BULLETIN DE LA S. M. F.

NIELS VIGAND PEDERSEN

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Bulletin de la S. M. F., tome 112 (1984), p. 423-467
<http://www.numdam.org/item?id=BSMF_1984__112__423_0>

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Bull. Soc. math. France,
112, 1984, p. 423-467.

ON THE INFINITESIMAL KERNEL OF IRREDUCIBLE REPRESENTATIONS OF NILPOTENT LIE GROUPS

BY

NIELS VIGAND PEDERSEN (*)

RÉSUMÉ. — Soit G un groupe de Lie nilpotent, connexe et simplement connexe d'algèbre de Lie $\mathfrak{g}$. Pour une représentation irréductible π de G, on dénote $\ker(d\pi)$ le noyau de la différentielle $d\pi$ de π considérée comme représentation de l'algèbre universelle enveloppante $U(\mathfrak{g}_{\mathbb{C}})$ de la complexification $\mathfrak{g}_{\mathbb{C}}$ de $\mathfrak{g}$. Dans cet article nous donnons pour chaque représentation irréductible π de G une formule explicite de $\ker(d\pi)$ en termes de l'orbite coadjointe associée par la théorie de Kirillov à π. Ensuite nous donnons un algorithme algébrique permettant de trouver l'orbite coadjointe associée à une représentation irréductible donnée. Finalement, nous prouvons, que la C^*-algèbre $C^*(G)$ de G est de trace continue généralisée par rapport à la $*$-sous algèbre $C_c^\infty(G)$ de $C^*(G)$ (cette notion est définie dans l'article) et que la suite de composition canonique correspondante est de longueur finie, ainsi améliorant un résultat de J. Dixmier.

ABSTRACT. — Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. For an irreducible representation π of G denote by $\ker(d\pi)$ the kernel of the differential $d\pi$ of π considered as a representation of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. In this paper we give first for each irreducible representation π of G an explicit formula for $\ker(d\pi)$ in terms of the coadjoint orbit associated by the Kirillov theory with π. Next we give an algebraic algorithm for finding the orbit associated with a given irreducible representation. Finally we show that the group C^*-algebra $C^*(G)$ of G is with generalized continuous trace with respect to the *-subalgebra $C_c^\infty(G)$ of $C^*(G)$ (the meaning of this is defined in the paper), and that the corresponding canonical composition series is of finite length, thus sharping a result of J. Dixmier.

(*) Texte reçu le 10 décembre 1983.

Niels V. PEDERSEN, Mathematical Institute, The Technical University of Denmark, Building 303, DK 2800 LYNGBY, Denmark.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE — 0037-9484/1984/04 423 45/\$ 6.50
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Introduction

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ denote the dual of the underlying vector space of $\mathfrak{g}$. For a strongly continuous, unitary representation (="a representation") π of G, let $d\pi$ denote the differential of π considered as a representation of $U(\mathfrak{g}_\mathbb{C})$, the universal enveloping algebra of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. In [3] DIXMIER showed that if π is an irreducible representation of G, then the kernel ker $(d\pi)$ of $d\pi$ is a selfadjoint primitive ideal in $U(\mathfrak{g}_\mathbb{C})$, and that the map $\pi \to \ker(d\pi)$ from the set of equivalence classes of irreducible representations of G to the space of selfadjoint primitive ideals in $U(\mathfrak{g}_\mathbb{C})$ is a bijection. In particular the kernel of $d\pi$ characterizes π. The first main result in this paper (Theorem 2.3.2) is an explicit formula for this kernel of $d\pi$ in terms of the coadjoint orbit associated by the Kirillov theory [7] with π. This formula establishes in algebraic terms a direct link between the coadjoint orbit space $\mathfrak{g}^*/G$, and the space $\hat{G}$ of equivalence classes of irreducible representations of G, and thus it serves a purpose analogous to the one of the Kirillov character formula ([7], Theorem 7.4 or [9], § 8, Théorème, p. 145). Probably our formula should be viewed as an algebraic counterpart of the latter, and it can presumably be used to establish the pairing between orbits and representations [or, if one prefers, between orbits and primitive ideals for e. g. complex nilpotent Lie algebras ([3], [5])] much like the way the Kirillov character formula was used to set up this pairing in [9].

We shall briefly describe our formula: Fix a Jordan-Hölder sequence

$$\mathfrak{g}=\mathfrak{g}_m \supset \mathfrak{g}_{m-1} \supset \ldots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0=\{0\}$$

in $\mathfrak{g}$, and a basis $X_1, \ldots, X_m$ with $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$. Let $l_1, \ldots, l_m$ be the basis in $\mathfrak{g}^*$ dual to the basis $X_1, \ldots, X_m$, and denote by ξ_j the coordinate of $l \in \mathfrak{g}^*$ with respect to the basis $l_1, \ldots, l_m$: $\xi_j = \langle l, X_j \rangle$. From [7] or [10], Lemma 1, p. 264 we extract the following. If O is a coadjoint orbit there exists a subset $e=\{j_1 < \ldots < j_d\}$ of $\{1, \ldots, m\}$ and polynomial functions $P_1, \ldots, P_m$ on $\mathfrak{g}^*$ uniquely determined by the following properties (identifying $\mathfrak{g}^*$ with $\mathbb{R}^m$ via the chosen basis):

(*a*) $P_{j_k}(\xi_1, \ldots, \xi_m)=\xi_{j_k}, k=1, \ldots, d$;

(*b*) $P_j(\xi_1, \ldots, \xi_m)$ depends only on the variables $\xi_{j_1}, \ldots, \xi_{j_k}$, where k is such that $j_k \leqslant j < j_{k+1}$;

(*c*) $O=\{l=(\xi_1, \ldots, \xi_m) \mid \xi_j=P_j(\xi_1, \ldots, \xi_m),\ 1 \leq j \leq m\}$.

Set then $Q_j(\xi_1, \ldots, \xi_m)=\xi_j-P_j(\xi_1, \ldots, \xi_m)$, let u_j be the element in $U(\mathfrak{g}_C)$ corresponding to the polynomial function $l \to Q_j(-il)$ on $\mathfrak{g}^*$ via symmetrization (note that $u_{j_k} \equiv 0$), and let π be the irreducible representation of G corresponding to the orbit O. Our formula for the kernel of $d\pi$ then reads

$$\ker(d\pi)=\sum_{j \notin e,\, j=1}^{m} u_j \cdot U(\mathfrak{g}_C);$$

in other words, ker $(d\pi)$ is the right ideal generated by the elements $(u_j)_{j \notin e}$.

Our second main result (Section 3) is concerned with the problem of determining algebraically the coadjoint orbit associated with a given irreducible representation of G. In this connection, let us recall that e. g. for compact semisimple Lie groups an irreducible representation is completely determined by its infinitesimal character, but that this is far from true for nilpotent Lie groups (although it is known, [7], that for representations corresponding to orbits in general position (in some specific sense) the infinitesimal characters do determine the representation). We present here for nilpotent Lie groups an approach—not based on infinitesimal characters, but on the results of Section 2 and certain parts of the results of [8]—to the solution of the problem. Our method consists of checking the differential of the given irreducible representation on a finite, explicitly constructible family of elements in the universal enveloping algebra of $\mathfrak{g}_C$. As a corollary we get an algebraic criterion for a representation π of G to be factorial (i. e. a multiple of an irreducible representation).

In the last part of the paper we consider a question concerning the continuity of the trace. In [4] DIXMIER showed that the group C^*-algebra $C^*(G)$ of G is with generalized continuous trace (GCT), and that the canonical composition series of $C^*(G)$ is of finite length (for definitions, *see* Section 4.1, *cf.* [2]). Here we show—using in an essential way the results of Section 3—that such a finite composition series can be found even in the $*$-algebra $C_c^\infty(G)$, the space of infinitely differentiable functions on G with compact support, and not just in $C^*(G)$.

1. Preliminaries

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}=\mathfrak{g}_m \supset \mathfrak{g}_{m-1} \supset \ldots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0=\{0\}$ be a Jordan-

Hölder sequence for $\mathfrak{g}$, i. e. a decreasing sequence of ideals such that $\dim \mathfrak{g}_j = j$, $j=0, \ldots, m$.

We let G act in $\mathfrak{g}^*$ via the coadjoint representation. For $g \in \mathfrak{g}^*$ we have the skewsymmetric bilinear form $B_g : \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ given by

$$B_g(X, Y) = \langle g, [X, Y] \rangle, \qquad X, Y \in \mathfrak{g}.$$

The radical of B_g is equal to the Lie algebra $\mathfrak{g}_g$ of the stabilizer G_g of g:

$$\mathfrak{g}_g = \{ X \in \mathfrak{g} \mid B_g(X, Y) = 0 \quad \text{for all } Y \in \mathfrak{g} \}.$$

1.1. For $g \in \mathfrak{g}^*$ we define J_g to be the set

$$J_g = \{ 1 \leqslant j \leqslant m \mid \mathfrak{g}_j \nsubseteq \mathfrak{g}_{j-1} + \mathfrak{g}_g \}.$$

Let $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$, $j=1, \ldots, m$. Then $X_1, \ldots, X_m$ is a basis in $\mathfrak{g}$, and we have $j \in J_g \Leftrightarrow X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}_g$.

If $g \in \mathfrak{g}^*$ with $J_g \neq \varnothing$ ($\Leftrightarrow \mathfrak{g}_g \neq \mathfrak{g}$) and if $J_g = \{ j_1 < \ldots < j_d \}$, then $X_{j_1}, \ldots, X_{j_d}$ is a basis for $\mathfrak{g} (\operatorname{mod} \mathfrak{g}_g)$.

Set $\mathscr{E} = \{ J_g \mid g \in \mathfrak{g}^* \}$, and set, for $e \in \mathscr{E}$,

$$\Omega_e = \{ g \in \mathfrak{g}^* \mid J_g = e \}.$$

We have $\mathfrak{g} = \bigcup_{e \in \mathscr{E}} \Omega_e$ as a (finite) disjoint union.

If α is an automorphism of $\mathfrak{g}$ leaving invariant the Jordan-Hölder sequence $\mathfrak{g} = \mathfrak{g}_m \supset \ldots \supset \mathfrak{g}_0 = \{ 0 \}$, then clearly $J_{\alpha g} = J_g$ for all $g \in \mathfrak{g}^*$, so Ω_e is α-invariant for all $e \in \mathscr{E}$. In particular, Ω_e is G-invariant for all $e \in \mathscr{E}$.

Let $e \in \mathscr{E}$. If $e \neq \varnothing$ with $e = \{ j_1 < \ldots < j_d \}$ we define the skewsymmetric $d \times d$-matrix $M_e(g)$, $g \in \mathfrak{g}^*$, by

$$M_e(g) = [B_g(X_{j_r}, X_{j_s})]_{1 \leqslant r, s \leqslant d},$$

and let $P_e(g)$ denote the Pfaffian of $M_e(g)$. If $e = \varnothing$ we set $M_e(g) = 1$ and $P_e(g) = 1$.

