C}^n$.

In section 3 below we show that in the above result it is sufficient to suppose that $\log^+ | f |$ *has a harmonic majorant in $G \setminus E$*. Our result gives thus a positive answer to a question posed by Cima and Graham ([3]; Remarks 7.4, p. 255).

The results for subharmonic functions are due to the first author, the results for n-hypoharmonic, i. e. separately hypoharmonic functions (except Remark 2.8) and for functions in the Nevanlinna class are due to the second author.

1.2. We use mainly the same notation as in [8]. *See* also [16]. However, we recall the following.

If $a \in \mathbb{R}^k$, $k \geqslant 1$, and $r > 0$, we write

$$B^k(a, r) = \{ x \in \mathbb{R}^k \mid | x - a | < r \}, \qquad U = B^2(0, 1).$$

The complex space $\mathbb{C}^n$, $n \geqslant 1$, will be identified with the real space $\mathbb{R}^{2n}$. If $z_0 \in \mathbb{C}$ and $r > 0$, we write $S^1(z_0, r) = \partial B^2(z_0, r)$. If

$z=(z_1, \ldots, z_n) \in \mathbf{C}^n$, $n>1$, we set for each j, $1 \leqslant j \leqslant n$,

$$Z_j=(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbf{C}^{n-1} \quad \text{and} \quad (z_j, Z_j)=z.$$

If $G \subset \mathbf{C}^n$ and $z_0=(z_j^0, Z_j^0)$ we write

$$G(z_j^0)=\{ Z_j \in \mathbf{C}^{n-1} \mid (z_j^0, Z_j) \in G \},$$
$$G(Z_j^0)=\{ z_j \in \mathbf{C} \mid (z_j, Z_j^0) \in G \}.$$

If $r=(r_1, \ldots, r_n) \in \mathbf{R}_+^n$, we write $R_1=(r_2, \ldots, r_n)$ and

$$D^n(z_0, r)=B^2(z_1^0, r_1) \times D^{n-1}(Z_1^0, R_1),$$

where

$$D^{n-1}(Z_1^0, R_1)=B^2(z_2^0, r_2) \times \ldots \times B^2(z_n^0, r_n).$$

If $G \subset \mathbf{C}^n$ is open and $f: G \to \mathbf{C}$ (resp. $[-\infty, \infty]$) we write for each $Z_1 \in \mathbf{C}^{n-1}$ $f_{Z_1} : G(Z_1) \to \mathbf{C}$ (resp. $[-\infty, \infty]$),

$$f_{Z_1}(z_1)=f(z_1, Z_1).$$

For the Laplace of f (in the distribution sense) we write

$$\Delta f=\textstyle\sum_{j=1}^n \Delta_j f,$$

where

$$\Delta_j f=4\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}.$$

For the definition of n-hyperharmonic, i.e. separately hyperharmonic functions *see* [8]. A function $u: G \to [-\infty, \infty)$ is n-hypoharmonic, if $-u$ is n-hyperharmonic. Note that a function $u: G \to (-\infty, \infty]$ (resp. $[-\infty, \infty)$) is superharmonic (resp. subharmonic) if u is hyperharmonic and $\not\equiv \infty$ (resp. hypoharmonic and $\not\equiv -\infty$) on each component of G.

The k-dimensional Hausdorff measure is denoted by H_k (note the difference between the Hardy class H^p), the k-dimensional Lebesgue measure by m_k.

For the theory of holomorphic functions, Hardy classes and Nevanlinna class *see* [18] and [21]. For potential theory *see* [8] and [6].

2. On the extension in the Hardy classes

2.1. Let G be an open set in $\mathbf{R}^n$, $n \geqslant 2$. Let $p>0$. Set $h^p(G)=\{ u: G \to \mathbf{R}_+ \mid u$ is subharmonic and u^p has a harmonic majorant in $G \}$.

If G is an open set in $\mathbf{C}^n$, $n \geqslant 1$, set

$h^p_n(G) = \{ u : G \to \mathbf{R}_+ \mid u$ is n-hypoharmonic and u^p has an n-hyperharmonic majorant in G which is $\not\equiv \infty$ on each component of $G\}$;

$H^p(G) = \{ f : G \to \mathbf{C} \mid f$ is holomorphic and $|f|^p$ has a harmonic majorant in $G\}$;

$H^p_n(G) = \{ f : G \to \mathbf{C} \mid f$ is holomorphic and $|f|^p$ has an n-hyperharmonic majorant in G which is $\not\equiv \infty$ on each component of $G\}$.