The map $g \to P_e(g)$ is a real valued polynomial function on $\mathfrak{g}^*$. $P_e(g)$ has the property that $P_e(g)^2 = \det M_e(g)$.

Let α be an automorphism of $\mathfrak{g}$ respecting the given Jordan-Hölder sequence, and let μ_j be the (non-zero) real number such that $\alpha(X_j)=\mu_j X_j (\mathrm{mod}\, \mathfrak{g}_{j-1})$, $j=1, \ldots, m$. For $e \in \mathscr{E}$, set $\mu_e=\prod_{j \in e} \mu_j$.

LEMMA 1.1.1. — *Let $e \in \mathscr{E}$. If $g \in \Omega_e$, then $P_e(g) \neq 0$ and $P_e(\alpha g)=\mu_e^{-1} P_e(g)$. In particular $P(sg)=P(g)$ for all $s \in G$.*

Proof. — Write $e=\{j_1<\ldots<j_d\}$ (the case $e=\varnothing$ is trivial). Since $X_{j_1}, \ldots, X_{j_d}$ is a basis for $\mathfrak{g}(\mathrm{mod}\, \mathfrak{g}_g)$ we have that $M_e(g)$ is a regular matrix, hence $P_e(g)^2=\det M_e(g) \neq 0$.

Next, write

$$\alpha^{-1}(X_{j_v})=\sum_{u=1}^d a_{uv} X_{j_u}+c_v,$$

where $c_v \in \mathfrak{g}_g$, $v=1, \ldots, d$. Then $a_{uv}=0$ for $u>v$, $a_{uu}=\mu_u^{-1}$ and

$$\begin{aligned} B_{\alpha g}(X_{j_u}, X_{j_v}) &= \langle \alpha g, [X_{j_u}, X_{j_v}] \rangle \\ &= \langle g, [\alpha^{-1}(X_{j_u}), \alpha^{-1}(X_{j_v})] \rangle \\ &= \sum_{p, q=1}^d a_{pu} \langle g, [X_{j_p}, X_{j_q}] \rangle a_{qv}=({}^t A M_e(g) A)_{u, v}, \end{aligned}$$

where A is the matrix $[a_{uv}]_{1 \leqslant u, v \leqslant d}$. This shows that $M_e(\alpha g)={}^t A M_e(g) A$, and since $\det A=\mu_e^{-1}$ we find that

$$P_e(\alpha g)=Pf(M_e(\alpha g))=Pf({}^t A M_e(g) A)=\det A\, Pf(M_e(g))=\mu_e^{-1} P_e(g).$$

This ends the proof of the lemma.

Remark 1.1.2. — Our definitions agree with those given by PUKANSZKY in [11], p. 525 ff., *cf.* also [10] and [8].

1.2. Recall the following facts (*cf.* [11], Proposition 1.1, p. 513 and Proposition 4.1, p. 525, *cf.* also [9], [10]):

Let $e \in \mathscr{E}$ and write (for $e \neq \varnothing$) $e=\{j_1<\ldots<j_d\}$. There exists functions $R_j^e: \Omega_e \times \mathbb{R}^d \rightarrow \mathbb{R}$, $j=1, \ldots, m$, such that:

(*a*) the function $x=(x_1, \ldots, x_d) \rightarrow R_j^e(g, x): \mathbb{R}^d \rightarrow \mathbb{R}$ is (for fixed $g \in \Omega_e$) a polynomial function depending only on the variables $x_1, \ldots, x_k$, where k is such that $j_k \leqslant j<j_{k+1}$;

(*b*) $R^e_{j_k}(g, x)=x_k$ for $g\in\Omega_e$, $k=1, \ldots, d$;

(*c*) for each $g\in\Omega_e$ the coadjoint orbit $G.g$ through g is given by

$$G.g=\{\sum_{j=1}^m R^e_j(g,x)\,l_j \mid x\in\mathbf{R}^d\},$$

where $l_1, \ldots, l_m$ is the basis in $\mathfrak{g}^*$ dual to $X_1, \ldots, X_m$.

The functions $R^e_j:\Omega_e\times\mathbf{R}^d\to\mathbf{R}$ are characterised by the three properties (*a*), (*b*) and (*c*), and they have the following further properties:

(*d*) there exists an integer N such that the function $(g, x)\to P_e(g)^N R^e_j(g, x)$ is the restriction to $\Omega_e\times\mathbf{R}^d$ of a polynomial function on $\mathfrak{g}^*\times\mathbf{R}^d$;

(*e*) $R^e_j(sg, x)=R^e_j(g, x)$ for all $g\in\Omega_e$, $x\in\mathbf{R}^d$ and $s\in G$.

For $\alpha=(\alpha_1, \ldots, \alpha_d)$ a d-multi-index of non-negative integers and $x=(x_1, \ldots, x_d)\in\mathbf{R}^d$ we write $x^\alpha=x_1^{\alpha_1}\ldots x_d^{\alpha_d}$. From the properties above it then follows that we can write

$$R^e_j(g, x)=\sum_\alpha a^e_{j,\alpha}(g)\,x^\alpha,$$

where $a^e_{j,\alpha}:\Omega_e\to\mathbf{R}$ are G-invariant functions on Ω_e which are identically zero, except for finitely many α. The function $a^e_{j,\alpha}$ has the property that there exists an integer N such that $g\to P_e(g)^N a^e_{j,\alpha}(g)$ is the restriction to Ω_e of a polynomial function on $\mathfrak{g}^*$.

1.3. In the following we shall make repeated use of the following facts [5]: There exists an isomorphism ω (the symmetrization map) between the complex vector space $S(\mathfrak{g}_\mathbf{C})$ (the symmetric algebra of $\mathfrak{g}_\mathbf{C}$), and the complex vector space $U(\mathfrak{g}_\mathbf{C})$ (the universal enveloping algebra of $\mathfrak{g}_\mathbf{C}$), characterised by the following property: If $Y_1, \ldots, Y_p$ are elements in $\mathfrak{g}_\mathbf{C}$, then the image of the element $Y_1\ldots Y_p$ in $S(\mathfrak{g}_\mathbf{C})$ by ω is the element

$$(p!)^{-1}\sum_{\sigma\in S_p} Y_{\sigma(1)}\ldots Y_{\sigma(p)}$$

in $U(\mathfrak{g}_\mathbf{C})$, where S_p is the group of permutations of p elements. Moreover we have the following lemma (*cf.* [8], Lemma 1.2.1).

LEMMA 1.3.1. — *If Z is a central element in $\mathfrak{g}_\mathbf{C}$, then $\omega(Zu)=Z\,\omega(u)$ for all $u\in S(\mathfrak{g}_\mathbf{C})$.*

1.4. Let $e \in \mathcal{E}$ with $e \neq \emptyset$ and write $e = \{j_1 < \ldots < j_d\}$. For $1 \leqslant j \leqslant m$ we let $r_j^e(g)$, $g \in \Omega_e$, be the image in $U(\mathfrak{g}_{\mathbf{C}})$ by ω of the element

$$R_j^e(g, -iX_{j_1}, \ldots, -iX_{j_d})$$

in $S(\mathfrak{g}_{\mathbf{C}})$ (what we have done here is that we have replaced the variable x_k in $R_j^e(g, x) = R_j^e(g, x_1, \ldots, x_d)$ by $-iX_{j_k}$). If $e = \emptyset$ we set $r_j^e(g) = \langle g, X_j \rangle . 1 (= \omega(R_j^e(g, x))$, since $R_j^e(g, x) = \langle g, X_j \rangle)$. Note that in particular $r_{j_k}^e(g) = -iX_{j_k}$.

2. A formula for the infinitesimal kernel of the irreducible representations

2.1. Our first result shows the relevance of the elements $r_j^e(g) \in U(\mathfrak{g}_{\mathbf{C}})$ introduced in Section 1.4.

THEOREM 2.1.1. — *Let $g \in \Omega_e$ and let π be the irreducible representation of G corresponding to the orbit $G.g$. Then*

$$d\pi(X_j) = i d\pi(r_j^e(g))$$

for $1 \leqslant j \leqslant m$.

Remark 2.1.2. — For $j = j_k \in e$ the statement of the theorem is empty since $r_{j_k}^e(g) = -iX_{j_k}$.

Proof. — The proof is by induction on the dimension of $\mathfrak{g}$. Suppose first that $\dim \mathfrak{g} = 1$. Then $e = \emptyset$, $R_1^e(g, x) = \langle g, X_1 \rangle$ and $r_1^e(g) = \langle g, X_1 \rangle . 1$. But $d\pi(X_1) = i \langle g, X_1 \rangle I = i d\pi(r_1^e(g))$ so this shows the validity of the result in this case.

Suppose then that the result has been proved for all dimensions of the group less than $m > 1$. Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$, and set $\mathfrak{z}_0 = \ker g \mid \mathfrak{z}$ which is an ideal in $\mathfrak{g}$. We distinguish two cases: case (*a*): $\dim \mathfrak{z}_0 > 0$ and case (*b*): $\dim \mathfrak{z}_0 = 0$.

Case (*a*). — Set $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}_0$, and let $c : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be the quotient map. We let also c denote the quotient map $c : G \to \tilde{G} = G/Z_0$, where $Z_0 = \exp \mathfrak{z}_0$. There exists an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ such that $\tilde{\pi} \circ c = \pi$, and the orbit of $\tilde{\pi}$ is determined by the functional $\tilde{g} \in \tilde{\mathfrak{g}}^*$ defined by $\tilde{g} \circ c = g$.

We set $I = \{ 1 \leqslant j \leqslant m \mid \mathfrak{g}_j \nsubseteq \mathfrak{g}_{j-1} + \mathfrak{z}_0 \}$, and write $I = \{ i_1 < \ldots < i_n \}$ and $\tilde{\mathfrak{g}}_j = (\mathfrak{g}_{i_j} + \mathfrak{z}_0)/\mathfrak{z}_0$. Then $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_n \supset \ldots \supset \tilde{\mathfrak{g}}_0 = \{0\}$ is a Jordan-Hölder sequence in $\tilde{\mathfrak{g}}$, and setting $\tilde{X}_j=c(X_{i_j})$ we have $\tilde{X}_j\in\tilde{\mathfrak{g}}_j\setminus\tilde{\mathfrak{g}}_{j-1}$. We next note that $\mathfrak{z}_0\subset\mathfrak{g}_g$ and that $\tilde{\mathfrak{g}}_{\tilde{g}}=\mathfrak{g}_g/\mathfrak{z}_0$. Moreover, $J_g\subset I$ since $j\notin I \Rightarrow X_j\in\mathfrak{g}_{j-1}+\mathfrak{z}_0 \Rightarrow X_j\in\mathfrak{g}_{j-1}+\mathfrak{g}_g \Rightarrow j\notin J_g$.

Writing $e=J_g=\{j_1<\ldots<j_d\}$ and $\tilde{e}=J_{\tilde{g}}=\{\tilde{j}_1<\ldots<\tilde{j}_d\}$ we have that $i_{\tilde{j}_k}=j_k$, $k=1,\ldots,d$.

Let $x\in\mathbb{R}^d$, and set $\tilde{l}=\sum_{j=1}^{n}R_j^{\tilde{e}}(\tilde{g},x)\tilde{l}_j$, where $\tilde{l}_1,\ldots,\tilde{l}_n$ is the basis in $\mathfrak{g}^*$ dual to $\tilde{X}_1,\ldots,\tilde{X}_n$. Then setting $l=\tilde{l}\circ c$ we have

$$R_j^{\tilde{e}}(g,x)=\langle\tilde{l},\tilde{X}_j\rangle=\langle l,X_{i_j}\rangle;$$

in particular $x_k=\langle l,X_{i_{\tilde{j}_k}}\rangle=\langle l,X_{j_k}\rangle$, and this implies that

$$l=\sum_{j=1}^{m}R_j^e(g,x)\,l_j,\qquad\text{so}\quad R_j^e(g,x)=\langle l,X_j\rangle,$$

$j=1,\ldots,m$. We conclude from this that

$$R_{i_j}^e(g,x)=R_j^{\tilde{e}}(\tilde{g},x)\qquad\text{for}\quad 1\leqslant j\leqslant n,$$

and therefore $c(r_{i_j}^e(g))=r_j^{\tilde{e}}(\tilde{g})$, $1\leqslant j\leqslant n$, hence, by the induction hypothesis,

$$d\pi(X_{i_j})=d\tilde{\pi}(\tilde{X}_j)=id\tilde{\pi}(r_j^{\tilde{e}}(\tilde{g}))=id\pi(r_{i_j}^e(g))\qquad\text{for}\quad 1\leqslant j\leqslant n.$$

Suppose then that $j\notin I$. We can write $X_j=\sum_{p=1}^{n}a_{jp}X_{i_p}+Z_j$, where $Z_j\in\mathfrak{z}_0$, since $X_{i_1},\ldots,X_{i_n}$ is a basis in $\mathfrak{g}$ (mod $\mathfrak{z}_0$). Let then $x\in\mathbb{R}^d$, and set $l=\sum_{k=1}^{m}R_k^e(g,x)\,l_j$. We have $R_j^e(g,x)=\langle l,X_j\rangle$, and since $l\in G.g$ and therefore $l\,|\,\mathfrak{z}_0\equiv 0$, we have

$$R_j^e(g,x)=\sum_{p=1}^{n}a_{jp}R_{i_p}^e(g,x),$$

so that

$$r_j^e(g)=\sum_{p=1}^{n}a_{jp}r_{i_p}^e(g).$$

But since $Z_j\in\mathfrak{z}_0$ we have that $d\pi(Z_j)=0$, and therefore $d\pi(X_j)=\sum_{p=1}^{n}a_{jp}d\pi(X_{i_p})$. It follows that $d\pi(X_j)=id\pi(r_j^e(g))$, since we have already shown that $d\pi(X_{i_j})=id\pi(r^e_{i_j}(g))$ for $1\leqslant j\leqslant n$. This ends case (*a*).

Case (*b*). — In this case we have that $\dim \mathfrak{z}=1$ and $g|\mathfrak{z}\neq 0$, so $\mathfrak{z}=\mathfrak{g}_1$ and $\langle g, X_1\rangle\neq 0$. In particular $[\mathfrak{g}, \mathfrak{g}_2]=\mathfrak{g}_1$, and therefore $\mathfrak{g}_2\not\subset\mathfrak{g}_g$, hence $2\in J_g$ and $j_1=2$. Note also that $\mathfrak{g}_g\subset\mathfrak{h}=\ker \operatorname{ad} X_2$ (since otherwise $\mathfrak{g}_g+\mathfrak{h}=\mathfrak{g}$ and therefore

$$\langle g, \mathfrak{g}_1\rangle=\langle g, [\mathfrak{g}, X_2]\rangle=\langle \mathfrak{g}g, X_2\rangle=\langle \mathfrak{h}g, X_2\rangle=0$$

which is a contradiction). We then claim that we can assume that $\mathfrak{g}_{m-1}=\mathfrak{h}=\ker \operatorname{ad} X_2$.

Proof of claim. — Clearly $\mathfrak{h}=\ker \operatorname{ad} X_2$ is an ideal in $\mathfrak{g}$ of codimension 1. Set

$$p=\min\{1\leqslant j\leqslant m\,|\,X_j\notin\mathfrak{g}\}.$$

Then p is well-defined, $p\geqslant 3$ and $\mathfrak{g}=\mathfrak{h}\oplus\mathbf{R}X_p$. It is easily seen that $p\in J_g$ (in fact, if $p\notin J_g$, then $X_p\in\mathfrak{g}_g+\mathfrak{g}_{p-1}\subset\mathfrak{g}_g+\mathfrak{h}\subset\mathfrak{h}$ which is a contradiction). We then define a new basis $\hat{X}_1, \ldots, \hat{X}_m$ in $\mathfrak{g}$ in the following way: For $1\leqslant j\leqslant p-1$ we set $\hat{X}_j=X_j$, for $p\leqslant j\leqslant m-1$ we set $\hat{X}_j=X_{j+1}+c_{j+1}X_p$, where the $c_{j+1}\in\mathbf{R}$ are selected such that $\hat{X}_j\in\mathfrak{h}$ (which is possible since $\mathbf{R}X_p\oplus\mathfrak{h}=\mathfrak{g}$), and finally we set $\hat{X}_m=X_p$. We then define the linear subspaces $\hat{\mathfrak{g}}_j$, $j=1, \ldots, m$, in $\mathfrak{g}$ by

$$\hat{\mathfrak{g}}_j=\mathbf{R}\hat{X}_1\oplus\ldots\oplus\mathbf{R}\hat{X}_j.$$

We have

$$\hat{\mathfrak{g}}_j=\mathfrak{g}_j \qquad \text{for}\quad 1\leqslant j\leqslant p-1,$$

and

$$\hat{\mathfrak{g}}_{j+1}=\mathfrak{g}_j\oplus\mathbf{R}X_p \qquad \text{for}\quad p-1\leqslant j\leqslant m-1,$$

implying that

$$\mathfrak{g}_j=\hat{\mathfrak{g}}_{j+1}\cap\mathfrak{h} \qquad \text{for}\quad p-1\leqslant j\leqslant m-1.$$

This shows that $\hat{\mathfrak{g}}_1, \ldots, \hat{\mathfrak{g}}_m$ is a Jordan-Hölder sequence for $\mathfrak{g}$. By construction $\hat{\mathfrak{g}}_{m-1}=\mathfrak{h}$. We designate the objects associated with this new Jordan-Hölder sequence $\hat{J}_g=\hat{e}$, etc. We write $\hat{J}_g=\{\hat{j}_1<\ldots<\hat{j}_d\}$.

For $1\leqslant j\leqslant p-1$ we clearly have that $j\in J_g\Leftrightarrow j\in\hat{J}_j$. Furthermore $p\in J_g$ (*see* above) and $m\in\hat{J}_g$ (in fact, if $m\notin\hat{J}_g$, then $X_p=\hat{X}_m\in\hat{\mathfrak{g}}_{m-1}+\mathfrak{g}_g=\mathfrak{h}+\mathfrak{g}_g=\mathfrak{h}$

which is a contradiction). For $p+1\leqslant j\leqslant m$ we have

$$j\notin J_g \Leftrightarrow X_j\in \mathfrak{g}_{j-1}+\mathfrak{g}_g \Leftrightarrow X_j\in \hat{\mathfrak{g}}_{j-2}+\mathbf{R}X_p+\mathfrak{g}_g$$
$$\Leftrightarrow \hat{X}_{j-1}\in \hat{\mathfrak{g}}_{j-2}+\mathbf{R}X_p+\mathfrak{g}_g \Leftrightarrow \hat{X}_{j-1}\in \hat{\mathfrak{g}}_{j-2}+\mathfrak{g}_g$$
$$(\text{since } \mathfrak{g}_g\subset \mathfrak{h}) \Leftrightarrow j-1\notin \hat{J}_g.$$

Therefore, if $j_\alpha=p$ we have

$$\hat{j}_r=j_r \quad \text{for} \quad 1\leqslant r\leqslant \alpha-1,$$
$$\hat{j}_r+1=j_{r+1} \quad \text{for} \quad \alpha\leqslant r\leqslant d-1$$
$$\text{and} \quad \hat{j}_d=m.$$

Let then $x\in \mathbf{R}^d$ and set $l=\sum_{j=1}^m R_j^e(g, x)\,l_j$. We have

$$R_j^e(g,x)=\langle l, X_j\rangle \qquad \text{and} \qquad x_k=\langle l, X_{j_k}\rangle.$$

Now we can also write $l=\sum_{j=1}^m R_j^{\hat{e}}(g, \hat{x})$, where $\hat{x}\in \mathbf{R}^d$ and

$$R_j^{\hat{e}}(g,\hat{x})=\langle l, \hat{X}_j\rangle, \qquad \hat{x}_k=\langle l, \hat{X}_{\hat{j}_k}\rangle.$$

For $1\leqslant k\leqslant \alpha-1$ we have

$$x_k=\langle l, X_{j_k}\rangle=\langle l, \hat{X}_{\hat{j}_k}\rangle=\hat{x}_k,$$

and for $\alpha\leqslant k\leqslant d-1$ we have

$$\hat{x}_k=\langle l, \hat{X}_{\hat{j}_k}\rangle=\langle l, X_{\hat{j}_k+1}+c_{\hat{j}_k+1}X_p\rangle$$
$$=\langle l, X_{j_{k+1}}+c_{j_{k+1}}X_p\rangle=x_{k+1}+c_{j_{k+1}}x_\alpha,$$

and

$$\hat{x}_d=\langle l, \hat{X}_{\hat{j}_d}\rangle=\langle l, \hat{X}_m\rangle=\langle l, X_p\rangle=x_\alpha.$$