In Theorem 2.5 below we give extension results for the classes h^p and h^p_n. In the case of the class h^p the exceptional set is polar and the proof is based on the well-known result which states that polar sets are removable singularities for subharmonic functions which are locally bounded above (*see* [8]; Theorem 2, p. 25). In the case of the class h^p_n the exceptional set is n-small (*see* [16]; Definition 2.2 and 2.2 below) and the proof is based on a corresponding result according to which n-hypoharmonic functions which are locally bounded above can be extended across n-small sets (*see* [16]; Theorem 4.1). We recall here, however, the definition of n-small sets and give a property of these sets.

2.2. For each set $E \subset \mathbf{C}$ we define $\mathscr{C}^1(E) = \operatorname{cap}^* E$, where cap^* denotes the outer logarithmic capacity in $\mathbf{C}$. If $n \geqslant 2$, $1 \leqslant j \leqslant n$ and $\mathscr{C}^{n-1}$ is defined for subsets of $\mathbf{C}^{n-1}$, we define for $E \subset \mathbf{C}^n$

$$\mathscr{C}^n_j(E) = H_2\{ z_j \in \mathbf{C} \mid \mathscr{C}^{n-1}\{ Z_j \in \mathbf{C}^{n-1} \mid (z_j, Z_j) \in E\} > 0\}.$$

Finally, set

$$\mathscr{C}^n(E) = \max_{1 \leqslant j \leqslant n} \mathscr{C}^n_j(E).$$

We say that $E \subset \mathbf{C}^n$ is *n-small*, if $\mathscr{C}^n(E) = 0$.

2.3. Proposition. — *Let $E \subset \mathbf{C}^n$, $n \geqslant 2$. Then E is n-small, if for each k, $1 \leqslant k \leqslant n$, $H_{2n-2}(E_k) = 0$, where*

$$E_k = \{ Z_k \in \mathbf{C}^{n-1} \mid \operatorname{cap}^*\{ z_k \in \mathbf{C} \mid (z_k, Z_k) \in E\} > 0\}.$$

Conversely, if E is n-small and an $\mathscr{F}_\sigma$-set, then $H_{2n-2}(E_k) = 0$ for each k, $1 \leqslant k \leqslant n$.

Proof. — The first part of the Proposition is proved in [16]; Proposition 2.4. Note that there are (at least Lebesgue nonmeasurable) n-small sets E for which $H_{2n-2}(E_n) > 0$. *See* [16]; Remark 2.8. We give an induction proof for the second part. If $n = 2$ then the assertion clearly holds. Suppose then that $n\geqslant 3$ and take k, $1\leqslant k\leqslant n$, arbitrarily. Since the outer logarithmic capacity and the Hausdorff outer measure are subadditive, we may suppose that E is compact. From [17]; Lemma 2.2.1, p. 87 it follows that E_k is an $\mathcal{F}_\sigma$-set and thus Lebesgue measurable.

Take $j\neq k$, $1\leqslant j\leqslant n$, arbitrarily. Since E is n-small, there is $B_j\subset \mathbf{C}$ such that $H_2(B_j)=0$ and that for each $z_j\notin B_j$ the set

$$E(z_j)=\{Z_j\in \mathbf{C}^{n-1}\,|\,(z_j, Z_j)\in E\}$$

is $(n-1)$-small. It follows from the induction hypothesis that $H_{2n-4}(E_k(z_j))=0$ for each $z_j\notin B_j$, where

$$E_k(z_j)=\{Z_{kj}\in \mathbf{C}^{n-2}\,|\,\mathrm{cap}^*\{z_k\in \mathbf{C}\,|\,(z_k, Z_{kj})\in E(z_j)\}>0\}.$$

If χ_{E_k} is the characteristic function of E_k, we get by Fubini's theorem

$$m_{2n-2}(E_k)=\int_{\mathbf{C}\setminus B_j}\left(\int_{\mathbf{C}^{n-2}\setminus E_k(z_j)}\chi_{E_k}(z_j, Z_{kj})\,dm_{2n-4}(Z_{kj})\right)dm_2(z_j)=0,$$

concluding the proof.