So for $1\leqslant j\leqslant p-1$ we get

$$R_j^e(g, x)=\langle l, X_j\rangle=\langle l, \hat{X}_j\rangle=R_j^e(g, \hat{x}),$$

and therefore

$$R_j^e(g, -iX_{j_1}, \ldots, -iX_{j_d})=R_j^e(g, -iX_{j_1}, \ldots, -iX_{j_{\alpha-1}}, 0, \ldots, 0)$$
$$=R_j^{\hat{e}}(g, -i\hat{X}_{\hat{j}_1}, \ldots, -i\hat{X}_{\hat{j}_{\alpha-1}}, 0, \ldots, 0)=R_j^{\hat{e}}(g, -i\hat{X}_{\hat{j}_1}, \ldots, -i\hat{X}_{\hat{j}_d}),$$

and this implis that $r_j^{\hat{e}}(g)=r_j^e(g)$ for $1\leqslant j\leqslant p-1$. For $p\leqslant j\leqslant m-1$ we get

$$R_j^{\hat{e}}(g, \hat{x}_1, \ldots, \hat{x}_d)=\langle l, \hat{X}_j\rangle=\langle l, X_{j+1}+c_{j+1}X_p\rangle$$
$$=R_{j+1}^e(g, x_1, \ldots, x_d)+c_{j+1}x_\alpha,$$

and therefore

$$R_{j+1}^e(g, x_1, \ldots, x_d)+c_{j+1}x_\alpha$$
$$=R_j^{\hat{e}}(g, x_1, \ldots, x_{\alpha-1}, x_{\alpha+1}+c_{j_{\alpha+1}}x_\alpha, \ldots, x_d+c_{j_d}x_\alpha, x_\alpha),$$

so

$$R_{j+1}^e(g, -iX_{j_1}, \ldots, -iX_{j_d})-ic_{j+1}X_p$$
$$=R_j^{\hat{e}}(g, -iX_{j_1}, \ldots, -iX_{j_{\alpha-1}}, -iX_{j_{\alpha+1}}$$
$$-ic_{j_{\alpha+1}}X_p, \ldots, -iX_{j_d}-ic_{j_d}X_p, -iX_p)$$
$$=R_j^{\hat{e}}(g, -i\hat{X}_{\hat{j}_1}, \ldots, -i\hat{X}_{\hat{j}_d}),$$

implying that $r_{j+1}^e(g)-ic_{j+1}X_p=r_j^{\hat{e}}(g)$, and therefore

$$X_{j+1}-ir_{j+1}^e(g)=\hat{X}_j-c_{j+1}X_p-i(r_j^{\hat{e}}(g)+ic_{j+1}X_p)=\hat{X}_j-ir_j^{\hat{e}}(g)).$$

We have thus reduced to the case where $\mathfrak{g}_{m-1}=\ker \operatorname{ad} X_2$, and proved our claim.

From now on we then assume that $\mathfrak{g}_{m-1}=\mathfrak{h}=\ker \operatorname{ad} X_2$, and set $g_0=g\,|\,\mathfrak{g}_{m-1}$. Set $H=\exp\mathfrak{h}$. The representation π can be realized as the induced representation $\pi=\operatorname{ind}_{H\uparrow G}\pi_0$ on the space $L^2(G, \pi_0)$, where π_0 is the irreducible representation associated with the H-orbit through g_0. For a differentiable vector $\varphi\in L^2(G, \pi_0)$ and an element $u\in U(\mathfrak{g}_{\mathbf{C}})$ we have $(d\pi(u)\varphi)(s)=d\pi_0(\operatorname{Ad}(s^{-1})u)\varphi(s)$.

We designate the objects associated with the Jordan-Hölder sequence $\mathfrak{h}=\mathfrak{g}_{m-1}\supset\ldots\supset\mathfrak{g}_0=\{0\}$ by $J_{g0}=e^0$, etc. Since $\mathscr{G}_{g0}=\mathfrak{g}_g\oplus\mathbf{R}X_2$ we have that

$$j_r^0=j_{r+1}, \qquad r=1, \ldots, d-2.$$

Let then $1\leqslant j\leqslant m-1$, $x\in\mathbf{R}^d$, write $l=\sum_{j=1}^m \mathfrak{g}_g(g, x)l_j$ and set $l_0=l\,|\,\mathfrak{h}$. We can write $l=sg$ with

$$s=s_0\exp tX_m, \qquad s_0\in H, \qquad t\in\mathbf{R},$$

implying that l_0 is in the H-orbit of $\exp t X_m g_0$. Therefore

$$l_0=\sum_{j=1}^{m-1} R_j^{e^0}(\exp t X_m g_0, x^0)\, l_j^0,$$

so

$$R_j^e(g, x)=R_j^{e^0}(\exp t X_m g_0, x^0) \qquad \text{for} \quad 1\leqslant j\leqslant m-1.$$

Now for $1\leqslant r\leqslant d-2$ we have

$$x_r^0=\langle l, X_{j_r^0}\rangle=\langle l, X_{j_{r+1}}\rangle=x_{r+1},$$

and

$$x_1=\langle l, X_{j_1}\rangle=\langle l, X_2\rangle=R_2^e(g, x)=R_2^{e^0}(\exp t X_m g_0, x^0)$$
$$=\langle \exp t X_m g_0, X_2\rangle=\langle g_0, X_2-t[X_m, X_2]\rangle,$$

and therefore $t=(\langle g, [X_m, X_2]\rangle)^{-1}(\langle g, X_2\rangle-x_1)$.

The conclusion is that for $1\leqslant j\leqslant m-1$ we have:

$$R_j^e(g, x_1, x_2, \ldots, x_{d-1}, x_d)=R_j^{e^0}\left(\exp\frac{\langle g, X_2\rangle-x_1}{\langle g, [X_m, X_2]\rangle}X_m g_0, x_2, \ldots, x_{d-1}\right).$$

We then write (*cf.* 1.3) for $1\leqslant j\leqslant m-1$:

$$R_j^{e^0}(l_0, x^0)=\sum_{\alpha_0} a_{j,\alpha_0}^{e^0}(l_0)(x^0)^{\alpha_0}, \qquad l_0\in\Omega_{e^0},$$

and get

$$R_j^e(g, x)=\sum_{\alpha_0} a_{j,\alpha_0}^{e^0}\left(\exp\frac{\langle g, X_2\rangle-x_1}{\langle g, [X_m, X_2]\rangle}X_m g_0\right)x_2^{\alpha_1^0}\ldots x_{d-1}^{\alpha_{d-2}^0}.$$

Now $a_{j,\alpha_0}^{e^0}(l_0)$ has the form $P(l_0)P_{e^0}(l_0)^{-N}$, where P is a polynomial function on $\mathfrak{h}^*$, and since P_{e^0} is G-invariant (Lemma 1.1.1) we get that

$$x_1\rightarrow a_{j,\alpha_0}^{e^0}\left(\exp\frac{\langle g, X_2\rangle-x_1}{\langle g, [X_m, X_2]\rangle}X_m g_0\right)$$

is a polynomial function in x_1 which we denote $T_{\alpha_0}(x_1)$. We set $P_{\alpha_0}(x)=x_2^{\alpha_1^0}\ldots x_{d-1}^{\alpha_{d-2}^0}$ and so we get

$$R_j^e(g, x)=\sum_{\alpha_0} T_{\alpha_0}(x_1)P_{\alpha_0}(x_2, \ldots, x_{d-1}),$$

and therefore

$$R_j^e(g, -iX_{j_1}, \ldots, -iX_{j_d}) = \sum_{\alpha_0} T_{\alpha_0}(-iX_{j_1}) P_{\alpha_0}(-iX_{j_2}, \ldots, -iX_{j_d - 1}),$$

and since $X_{j_1} = X_2$ is central in $\mathfrak{h}$ we get that

$$r_j^e(g) = \sum_{\alpha_0} t_{\alpha_0} \cdot p_{\alpha_0},$$

where t_{α_0} is the symmetrization of $T_{\alpha_0}(-iX_{j_1})$ and p_{α_0} is the symmetrization of $P_{\alpha_0}(-iX_{j_2}, \ldots, -iX_{j_d - 1})$ (Lemma 1.3.1).

But then

$$d\pi_0(r_j^e(g)) = \sum_{\alpha_0} d\pi_0(t_{\alpha_0}) d\pi_0(p_{\alpha_0}) = \sum_{\alpha_0} a_{j,\alpha_0}^{e^0}(g_0) d\pi_0(p_{\alpha_0}),$$

and since

$$r_j^{e^0}(g_0) = \sum_{\alpha_0} a_{j,\alpha_0}^{e^0}(g_0) p_{\alpha_0},$$

we have showed that

$$d\pi_0(r_j^e(g)) = d\pi_0(r_j^{e^0}(g_0)),$$

and using the induction hypothesis we then get that $d\pi_0(X_j) = id\pi_0(r_j^e(g))$. Applying this to the functional sg, $s \in G$, we get

$$d(s\pi_0)(X_j) = id(s\pi_0)(r_j^e(sg)) = id(s\pi_0)(r_j^e(g)),$$

so that

$$d\pi_0(\mathrm{Ad}(s^{-1}))X_j) = id\pi_0(\mathrm{Ad}(s^{-1}) r_j^e(g)),$$

and therefore finally $d\pi(X_j) = id\pi(r_j^e(g))$. This ends the proof of the theorem.

Remark 2.1.3. — Certain points in the reasoning above can be found already in our previous publication [8]. However, for the convenience of the reader we have repeated them here, since the present context is much simpler than the one in [8].

2.2. If $g \in \mathfrak{g}^*$ and if π is the irreducible representation of G associated with the orbit $0 = Gg$, we let $I(g)$ denote the kernel of the differential $d\pi$ of π considered as a representation of $U(\mathfrak{g}_{\mathbf{C}})$.

For $e \in \mathscr{E}$ with $e \neq \varnothing$, let G_e denote the linear span in $S(\mathfrak{g}_{\mathbf{C}})$ of the elements of the form $X_{j_1}^{\alpha_1} \ldots X_{j_d}^{\alpha_d}$, where $e = \{j_1 < \ldots < j_d\}$, and where $\alpha_1, \ldots, \alpha_d$ are non-negative integers, and set F_e to be the image in $U(\mathfrak{g}_{\mathbf{C}})$ of G_e by the symmetrization map ω. Moreover, let E_e denote the linear span in $U(\mathfrak{g}_{\mathbf{C}})$ of the elements of the form $X_{j_1}^{\alpha_1} \ldots X_{j_d}^{\alpha_d}$. If $e = \varnothing$, set $G_e = \mathbf{C}\,1$, $F_e = \mathbf{C}\,1 = \omega(G_e)$, $E_e = \mathbf{C}\,1$.

Set, for $e \in \mathscr{E}$, $g \in \Omega_e$ and $1 \leqslant j \leqslant m$, $u_j^e(g)$ to be equal to $X_j - ir_j^e(g)$ (note that $u_j^e(g) \equiv 0$ if $j \in e$).

The following theorem not only gives an explicit finite set of generators for the ideal $I(g)$, but also an explicit (in fact two) supplementary subspace (s) of $I(g)$ in $U(\mathfrak{g}_{\mathbf{C}})$.

THEOREM 2.2.1. — *If $g \in \Omega_e$, then $I(g)$ is generated by the elements $(u_j^e(g))_{j \notin e}$, and*

$$U(\mathfrak{g}_{\mathbf{C}}) = I(g) \oplus E_e = I(g) \oplus F_e.$$

Remark 2.2.2. — M. Duflo has kindly made me aware of the paper [6] of Godfrey, where it is proved, in the language of enveloping algebras, that there exists, for a given coadjoint orbit O, polynomial functions $P_1, \ldots, P_n$ on $\mathfrak{g}^*$ defining O such that the elements $u_1, \ldots, u_n$ in $U(\mathfrak{g}_{\mathbf{C}})$ corresponding by symmetrization to the polynomial functions $l \to P_j(-il)$, $j = 1, \ldots, n$, generate ker $(d\pi)$, where π is the irreducible representation associated with O.