2.4. *Remark*. — From [19]; Lemma 6, p. 115 (*see* also [12]; Corollary 3.3) and Proposition 2.2 it follows that polar sets are n-small. Note that Lebesgue measurable n-small sets are of Lebesgue measure zero ([16]; Remark 2.3).

2.5. THEOREM. — *Let G be an open set in $\mathbf{R}^n$, $n\geqslant 2$ (resp. in $\mathbf{C}^n$, $n\geqslant 1$). Let $E\subset G$ be closed in G and polar (resp. n-small). Let $p>1$. If $u\in h^p(G\setminus E)$ [resp. $h^p_n(G\setminus E)$], then u has a unique extension $u^*\in h^p(G)$ [resp. $h^p_n(G)$].*

Proof. — Let h be a harmonic majorant (resp. an n-hyperharmonic majorant which is $\not\equiv\infty$ on each component of $G\setminus E$) of u in $G\setminus E$. By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) h has a unique superharmonic (resp. n-hyperharmonic which by [8]; Theorem, p. 31 is superharmonic) extension $h^*: G\to(-\infty, \infty]$. Thus the greatest harmonic minorant v of h^* in G exists by [8]; Corollary 1, p. 10.

Take q, $1<q<p$, and $\varepsilon>0$ arbitrarily. Then the function $u_\varepsilon: G\setminus E\to[-\infty, \infty)$,

$$u_\varepsilon(z)=u(z)^q-\varepsilon\, h(z),$$

is subharmonic (resp. n-hypoharmonic). Since

$$u_\varepsilon(z)\leqslant u(z)^q-\varepsilon\, u(z)^p$$

for each $z \in G \setminus E$ and $q<p$, u_ε is bounded above in G. By [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) u_ε has a unique subharmonic (resp. n-hypoharmonic which by [8]; Theorem, p. 31 is subharmonic) extension $u_\varepsilon^* : G \to [-\infty, \infty)$.

For each $z \in G \setminus E$ we have

$$h^*(z) - u_\varepsilon^*(z) = h(z) - u(z)^q + \varepsilon h(z) \geqslant u(z)^p - u(z)^q + \varepsilon h(z) \geqslant -1.$$

Since E is of Lebesgue measure zero, it follows that

$$u_\varepsilon^*(z) \leqslant h^*(z) + 1$$

for each $z \in G$. Thus by [8]; Corollary 1, p. 10

$$u_\varepsilon^*(z) \leqslant v(z) + 1$$

for each $z \in G$. But then

$$u(z)^q - \varepsilon h(z) \leqslant v(z) + 1$$

for each $z \in G \setminus E$. Since $\varepsilon > 0$ was arbitrary, we get

$$\text{(A)} \qquad u(z)^q \leqslant v(z) + 1$$

for each $z \in G \setminus E$. Therefore u is locally bounded above in G and thus by [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) has a unique subharmonic (resp. n-hypoharmonic) extension $u^* : G \to [0, \infty)$. Since (A) holds for all $q<p$ and E is of Lebesgue measure zero,

$$u^*(z)^p \leqslant v(z) + 1$$

for all $z \in G$. Thus $u^* \in h^p(G)$. [Resp. it follows directly that $u^*(z)^p \leqslant h^*(z)$ for each $z \in G$. Thus $u^* \in h_n^p(G)$.]

2.6. Corollary ([9]; Theorem 1, p. 597 and [16]; Theorem 5.1). — *Let G be an open set in $\mathbb{C}^n$, $n \geqslant 1$. Let $E \subset G$ be closed in G and polar (resp. n-small). Let $p>0$. If $f \in H^p(G \setminus E)$ [resp. $H_n^p(G \setminus E)$], then f has a unique extension $f^* \in H^p(G)$ [resp. $H_n^p(G)$].*

Proof. — Choose $u = |f|^{p/2}$ and observe that $u \in h^2(G \setminus E)$ [resp. $h_n^2(G \setminus E)$]. By Theorem 2.5 u has a unique extension $u^* \in h^2(G)$ [resp. $h_n^2(G)$]. Using then [8]; Theorem 2, p. 25 (resp. [16]; Theorem 4.1) to the harmonic functions $\operatorname{Re} f$ and $\operatorname{Im} f$ locally bounded in G we see that f has a unique extension $f^* \in H^p(G)$ [resp. $H_n^p(G)$].