Proof. — For simplicity we set $Y_r = X_{j_r}$, $1 \leqslant r \leqslant d$. We denote by $\tilde{E}_e$ the linear span in $U(\mathfrak{g}_{\mathbf{C}})$ of the elements of the form $Y_{r_1} \ldots Y_{r_n}$, where $1 \leqslant r_k \leqslant d$ (in other words, $\tilde{E}_e$ is the subalgebra spanned by $Y_1, \ldots, Y_d$), and set I_0 to be the ideal generated by $(u_j^e(g))_{j \notin e}$. We already know that $I_0 \subset I(g)$ (Theorem 2.1.1).

LEMMA 2.2.3. — $U(\mathfrak{g}_{\mathbf{C}}) = I_0 + \tilde{E}_e$.

Proof. — We have $u_j^e(g) = X_j - ir_j^e(g)$, so $X_j = u_j^e(g) + ir_j^e(g)$. Let then $u \in U(\mathfrak{g}_{\mathbf{C}})$. We can write

$$u = \sum_\alpha a_\alpha X_1^{\alpha_1} \ldots X_m^{\alpha_m}$$

where $a_\alpha = 0$ except for finitely many multi-indices $\alpha = (\alpha_1, \ldots, \alpha_m)$.

But then

$$u = \sum_\alpha a_\alpha (u^e(g) + ir_1^e 2(g))^{\alpha_1} \ldots (u_m^e(g) + ir_m^e(g))\,\alpha_m = u_0 + \sum_\alpha i^{\alpha_1 + \ldots + \alpha_m} r_1^e(g)^{\alpha_1} \ldots r_m^e(g)^{\alpha_m},$$

where $u_0 \in I_0$. Now $r_j^e(g) \in \tilde{E}_e$, and we have thus shown that $u \in I_0 + \tilde{E}_e$. This ends the proof of the lemma.

We next proove the following two lemmas:

LEMMA 2.2.4. — $\tilde{E}_e \subset I_0 + E_e$.

LEMMA 2.2.5. — $\tilde{E}_e \subset I_0 + F_e$.

For the proof of these two lemmas we need a little preparation: Let A be the set of d-multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_1, \ldots, \alpha_d$ being non-negative integers. We define a total ordering on A in the following way: Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\alpha' = (\alpha'_1, \ldots, \alpha'_d)$ belong to A with $\alpha \neq \alpha'$; then

$$\alpha < \alpha' \quad \Leftrightarrow \quad \alpha_p < \alpha'_p,$$

where:

$$p = \max\{1 \leqslant k \leqslant d \,|\, \alpha_k \neq \alpha'_k\}.$$

In this way A is well-ordered.

For $\alpha = (\alpha_1, \ldots, \alpha_d)$, let G_e^α be the linear span in $S(\mathfrak{g}_{\mathbf{C}})$ of elements of the form $Y_1^{\beta_1} \ldots Y_d^{\beta_d}$, where $\beta_j \leqslant \alpha_j$ for $1 \leqslant j \leqslant d$, and set $F_e^\alpha = \omega(G_e^\alpha)$.

Moreover, let E_e^α be the linear span in $U(\mathfrak{g}_{\mathbf{C}})$ of elements of the form $Y_1^{\beta_1} \ldots Y_d^{\beta_d}$, where $\beta_j \leqslant \alpha_j$ for $1 \leqslant j \leqslant d$.

Finally set $\tilde{E}_e^\alpha$ to be the linear span of elements of the form $Y_{r_1} \ldots Y_{r_n}$, where Y_k appears at most α_k times in the product, i. e. such that $\#\{1 \leqslant t \leqslant n \,|\, r_t = k\} \leqslant \alpha_k$.

SUBLEMMA 2.2.6. — *For $\alpha \in A$ we have*

$$\tilde{E}_e^\alpha \subset I_0 + E_e^\alpha + \sum_{\beta < \alpha} \tilde{E}_e^\beta.$$

Proof. — The proof is by transfinite induction. Write $\alpha=(\alpha_1, \ldots, \alpha_d)$. If all α_j are zero except possibly for one value of j, then the lemma is clearly valid. So suppose α is not of this type, and that the result has been proved for all elements in A smaller than α, and let $Y_{r_1} \ldots Y_{r_n} \in \tilde{E}_e^\alpha$ be such that $\#\{t \mid r_t=j\}=\alpha_j, j=1, \ldots, d$.

Let k be such that $\alpha_k>0$ *and* $\alpha_j=0$ for $j>k$. Choose $1 \leqslant t \leqslant n$ such that $r_t=k$. We now claim that the element

$$Y_{r_1} \ldots Y_{r_t} \ldots Y_{r_n}-Y_{r_1} \ldots \not{Y}_{r_t} \ldots Y_{r_n} Y_{r_t}$$

belongs to $I_0+\sum_{\beta<\alpha} \tilde{E}_e^\beta$. If $t=n$ this is clear, so suppose that $t<n$. We can then write

$$\begin{aligned} Y_{r_t} Y_{r_{t+1}} = Y_{r_{t+1}} Y_{r_t}-[Y_{r_{t+1}}, Y_{r_t}] = Y_{r_{t+1}} Y_{r_t}-\textstyle\sum_{j<j_k} a_j X_j=(a_j \in \mathbf{R}) \\ = Y_{r_{t+1}} Y_{r_t}-\textstyle\sum_{j<j_k} a_j(u_j^e(g)+i r_j^e(g)) \\ = Y_{r_{t+1}} Y_{r_t}-\textstyle\sum_{j<j_k} i a_j r_j^e(g)-\sum_{j<j_k} a_j u_j^e(g). \end{aligned}$$

Now since $u_j^e(g) \in I_0$ and since an element

$$Y_{r_1} \ldots Y_{r_{t-1}} r_j^e(g) Y_{r_{t+2}} \ldots Y_{r_n}$$

clearly belongs to $\sum_{\beta<\alpha} \tilde{E}_e^\beta$ for all $j<j_k$ we see that the element $u=Y_{r_1} \ldots Y_{r_t} Y_{r_{t+1}} \ldots Y_{r_n}$ is equal to $Y_{r_1} \ldots Y_{r_{t+1}} Y_{r_t} \ldots Y_{r_n}+v+u_0$, where $v \in \sum_{\beta<\alpha} \tilde{E}_e^\beta$ and where $u_0 \in I_0$. Therefore, by moving Y_{r_t} one step to the right in the expression $Y_{r_1} \ldots Y_{r_t} \ldots Y_{r_n}$ we have perturbed only by an element in $I_0+\sum_{\beta<\alpha} \tilde{E}_e^\beta$. Continuing like this in finitely many steps we see that

$$Y_{r_1} \ldots Y_{r_t} \ldots Y_{r_n}-Y_{r_1} \ldots \not{Y}_{r_t} \ldots Y_{r_n} Y_{r_t}$$

belongs to $I_0+\sum_{\beta<\alpha} \tilde{E}_e^\beta$, and this establishes the validity of our claim. Now the element $u'=Y_{r_1} \ldots \not{Y}_{r_t} \ldots Y_{r_n}$ belongs to $\tilde{E}_e^{\alpha'}$, where $\alpha'=(\alpha_1, \ldots, \alpha_k-1, 0, \ldots, 0)$, and therefore, by the induction hypothesis,

$$u' \in I_0+E_e^{\alpha'}+\textstyle\sum_{\beta<\alpha'} \tilde{E}_e^\beta.$$

Moreover, if an element v belongs to $\tilde{E}_e^{\beta'}$, where $\beta'=(\beta'_1, \ldots, \beta'_k, 0, \ldots, 0)<\alpha'$, then $v\,Y_k$ belongs to $\tilde{E}_e^{\beta}$, where $\beta=(\beta'_1, \ldots, \beta'_k+1, 0 \ldots, 0)$, and clearly $\beta<\alpha$. But this shows that

$$u'\,Y_k \in I_0+E_e^{\alpha}+\textstyle\sum_{\beta<\alpha}\tilde{E}_e^{\beta},$$

since clearly $E_e^{\alpha'}.\,Y_k \subset E_e^{\alpha}$. This ends the proof of the sublemma.

SUBCOROLLARY 2.2.7. – *For $\alpha\in A$ we have*

$$\tilde{E}_e^{\alpha} \subset I_0+\textstyle\sum_{\beta\leqslant\alpha}E_e^{\beta}.$$

Proof. – Again by transfinite induction. The result is trivial for the minimal element. Suppose then that the corollary has been proved for all elements in A smaller than α. Then by the sublemma

$$\tilde{E}_e^{\alpha} \subset I_0+E_e^{\alpha}+\textstyle\sum_{\beta<\alpha}\tilde{E}_e^{\beta},$$

and the induction hypothesis gives that

$$\tilde{E}_e^{\beta} \subset I_0+\textstyle\sum_{\gamma\leqslant\beta}E_e^{\gamma} \qquad \text{for} \quad \beta<\alpha,$$

and therefore $\tilde{E}_e^{\alpha} \subset I_0+\sum_{\beta<\alpha}E_e^{\beta}$. This ends the proof of the subcorollary.

Now the validity of Lemma 2.2.4 follows immediately from Subcorollary 2.2.7. To prove Lemma 2.2.5 we need the following.

SUBLEMMA 2.2.8. – *If $Y_{r_1} \ldots Y_{r_n}$ belongs to E_e^{α}, then*

$$Y_{r_1} \ldots Y_{r_n} - \frac{1}{n!}\textstyle\sum_{\sigma\in S_n} Y_{r_{\sigma(1)}} \ldots Y_{r_{\sigma(n)}}$$

belongs to $I_0+\sum_{\beta<\alpha}\tilde{E}_e^{\beta}$.

Proof. – The proof is by transfinite induction. The results is clearly valid for the minimal element. Suppose we have proved the result for all elements in A smaller than α, where α is not the minimal element. Let k be the number such that $\alpha_j=0$ for $j>k$ and $\alpha_k\geqslant 1$ (so that $r_n=k$). Set, for $1\leqslant p\leqslant n$, $S_n^p=\{\sigma\in S_n \mid \sigma(p)=n\}$.