2.7. COROLLARY ([16]; Theorem 5.2). — *Let $E \subset U^n$, $n \geqslant 1$, be closed in U^n and n-small. Let $f: U^n \setminus E \to \mathbf{C}$ be a holomorphic function such that for some $p>0$, $|f|^p$ has an n-harmonic majorant in $U^n \setminus E$. Then f has a unique holomorphic extension $f^*: U^n \to \mathbf{C}$ such that $|f^*|^p$ has an n-harmonic majorant in U^n.*

Proof. — To see that $|f^*|^p$ has an n-harmonic majorant in U^n just proceed as in [16]; proof of Theorem 5.2 (*see* also [15]; p. 287).

2.8. *Remark.* — Note that in Corollary 2.7 it is not possible to replace the polydisc U^n by an arbitrary open set G.

For example, let

$$G=\{(z_1, z_2)\in B^4(0, 1) \mid 1/|1-z_1| + 1/|1-z_2| < \log(1/|z_1|)+\log(1/|z_2|)\},$$

where conventionally $\log \infty = \infty$. The function $f: G \to \mathbf{C}$,

$$f(z)=1/(1-z_1)+1/(1-z_2),$$

belongs to the class $H^1_2(G)$. Moreover, $|f|$ has a 2-harmonic majorant outside the 2-small set

$$E=\{z\in G \mid z_1=0 \text{ or } z_2=0\}.$$

Since $G \cap (\mathbf{C}\times\{0\})=U\times\{0\}$ and the function

$$U \ni z \mapsto 1/(1-z) \in \mathbf{C}$$

does not belong to $H^1(U)$, it follows that $|f|$ has no 2-harmonic majorant in G.

2.9. COROLLARY. — *Let G be an open set in $\mathbf{C}^n$, $n \geqslant 1$. Let $E \subset G$ be closed in G and polar (resp. n-small). Let $f: G\setminus E \to \mathbf{C}$ be a holomorphic function such that for some $p>1$, $(\log^+|f|)^p$ has a harmonic majorant in $G\setminus E$ (resp. n-hyperharmonic majorant which is $\not\equiv \infty$ on each component of $G\setminus E$). Then f has a unique holomorphic extension $f^*: G \to \mathbf{C}$.*

Proof. — Observe that the subharmonic (resp. n-hypoharmonic) function $u: G\setminus E \to [-\infty, \infty)$,

$$u(z)=\log^+|f(z)|,$$

has by Theorem 2.5 a unique extension $u^* \in h^p(G)$ [resp. $h^p_n(G)$].

Therefore $|f|$ is locally bounded in G. Proceeding then as in the proof of Corollary 2.6 we see that f has a unique holomorphic extension $f^*: G \to \mathbb{C}$.

3. On the extension in the Nevanlinna class

3.1. Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let f be meromorphic in G. It is easy to see that for each point $z_0 \in G$ there is a neighborhood U_{z_0} in G and an analytic subvariety E_{z_0} in U_{z_0} such that f is holomorphic in $U_{z_0} \setminus E_{z_0}$ and $\log^+ |f|$ has a pluriharmonic majorant in $U_{z_0} \setminus E_{z_0}$.

In Theorem 3.4 below we consider the converse situation. To be more precise, we show that if $E \subset G$ is closed in G and polar and if $f: G \setminus E \to \mathbb{C}$ is holomorphic such that $\log^+ |f|$ has a harmonic majorant in $G \setminus E$, then f has a unique meromorphic extension to G. For the proof of this result we give two definitions and one Lemma.

3.2. Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in G. Let $\varphi : G \setminus E \to [-\infty, \infty)$ be subharmonic. By [8]; Theorem 1, p. 11 $\Delta\varphi$ is a measure in $G \setminus E$. We say that $\Delta\varphi$ has *locally finite mass near E*, if $\Delta\varphi(K \setminus E)$ is finite for each compact set $K \subset G$. Moreover, we say that φ has *locally a harmonic majorant near E*, if for each $z_0 \in E$ there is $R > 0$ such that $\bar{B}^{2n}(z_0, R) \subset G$ and a harmonic function $h : B^{2n}(z_0, R) \setminus E \to \mathbb{R}$ such that $\varphi(z) \leq h(z)$ for each $z \in B^{2n}(z_0, R) \setminus E$.

3.3. Lemma (*cf.* [2]; p. 283). — *Let G be an open set in* $\mathbb{C}^n$, $n \geq 1$. *Let* $\varphi : G \setminus E \to [-\infty, \infty)$ *be subharmonic. If* $\Delta\varphi$ *has locally finite mass near E, then* φ *has locally a harmonic majorant near E.*

Proof. — Take $z_0 \in E$ arbitrarily. Choose R and R_1 such that $0 < R < R_1$ and $\bar{B}^{2n}(z_0, R_1) \subset G$. Set $\mu = (1/c_{2n}) \Delta\varphi | (B^{2n}(z_0, R) \setminus E)$, where c_{2n} is the Poisson constant (*see* [8]; *p.* 4). Proceeding as Cegrell in [2]; proof of Proposition, (ii) $\Rightarrow$ (i), p. 283 one gets the desired harmonic majorant as follows. Define ψ,

$$\psi(z) = \varphi(z) + G_\mu(z)$$

where G_μ is the Green potential of μ in $B^{2n}(z_0, R_1)$. By [8]; p. 4 one sees that $\Delta\psi = 0$ in $B^{2n}(z_0, R) \setminus E$ in the distribution sense. Using then Weyl's lemma ([8]; Corollary, p. 3) one finds a harmonic function $h : B^{2n}(z_0, R) \setminus E \to \mathbb{R}$ such that $h = \psi$ Lebesgue almost everywhere in $B^{2n}(z_0, R) \setminus E$. Since G_μ is positive, h gives a harmonic majorant to φ in $B^{2n}(z_0, R) \setminus E$.

3.4. THEOREM. — *Let G be an open set in $\mathbb{C}^n$, $n \geq 1$. Let $E \subset G$ be closed in G and polar. Let $f: G \setminus E \to \mathbb{C}$ be a holomorphic function such that* $\log^+ | f |$ *has a harmonic majorant u in $G \setminus E$. Then f has a unique meromorphic extension f^* to G.*

Proof. — Since E is polar, int $E = \emptyset$. Therefore it is sufficient to show that each point $z_0 \in E$ has a neighborhood U_{z_0} in G such that $f | U_{z_0} \setminus E$ has a meromorphic extension to U_{z_0}.

Since E is polar, $H_{2n-1}(E) = 0$ by [6]; Theorem 5.13, p. 225. Thus we find by [4] (*see* also [20]; Lemma 2, p. 114) a complex line P through the point $z_0 = (z_1^0, Z_1^0)$ such that $H_1(E \cap P) = 0$. By [10]; Proposition 2, p. 266 (*see* also [11]; Theorem 2, p. 35) and by [6]; p. 55 we may rotate the coordinate system and thus assume that $P = \mathbb{C} \times \{ Z_1^0 \}$.

Using the facts that $H_1(E \cap (\mathbb{C} \times \{ Z_1^0 \})) = 0$ and E is closed in G, we find

$$r_1, r_1' \in \mathbf{R}_+, \quad 0 < r_1' < r_1 \qquad \text{and} \qquad R_1 = (r_2, \ldots, r_n) \in \mathbf{R}_+^{n-1}$$

such that

$$\bar{B}^2(z_1^0, r_1) \times \bar{D}^{n-1}(Z_1^0, R_1) \subset G$$

and

$$(\bar{B}^2(z_1^0, r_1) \setminus B^2(z_1^0, r_1')) \times \bar{D}^{n-1}(Z_1^0, R_1) \subset G \setminus E.$$

Therefore

$$f | (B^2(z_1^0, r_1) \setminus \bar{B}^2(z_1^0, r_1')) \times D^{n-1}(Z_1^0, R_1)$$

is holomorphic.