Suppose that $\sigma \in S_n^p$. Then

$$Y_{r_{\sigma(1)}} \dots Y_{r_{\sigma(p)}} Y_{r_{\sigma(p+1)}} \dots Y_{r_{\sigma(n)}}$$
$$= Y_{r_{\sigma(1)}} \dots Y_{r_{\sigma(p+1)}} Y_{r_{\sigma(p)}} \dots Y_{r_{\sigma(n)}}$$
$$- Y_{r_{\sigma(1)}} \dots [Y_{r_{\sigma(p+1)}}, Y_{r_{\sigma(p)}}] \dots Y_{r_{\sigma(n)}}.$$

Now we can write

$$[Y_{r_{\sigma(p+1)}}, Y_{r_{\sigma(p)}}] = \sum_{j<j_k} a_j X_j = \sum_{j<j_k} i a_j r_j^e(g) - \sum_{j<j_k} a_j u_j^e(g),$$

and since clearly an element of the form

$$Y_{r_{\sigma(1)}} \dots Y_{r_{\sigma(p-1)}} r_j^e(g) Y_{r_{\sigma(p+2)}} \dots Y_{r_{\sigma(n)}}$$

for $j<j_k$ belongs to $\tilde{E}_e^\beta$ with $\beta<\alpha$, we see that moving $Y_{r_{\sigma(p)}} = Y_{r_n} = Y_k$ one step to the right in the expression

$$u_\sigma = Y_{r_{\sigma(1)}} \dots Y_{r_{\sigma(p)}} \dots Y_{r_{\sigma(n)}}$$

only perturbs u_σ by an element from $I_0 + \sum_{\beta<\alpha} \tilde{E}_e^\beta$. Continuing like this in finitely many steps we see that element

$$Y_{r_{\sigma(1)}} \dots Y_{r_{\sigma(p)}} \dots Y_{r_{\sigma(n)}} - Y_{r_{\sigma(1)}} \dots \hat{Y}_{r_{\sigma(p)}} \dots Y_{r_{\sigma(n)}} Y_{r_{\sigma(p)}}$$

belongs to $I_0 + \sum_{\beta<\alpha} \tilde{E}_e^\beta$. We conclude from this that

$$\sum_{\sigma \in S_n^p} Y_{r_{\sigma(1)}} \dots Y_{r_{\sigma(p)}} \dots Y_{r_{\sigma(n)}} - \sum_{\sigma \in S_n^p} Y_{r_{\sigma(1)}} \dots \hat{Y}_{r_{\sigma(p)}} \dots Y_{r_{\sigma(n)}} Y_k$$

belongs to $I_0 + \sum_{\beta<\alpha} \tilde{E}_e^\beta$ for all $1 \leqslant p \leqslant n$.

Now clearly

$$\sum_{\sigma \in S_n^p} Y_{r_{\sigma(1)}} \dots \hat{Y}_{r_{\sigma(p)}} \dots Y_{r_{\sigma(n)}} = \sum_{\sigma \in S_{n-1}} Y_{r_{\sigma(1)}} \dots Y_{r_{\sigma(n-1)}},$$

so we find that

$$\frac{1}{n!}\sum_{\sigma\in S_n} Y_{r_{\sigma(1)}}\dots Y_{r_{\sigma(n)}}-\frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}} Y_{r_{\sigma(1)}}\dots Y_{r_{\sigma(n-1)}} Y_k$$

belongs to $I_0+\sum_{\beta<\alpha}\tilde{E}_e^{\beta}$. So we just have to show that

$$Y_{r_1}\dots Y_{r_{n-1}} Y_k-\frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}} Y_{r_{\sigma(1)}}\dots Y_{r_{\sigma(n-1)}} Y_k$$

belongs to $I_0+\sum_{\beta<\alpha}\tilde{E}_e^{\beta}$. But clearly $Y_{r_1}\dots Y_{r_{n-1}}$ belongs to $E_e^{\alpha'}$, where $\alpha'=(\alpha_1,\dots,\alpha_{k-1},\alpha_k-1,0\dots,0)$ and $\alpha'<\alpha$, and therefore, by the inductions hypothesis,

$$Y_{r_1}\dots Y_{r_{n-1}}-\frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}} Y_{r_{\sigma(1)}}\dots Y_{r_{\sigma(n-1)}}$$

belongs to $I_0+\sum_{\beta<\alpha'}\tilde{E}_e^{\beta}$. So to finish the proof we just have to note that if $u\in\tilde{E}_e^{\beta'}$ with $(\beta'_1,\dots,\beta'_k,0,\dots,0)=\beta'<\alpha'$, then $u\,Y_k\in\tilde{E}_e^{\beta}$, where $\beta=(\beta'_1,\dots,\beta'_{k-1},\beta_k+1,0\dots,0)$, and $\beta<\alpha$. This ends the proof of the sublemma.

Using Sublemma 2.2.6 we get as an immediate corollary:

SUBCOROLLARY 2.2.9. — *For $\alpha\in A$ we have*

$$\tilde{E}_e^{\alpha}\subset I_0+F_e^{\alpha}+\sum_{\beta<\alpha}\tilde{E}_e^{\beta}.$$

SUBCOROLLARY 2.2.10. — *For $\alpha\in A$ we have*

$$\tilde{E}_e^{\alpha}\subset I_0+\sum_{\beta\leqslant\alpha}F_e^{\beta}.$$

Proof. — We proceed by transfinite induction: The lemma is clearly valid for the minimal element. So suppose we have proved the lemma for all elements in A smalier than α. Then for $\beta<\alpha$ we have $\tilde{E}_e^{\beta}\subset I_0+\sum_{\gamma\leqslant\beta}F_e^{\gamma}$, and therefore, using Subcorollary 2.2.9.

$$\tilde{E}_e^{\alpha}\subset I_0+F_e^{\alpha}+\sum_{\beta<\alpha}\sum_{\gamma\leqslant\beta}F_e^{\gamma}=I_0+\sum_{\beta\leqslant\alpha}F_e^{\beta}.$$

This proves the subcorollary.

Lemma 2.2.5. now follows immediately from Subcorollary 2.2.10. Combining Lemma 2.2.3, Lemma 2.2.4 and Lemma 2.2.5 we get (since $E_e, F_e \subset \tilde{E}_e$):

LEMMA 2.2.11. — $U(\mathfrak{g}_{\mathbf{C}}) = I_0 + F_e = I_0 + E_e = I_0 + \tilde{E}_e$.

LEMMA 2.2.12. — *The restriction of $d\pi$ to F_e is faithful.*

Proof. — The proof is by induction on the dimension of $\mathfrak{g}$. The lemma is clearly valid for $\dim \mathfrak{g} = 1$ (in which case $e = \varnothing$ and $F_e = E_e = \mathbf{C}\, 1$). Assume then that the lemma has been proved for all dimensions less than or equal to $m-1$ and that $\dim \mathfrak{g} = m$. The case $e = \varnothing$ being trivial we can assume that $e \neq \varnothing$, and write $e = \{j_1 < \ldots < j_d\}$.

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and let $g \in \mathcal{O}$. Set $\mathfrak{z}_0 = \ker g \mid \mathfrak{z}$. We consider two cases: case (*a*): $\dim \mathfrak{z}_0 > 0$ and case (*b*): $\dim \mathfrak{z}_0 = 0$.

Case (*a*). — We use all the notation from the proof of Theorem 2.1.1. Suppose $u \in F_e$ and let $v = \omega^{-1}(u)$. Write

$$v = \sum_\alpha a_\alpha X_{j_1}^{\alpha_1} \ldots X_{j_d}^{\alpha_d}.$$

We have

$$\tilde{v} = c(v) = \sum_\alpha a_\alpha c(X_{j_1}^{\alpha_1}) \ldots c(X_{j_d}^{\alpha_d}) = \sum_\alpha a_\alpha \tilde{X}_{j_1}^{\alpha_1} \ldots \tilde{X}_{j_d}^{\alpha_d},$$

and

$$\omega(\tilde{v}) = \omega(c(v)) = c(\omega(v)) = c(u) = \tilde{u}.$$

If now $d\pi(u) = 0$, then $d\tilde{\pi}(\tilde{u}) = 0$, and therefore, by the induction hypothesis, $\tilde{u} = 0$, hence $\tilde{v} = 0$ and therefore $a_\alpha = 0$ for all α. But this shows that $v = 0$ and therefore $u = 0$. This settles case (*a*).

Case (*b*). — Again we use the notation from the proof of Theorem 2.1.1. Since clearly $G_e = G_{e'}$ we have that $F_e = F_{e'}$. We have therefore reduced to the case where $\mathfrak{g}_{m-1} = \ker \operatorname{ad} X_2 = \mathfrak{h}$. We assume that this is the case from now on.

We write again

$$v = \sum_\alpha a_\alpha X_{j_1}^{\alpha_1} \ldots X_{j_d}^{\alpha_d}$$

and $u=\omega(v)$. Suppose first that $a_\alpha\neq 0$ implies that $\alpha_d=0$, so that we can write

$$v=\sum_{\alpha'} a_{(\alpha',\,0)} X_{j_1}^{\alpha_1}\ldots X_{j_{d-1}}^{\alpha_{d-1}}.$$

For $p\geqslant 0$ we set

$$v_p=\sum_{\alpha^0} a_{(p,\,\alpha^0,\,0)} X_{j_1^0}^{\alpha_1^0}\ldots X_{j_{d-2}^0}^{\alpha_{d-2}^0}\in G_{e^0}.$$

We have $v=\sum_p X_2^p v_p$, and setting $u_p=\omega(v_p)$ we also have $u=\sum_p X_2^p u_p$ since X_2 is central in $\mathfrak{g}_{m-1}$. For $z\in\mathbf{C}$ we set $v_z=\sum_p z^p v_p\in G_{e^0}$, and $u_z=\omega(v_z)=\sum_p z^p u_p\in F_{e^0}$.

Setting $\mu=\langle g,[X_m,X_2]\rangle$ we get

$$\begin{aligned} d(\exp t X_m \pi_0)(u)&=d\pi_0(\mathrm{Ad}(\exp -t X_m)u)\\ &=\sum_p d\pi_0(\mathrm{Ad}(\exp -t X_m)X_2^p)\,d\pi_0(\mathrm{Ad}(\exp -t X_m)u_p)\\ &=\sum_p(-i\mu t)^p d(\exp t X_m \pi_0)(u_p)=d(\exp t X_m\pi_0)(u_{-i\mu t}).\end{aligned}$$

Now for a differentiable vector $\varphi\in L^2(G,\pi_0)$ we have

$$\begin{aligned}0&=d\pi(u)\varphi(\exp t X_m)\\ &=d\pi_0(\mathrm{Ad}(\exp -t X_m)u)\,\varphi(\exp t X_m)\quad\text{for all } t\in\mathbf{R},\end{aligned}$$

so

$$d\pi_0(\mathrm{Ad}(\exp -t X_m)u)=d(\exp t X_m\pi_0)(u)=0\quad\text{for all } t\in\mathbf{R},$$

hence, from what we saw above

$$d(\exp t X_m\pi_0)(u_{-i\mu t})=0\quad\text{for all } t\in\mathbf{R}.$$

Now $u_{-i\mu t}\in F_{e^0}$, and the induction hypothesis applied to the representation $\exp t X_m \pi_0$ then gives that $u_{-i\mu t}=0$ for all t. But this implies that $u_p=0$ for all $p\geqslant 0$, and therefore that $u=0$. We have thus shown that $d\pi$ is faithful on elements u of the special form considered.