Now we argue as in [2]; proofs of Proposition and Theorem, pp. 283-285. Since E is polar, we see using [8]; Theorem 2, p. 25 that the subharmonic functions $\log^+ | f | - u$ and $-u$ in $G \setminus E$ have subharmonic extensions $\varphi_1 : G \to [-\infty, \infty)$ and $\varphi_2 : G \to [-\infty, \infty)$, respectively. But then

$$\log^+ | f(z) | = \varphi_1(z) - \varphi_2(z)$$

for each $z \in G \setminus E$. Since $\Delta\varphi_1$ and $\Delta\varphi_2$ are measures in G, $\Delta \log^+ | f |$ *has locally finite mass near E.*

Set $r = (r_1, R_1)$ and take an increasing sequence of test functions

$$\chi_j \in D_+(D^n(z_0, r) \setminus E), \qquad j = 1, 2, \ldots,$$

such that $\chi_j(z) \to 1$ as $j \to \infty$ for each $z \in D^n(z_0, r) \setminus E$. Since $\Delta \log^+ |f|$ *has locally finite mass near* E, there is $M \in \mathbf{R}_+$ such that

$$\int \log^+ |f(z)| \Delta \chi_j(z) \, dm_{2n}(z) \leq M$$

for each $j=1, 2, \ldots$ Since $\log^+ |f|$ is n-hypoharmonic, we see by [8]; Proposition 1, p. 33 that

$$\int \log^+ |f(z)| \Delta_1 \chi_j(z) \, dm_{2n}(z) \leq \int \log^+ |f(z)| \Delta \chi_j(z) \, dm_{2n}(z) \leq M$$

for each $j=1, 2, \ldots$ From Fubini's theorem it follows that

$$\text{(B)} \qquad \int \left(\int \log^+ | f_{Z_1}(z_1) | \Delta \chi_{j Z_1}(z_1) \, dm_2(z_1) \right) dm_{2n-2}(Z_1) \leq M$$

for each $j=1, 2, \ldots$ Since the functions $\log^+ | f_{Z_1} |$, $Z_1 \in D^{n-1}(Z_1^0, R_1)$, are subharmonic, we see that the sequence

$$\int \log^+ | f_{Z_1}(z_1) | \Delta \chi_{j Z_1}(z_1) \, dm_2(z_1), \qquad j=1, 2, \ldots,$$

is increasing for each $Z_1 \in D^{n-1}(Z_1^0, R_1)$. Using then Monotone convergence in (B), we find a set $B_1 \subset D^{n-1}(Z_1^0, R_1)$ such that $H_{2n-2}(B_1)=0$ and that

$$\text{(C)} \qquad \lim_{j \to \infty} \int \log^+ | f_{Z_1}(z_1) | \Delta \chi_{j Z_1}(z_1) \, dm_2(z_1) < \infty$$

for each $Z_1 \in B_1$.

And now we continue with our previous techniques (*see* [14]; proof of Theorem 3.1, pp. 147-148). Since E is polar, there is by [19]; Lemma 6, p. 115 (*see* also [12]; Corollary 3.3) a set $B_2 \subset D^{n-1}(Z_1^0, R_1)$ such that $H_{2n-2}(B_2)=0$ and that for each $Z_1 \notin B_2$ the set $E(Z_1)$ is polar in $\mathbf{C}$.

Set $B = B_1 \cup B_2$. It follows from (C) that for each $Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B$ $\Delta \log^+ | f_{Z_1} |$ has locally finite mass near $E(Z_1) \cap B^2(z_1^0, r_1)$. From Lemma 3.2 it follows that for each

$$Z_1 \in D^{n-1}(Z_1^0, R_1) \setminus B, \qquad \log^+ | f_{Z_1} | \, | B^2(z_1^0, r_1) \setminus E(Z_1)$$

has locally a harmonic majorant near $E(Z_1)\cap B^2(z_1^0, r_1)$. Since then $E(Z_1)$ is polar in $\mathbb{C}$, it follows from [13]; Théorème 20, p. 182 (*see* also [1]; Corollary 2.10, p. 425) that f_{Z_1} has a unique meromorphic extension $f^*_{Z_1}$ to $D^n(z_0, r)(Z_1)=B^2(z_1^0, r_1)$. Since $H_{2n-2}(B)=0$, it follows from Levi's extension theorem ([5]; Theorem 2.1 (*b*), p. 710) (*see* also [14]; Lemma 2.4, p. 147) that $f\,|\,D^n(z_0, r)\setminus E$ has a unique meromorphic extension to $D^n(z_0, r)$.

3.5. *Remark.* — Using the fact that the Hardy classes are contained in the Nevanlinna class, Theorem 3.4 together with Cima's and Graham's rather difficult argument ([3]; pp. 251-252) we get another proof to the first part of Corollary 2.6 above.

REFERENCES

[1] ABRAHAMSSON (L.). — Meromorphic extension of holomomorphic functions with growth conditions, *Bull. Soc. Math. Fr.*, Vol. 110, 1982, pp. 417-427.

[2] CEGRELL (U.). — Removable singularity sets for analytic functions having modulus with bounded Laplace mass, *Proc. Amer. Math. Soc.*, Vol. 88, 1983, pp. 283-286.

[3] CIMA (J.) and GRAHAM (I.). — On the extension of holomorphic functions with growth conditions across analytic subvarieties, *Michigan Math. J.*, Vol. 28, 1981, pp. 241-256.

[4] FEDERER (H.). — *Geometric measure theory*, Berlin, Springer, 1969.

[5] HARVEY (R.) and POLKING (J.). — Extending analytic objects, *Comm. Pure Appl. Math.*, Vol. 28, 1975, pp. 701-727.

[6] HAYMAN (W.) and KENNEDY (P.). — *Subharmonic functions*, I, London, Academic Press, 1976.

[7] HEINS (M.). — *Hardy classes on Riemann surfaces*, Berlin, Springer, 1971.

[8] HERVÉ (M.). — *Analytic and plurisubharmonic functions in finite and infinite dimensional spaces*, Berlin, Springer, 1970.

[9] JÄRVI (P.). — Removable singularities for H^p-functions, *Proc. Amer. Math. Soc.*, Vol. 86, 1982, pp. 596-598.

[10] LELONG (P.). — Ensembles singuliers impropres des fonctions plurisousharmoniques, *J. Math. Pures Appl.*, Vol. 36, 1957, pp. 263-303.

[11] LELONG (P.). — *Plurisubharmonic functions and positive differential forms*, New York, Gordon and Breach, 1969.

[12] MATTILA (P.). — An integral inequality for capacities, *Math. Scand.*, Vol. 53, 1983, pp. 256-264.

[13] PARREAU (M.). — Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann, *Ann. Inst. Fourier*, Grenoble, Vol. 3, 1951, pp. 103-197.

[14] RIIHENTAUS (J.). — An extension theorem for meromorphic functions of several variables, *Ann. Acad. Sc. Fenn.*, Sér. A I, Vol. 4, 1978/1979, pp. 145-149.

[15] RIIHENTAUS (J.). — On the extension of holomorphic functions with growth conditions, *J. London Math. Soc.*, (2), Vol. 27, 1983, pp. 281-288.

[16] RIIHENTAUS (J.). — On the extension of separately hyperharmonic functions and H^p-functions, *Michigan Math J.*, Vol. 31, 1984, pp. 99-112.

[17] RONKIN (L.). — Introduction to the theory of entire functions of several variables, *Providence: Amer. Math. Soc.*, 1974.

[18] RUDIN (W.). — *Function theory in the unit ball of* $\mathbf{C}^n$, New York, Springer, 1980.

[19] SADULLAEV (A.). — Rational approximation and pluripolar sets (Russian), *Mat. Sbornik*, Vol. 119, 1982, pp. 96-118.

[20] SHIFFMAN (B.). — On the removal of singularities of analytic sets, *Michigan Math. J.*, Vol. 15, 1968, pp. 111-120.

[21] STEIN (E.). — *Boundary behavior of holomorphic functions of several complex variables*, Princeton, Princeton University Press, 1972.

Added in proof. — In the meantime D. Singman has proved extension results for Hardy classes in his article *Removable singularities for n-harmonic functions and Hardy classes in polydiscs*, Proc. Amer. Math. Soc., Vol. 90, 1984, pp. 299-302.