Suppose now that u is arbitrary, and define for $p\geqslant 0$ the element

$$v_p=\sum_{(\alpha',\,p)} X_{j_1}^{\alpha_1}\ldots X_{j_{d-1}}^{\alpha_{d-1}}$$

so that $v=\sum_p v_p X_m^p$. Suppose that there exists $p>0$ such that $v_p\neq 0$, and let q be the maximal such p. Then $(\mathrm{ad}\, X_2)^q v=q!\, v$, and

$$q!\,\omega(v_q)=\omega((\mathrm{ad}\, X_2)^q v)=(\mathrm{ad}\, X_2)^p u.$$

Since $d\pi(u)=0$ we also have that

$$\frac{1}{q!}d\pi((\mathrm{ad}\, X_2)^p u)=d\pi(v_q)=0,$$

hence that $v_q=0$, because v_q is of the special form considered above. But this is a contradiction, so $v=v_0$, and therefore, again appealing to the special case considered above, $u=0$. This ends the proof of the lemma.

LEMMA 2.2.13. — *The restriction of* $d\pi$ *to* E_e *is faithful.*

Proof. — We prove by transfinite induction on $\alpha\in A$ that the restriction of $d\pi$ to E_e^α is faithful. The result is clearly valid for the minimal element. So suppose we have proved the lemma for all elements in A smaller than α. For $\beta\in A$, let u_β denote the element $X_{j_1}^{\beta_1}\dots X_{j_d}^{\beta_d}$ in $U(\mathfrak{g}_C)$, let v_β be the element $X_{j_1}^{\beta_1}\dots X_{j_d}^{\beta_d}$ in $S(\mathfrak{g}_C)$, and set $\bar{u}_\beta=\omega(v_\beta)$. Let $u\in E_e^\alpha$, and write $u=\sum_{\beta\leq\alpha} a_\beta u_\beta$ and suppose that $d\pi(u)=0$. If $a_\alpha=0$ there exists $\alpha'<\alpha$ such that $u\in E_e^{\alpha'}$, so $u=0$ by the induction hypothesis. Assume therefore that $a_\alpha\neq 0$. It follows from Sublemma 2.2.8 and Subcorollary 2.2.10 that $u_\alpha-\bar{u}_\alpha\in I_0+\sum_{\beta<\alpha}F_e^\beta$. Therefore we can write $u=u_0+\bar{u}$, where $\bar{u}=a_\alpha\bar{u}_\alpha+\sum_{\beta<\alpha}\bar{a}_\beta\bar{u}_\beta$ and where $u_0\in I_0$.

Now $d\pi(\bar{u})=0$, and $\bar{u}\in F_e$, so $\bar{u}=0$ by Lemma 2.2.10. But then it follows that $a_\alpha=0$, since the system $(\bar{u}_\beta)_{\beta\in A}$ is linearly independent in $U(\mathfrak{g}_C)$. This is a contradiction and ends the proof of the lemma.

We can now end the proof of the theorem: From Lemma 2.2.11 we get that $U(\mathfrak{g}_C)=I_0+F_e=I_0+E_e$, and actually the sums are direct by Lemma 2.2.12 and 2.2.13. But since $I_0\subset I(g)$ we must have $I_0=I(g)$, and the theorem is proved.

2.3. We set $I_k(g)$, $1\leq k\leq m$, to be the kernel of the restriction of $d\pi$ to $U((\mathfrak{g}_k)_C)$, i. e. $I_k(g)=I(g)\cap U((\mathfrak{g}_k)_C)$. Moreover, we set

$$e(k)=\{j_1<\dots<j_{d'}\}\qquad\text{where}\quad d'=\max\{1\leq r\leq d\mid j_r\leq k\}.$$

Let $G_{e(k)}$, $F_{e(k)}$, $E_{e(k)}$ have the obvious meaning (*v.* the beginning of Section 2.2). Then using Subcorollary 2.2.7 and 2.2.10 we can prove the following result just like the way we proved Theorem 2.2.1:

PROPOSITION 2.3.1. — *If $g \in \Omega_e$, then $I_k(g)$ is generated by the elements $(u_j^e(g))_{j=1, j \notin e}^k$ and*

$$U((\mathfrak{g}_k)_{\mathbb{C}}) = I_k(g) \oplus E_{e(k)} = I_k(g) \oplus F_{e(k)}.$$

We now claim that we have

$$I_k(g) = \sum_{j \notin e\, j=1}^k u_j^e(g)\, U((\mathfrak{g}_k)_{\mathbb{C}}), \qquad k = 1, \ldots, m.$$

We prove this by induction on k. First we note that by Proposition 2.3.1 $I_k(g)$ is the set of finite linear combinations of elements $uu_j^e(g)v$, where u, $v \in U((\mathfrak{g}_k)_{\mathbb{C}})$ and $1 \leqslant j \leqslant k$.

Now $u_1^e(g) = X_1 - i\langle g, X_1\rangle$, and X_1 is central, so it follows immediately that $uu_1^e(g)v = u_1^e(g)uv$ so $I_1(g) = u_1^e(g)\, U((\mathfrak{g}_1)_{\mathbb{C}})$.

Suppose then that we have proved the result for all integers $\leqslant k(<m)$. Since we clearly have that

$$I_{k+1}(g) \supset \sum_{j \notin e,\, j=1}^{k+1} u_j^e(g)\, U((\mathfrak{g}_{k+1})_{\mathbb{C}}),$$

it suffices to show that

$$X u_j(g) \in \sum_{j \notin e,\, j=1}^{k+1} u_j^e(g)\, U((\mathfrak{g}_{k+1})_{\mathbb{C}})$$

for all $X \in \mathfrak{g}_{k+1}$, $1 \leqslant j \leqslant k+1$.

But $Xu_j^e(g) = u_j^e(g)X + [X, u_j^e(g)]$ and $u_j^e(g) = X_j - ir_j^e(g)$, so

$$[X, u_j^e(g)] = [X, X_j] - i \operatorname{ad} X(r_j^e(g)),$$

from which we see that $[X, u_j^e(g)]$ belongs to $U((\mathfrak{g}_k)_{\mathbb{C}})$. But obviously $d\pi([X, u_j^e(g)] = 0$, so by the induction hypothesis

$$[X, u_j^e(g)] \in I_k(g) = \sum_{j \notin e,\, j=1}^k u_j^e(g)\, U((\mathfrak{g}_k)_{\mathbb{C}}),$$

and therefore

$$X u_j^e(g) = u_j^e(g) X + [X, u_j^e(g)] \in \sum_{j \notin e,\, j=1}^{k+1} u_j^e(g)\, U((\mathfrak{g}_{k+1})_{\mathbf{C}}).$$

This ends the proof of the claim.

In particular for $k=m$ we get:

THEOREM 2.3.2. — *The ideal $I(g)$ coincides with the right (or left) ideal generated by $(u_j^e(g))_{j=1,\, j\notin e}^m$, i. e. we have the formula*

$$I(g) = \sum_{j \notin e,\, j=1}^{m} u_j^e(g) U(\mathfrak{g}_{\mathbf{C}}).$$

This is the formula alluded to in the heading of Section 2.

2.4. We end Section 2 by showing how one in principle can find in terms of a given irreducible representation π the element $e \in \mathscr{E}$ such that the orbit O associated with π is contained in Ω_e.

PROPOSITION 2.4.1. — *If $g \in \Omega_e$ and if π is the irreducible representation of G associated with the orbit $O = Gg$, then*

$$e = \{ 1 \leqslant j \leqslant m \mid d\pi(X_j) \notin d\pi(U((\mathfrak{g}_{j-1})_{\mathbf{C}})) \}.$$

Proof. — Suppose that $d\pi(X_j) \in d\pi(U((\mathfrak{g}_{j-1})_{\mathbf{C}}))$. Then $d\pi(X_j) = d\pi(u)$ where $u \in E_{e(j-1)}$ by Proposition 2.3.1. But then $X_j - u \in I(g)$, so if $j \in e$ this implies that $X_j - u = 0$, since then also $X_j - u \in E_e$ (Theorem 2.2.1). It follows that $X_j = u$, and this contradicts the fact that $u \in U((\mathfrak{g}_{j-1})_{\mathbf{C}})$. We have thus shown that $j \notin e$. Suppose conversely that $j \notin e$. Then

$$d\pi(X_j) = i d\pi(r_j^e(g)) \in d\pi(U((\mathfrak{g}_{j-1})_{\mathbf{C}})).$$

This ends the proof of the proposition.

3. An algebraic method for finding the orbit associated with a given irreducible representation

3.1. Given an irreducible representation π of G, how does one find the orbit associated with π? Using the results of Section 2 we shall in this section give a solution to this problem in algebraic terms (analytically one would, of course, use the Kirillov character formula).

We use all the notation from the Preliminaries (Section 1). In the following we shall often identify $g \in \mathfrak{g}^*$ with its coordinates $(\xi_1, \ldots, \xi_m)$

with respect to the basis $l_1, \ldots, l_m$ in $\mathfrak{g}^*$ dual to $X_1, \ldots, X_m$: $g=\sum_{j=1}^{m} \xi_j l_j$.

We start by noting that the function $g \to R_j^e(g, x): \Omega_e \to \mathbf{R}$ (for fixed $x \in \mathbf{R}^d$) only depends on the restriction of g to $\mathfrak{g}_j$ (in fact, the G-orbit in $\mathfrak{g}_j^*$ through $g_j = g \mid \mathfrak{g}_j$ is given by

$$G g_j = \{ \textstyle\sum_{p=1}^{j} R_p^e(g, x) l_p \mid x \in \mathbf{R}^d \}).$$

Moreover, since $[\mathfrak{g}, \mathfrak{g}_j] \subset \mathfrak{g}_{j-1}$, the function $R_j^e(g, x)$ for $j \in e$ actually has the form

$$R_j^e(g, x) = \xi_j + V_j^e(g, x),$$

where $V_j^e: \Omega_e \times \mathbf{R}^d \to \mathbf{R}$ is such that the function $g \to V_j^e(g, x)$ (for fixed $x \in \mathbf{R}^d$) only depends on the restriction of g to $\mathfrak{g}_{j-1}$. We write this symbolically:

$$R_j^e(g, x) = R_j^e(\xi_1, \ldots, \xi_j, x) = \xi_j + V_j^e(\xi_1, \ldots, \xi_{j-1}, x)$$

for $j \notin e$.

For $j \notin e$, let $v_j^e(g) = v_j^e(\xi_1, \ldots, \xi_{j-1})$ be the element in $U(\mathfrak{g}_{\mathbf{C}})$ corresponding by symmetrization to the element $V_j^e(g, -i X_{j_1}, \ldots, -i X_{j_d})$ in $S(\mathfrak{g}_{\mathbf{C}})$, so that

$$r_j^e(g) = \xi_j + v_j^e(g),$$

and set for $j \notin e$

$$t_j^e(g) = X_j - i v_j^e(g),$$

or

$$t_j^e(\xi_1, \ldots, \xi_{j-1}) = X_j - i v_j^e(\xi_1, \ldots, \xi_{j-1}).$$

With this notation we derive the following result from Theorem 2.1.1.

THEOREM 3.1.1. — *Let $\pi \in \hat{G}$, and suppose that the corresponding coadjoint orbit* O *is contained in Ω_e. We can determine an element $g = (\xi_1, \ldots, \xi_m)$ in* O *inductively as follows:*

(1) $i \xi_1 I = d\pi(X_1)$, 2) *if we have determined $\xi_1, \ldots \xi_j$ ($j < m$), then, if $j+1 \in e$ we can make an arbitrary choice of ξ_{j+1} (e. g. $\xi_{j+1} = 0$), and if*

$j+1 \notin e$ we have

$$i\xi_{j+1} I = d\pi(t^e_{j+1}(\xi_1, \ldots, \xi_j)).$$

Now the problem of determining, for a given irreducible representation π, the element $e \in \mathscr{E}$ such that $O \subset \Omega_e$ is solved by Proposition 2.4.1. The answer given there is, however, not of the same algorithmic nature as the one given in Theorem 3.1.1 and is therefore less satisfactory. In the following we shall remedy this situation. Our final goal is Theorem 3.4.6. First, however, a digression.

3.2. THE MAPS α_n AND A_n

In this section $\mathfrak{g}$ denotes a Lie algebra over $\mathbb{C}$. For $n \in \mathbb{N}$ we define the map $\alpha_n : \mathfrak{g} \times \ldots \times \mathfrak{g}\,(2n \text{ factors}) \to S(\mathfrak{g})$ by

$$\alpha_n(X_1, \ldots, X_{2n}) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sign} \sigma\, [X_{\sigma(1)}, X_{\sigma(2)}] \ldots [X_{\sigma(2n-1)}, X_{\sigma(2n)}].$$

It is immediately seen that α_n is an alternating $2n$-linear map from $\mathfrak{g} \times \ldots \times \mathfrak{g}\,(2n$ *factors) to* $S(\mathfrak{g})$.

An element in $S(\mathfrak{g})$ corresponds to an element in the algebra $\operatorname{Pol}(\mathfrak{g}^*)$ of complex valued polynomial functions on $\mathfrak{g}^*$. The polynomial function P corresponding to $\alpha_n(X_1, \ldots, X_{2n}) \in S(\mathfrak{g})$ is

$$P(l) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sign} \sigma \langle l, [X_{\sigma(1)}, X_{\sigma(2)}] \rangle \times \ldots \times \langle l, [X_{\sigma(2n-1)}, X_{\sigma(2n)}] \rangle,$$

$l \in \mathfrak{g}^*$, so we see that $P(l) = Pf(M(l))$, the Pfaffian of the skewsymmetric matrix

$$M(l) = [\langle l, [X_r, X_s] \rangle]_{1 \leq r, s \leq 2n}, \ l \in \mathfrak{g}^*.$$

In particular $P(l)^2 = \det M(l)$.

Let $C = [c_{rs}]_{1 \leq r, s \leq 2n}$ be a $2n \times 2n$-matrix, and set $X'_s = \sum_{r=1}^{2n} c_{rs} X_r$. Then we have (the proof is immediate):

LEMMA 3.2.1. — $\alpha_n(X'_1, \ldots, X'_{2n}) = \det C\, \alpha_n(X_1, \ldots, X_{2n})$.

LEMMA 3.2.2. — *Suppose that* X_1 *commutes with all* $X_2, \ldots, X_{2n-1}$. *Then*

$$\alpha_n(X_1, \ldots, X_{2n}) = [X_1, X_{2n}]\, \alpha_{n-1}(X_2, \ldots, X_{2n-1}).$$

Proof. — The matrix $M(l)$, $l \in \mathfrak{g}^*$, (v. above) has the form

$$M(l) = \left[\begin{array}{c|c|c} 0 & 0 \ldots 0 & \langle l, [X_1, X_{2n}]\rangle \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & M^0(l) & \Big| \\ \hline -\langle l, [X_1, X_{2n}]\rangle & \text{——} & 0 \end{array}\right]$$

so

$$Pf(M(l)) = \langle l, [X_1, X_{2n}]\rangle\, Pf(M_0(l)),$$

and therefore

$$\alpha_n(X_1, \ldots, X_{2n}) = [X_1, X_{2n}]\, \alpha_{n-1}(X_2, \ldots, X_{2n-1}).$$

COROLLARY 3.2.3. — *If* X_1 *commutes with all* $X_2, \ldots, X_{2n}$, *then* $\alpha_n(X_1, \ldots, X_{2n}) = 0$.

For $n \in \mathbb{N}$ we define the map $A_n : \mathfrak{g} \times \ldots \times \mathfrak{g}$ (n factors) $\to U(\mathfrak{g})$ by

$$A_n(X_1, \ldots, X_n) = \sum_{\sigma \in S_n} \operatorname{sign} \sigma\, X_{\sigma(1)} \ldots X_{\sigma(n)}.$$

It is immediately seen that A_n is an alternating n-linear map from $\mathfrak{g} \times \ldots \times \mathfrak{g}$ (n factors) to $U(\mathfrak{g})$.

Let $C = [c_{rs}]_{1 \leq r, s \leq n}$ be an $n \times n$-matrix and set $X'_s = \sum_{r=1}^{n} c_{rs} X_r$. Then we have

LEMMA 3.2.4. — $A_n(X'_1, \ldots, X'_n) = \det C\, A_n(X_1, \ldots, X_n)$.

The maps α_n and A_{2n} are connected in the following way:

PROPOSITION 3.2.5. — *For* $X_1, \ldots, X_{2n} \in \mathfrak{g}$ *we have*

$$\omega(\alpha_n(X_1, \ldots, X_{2n})) = \frac{1}{n!} A_{2n}(X_1, \ldots, X_{2n}).$$

Proof. — Writting, for $\sigma\in S_{2n}$, $Y_j^\sigma=[X_{\sigma(2j-1)}, X_{\sigma(2j)}]$ we have

$\omega(\alpha_n(X_1, \ldots, X_{2n}))$

$$=\frac{1}{2^n n!}\sum\nolimits_{\sigma\in S_{2n}}\operatorname{sign}\sigma\omega([X_{\sigma(1)}, X_{\sigma(2)}]\ldots[X_{\sigma(2n-1)}, X_{\sigma(2n)}])$$

$$=\frac{1}{2^n n!}\sum\nolimits_{\sigma\in S_{2n}}\operatorname{sign}\sigma\omega(Y_1^\sigma\ldots Y_n^\sigma).$$

Now

$$\omega(Y_1\ldots Y_n)=\frac{1}{n!}\sum\nolimits_{\rho\in S_n}Y_{\rho(1)}\ldots Y_{\rho(n)},$$

and defining for $\rho\in S_n$ the permutation

$$\sigma_\rho=(\sigma(2\rho(1)-1), \sigma(2\rho(1)), \ldots, \sigma(2\rho(j)-1),$$
$$\sigma(2\rho(j)), \ldots, \sigma(2\rho(n)-1), \sigma(2\rho(n))),$$

we have that the map $\sigma\to\sigma_\rho$ is a bijection of S_{2n} onto itself with $\operatorname{sign}\sigma_\rho=\operatorname{sign}\sigma$, and

$$Y_{\rho(j)}^\sigma=[X_{\sigma(2\rho(j)-1)}, X_{\sigma(2\rho(j))}]=[X_{\sigma_\rho(2j-1)}, X_{\sigma_\rho(2j)}]=Y_j^{\sigma_\rho},$$

so

$$\omega(Y_1^\sigma\ldots Y_n^\sigma)=\frac{1}{n!}\sum\nolimits_{\rho\in S_n}Y_1^{\sigma_\rho}\ldots Y_n^{\sigma_\rho},$$

and therefore

$\omega(\alpha_n(X_1, \ldots, X_{2n}))$

$$=\frac{1}{2^n(n!)^2}\sum\nolimits_{\rho\in S_n}\sum\nolimits_{\sigma\in S_{2n}}\operatorname{sign}\sigma_\rho Y_1^{\sigma_\rho}\ldots Y_n^{\sigma_\rho}$$

$$=\frac{1}{2^n(n!)^2}\sum\nolimits_{\rho\in S_n}\sum\nolimits_{\sigma\in S_{2n}}\operatorname{sign}\sigma Y_1^\sigma\ldots Y_n^\sigma$$

$$=\frac{1}{2^n n!}\sum\nolimits_{\sigma\in S_{2n}}\operatorname{sign}\sigma Y_1^\sigma\ldots Y_n^\sigma.$$

We next note that

$$\begin{aligned} Y_j^\sigma &= [X_{\sigma(2j-1)}, X_{\sigma(2j)}] \\ &= X_{\sigma(2j-1)} X_{\sigma(2j)} - X_{\sigma(2j)} X_{\sigma(2j-1)} \\ &= X_{\sigma(2j-1)} X_{\sigma(2j)} + \operatorname{sign} \tau_j X_{\sigma\circ\tau_j(2j-1)} X_{\sigma\circ\tau_j(2j)}, \end{aligned}$$

where τ_j is the transposition $\begin{bmatrix} 2j-1 & 2j \\ 2j & 2j-1 \end{bmatrix}$. For each subset $e \subset \{1, \ldots, 2n\}$ and permutation $\sigma \in S_{2n}$ define then the permutation σ^e by $\sigma^e = \sigma \circ \prod_{j \in e} \tau_j$. In this way $\sigma \to \sigma^e : S_{2n} \to S_{2n}$ is a bijection, and

$$\operatorname{sign} \sigma\, Y_1^\sigma \ldots Y_n^\sigma = \sum_e \operatorname{sign} \sigma^e X_{\sigma^e(1)} X_{\sigma^e(2)} \ldots X_{\sigma^e(2n-1)} X_{\sigma^e(2n)},$$

so

$$\begin{aligned} &\omega(\alpha_n(X_1, \ldots, X_{2n})) \\ &\quad = \frac{1}{2^n n!} \sum_2 \sum_{\sigma \in S_{2n}} \operatorname{sign} \sigma^e X_{\sigma^e(1)} \ldots X_{\sigma^e(n)} \\ &\quad = \frac{1}{2^n n!} \sum_e \sum_{\sigma \in S_{2n}} \operatorname{sign} \sigma X_{\sigma(1)} \ldots X_{\sigma(n)} \\ &\quad = \frac{1}{n!} \sum_{\sigma \in S_{2n}} \operatorname{sign} \sigma X_{\sigma(1)} \ldots X_{\sigma(n)} \\ &\quad = \frac{1}{n!} A_{2n}(X_1, \ldots, X_{2n}). \end{aligned}$$

This ends the proof of the proposition.

COROLLARY 3.2.6. – *Suppose that X_1 and $[X_1, X_{2n}]$ commute with all $X_2, \ldots, X_{2n-1}$.* Then

$$A_{2n}(X_1, \ldots, X_{2n}) = n [X_1, X_{2n}] A_{2(n-1)}(X_2, \ldots, X_{2n-1}).$$

Proof. – This follows from Lemma 3.2.2, Proposition 3.2.5 and Lemma 1.3.1.

COROLLARY 3.2.7. – *If X_1 commutes with all $X_2, \ldots, X_{2n}$, then $A_{2n}(X_1, \ldots, X_{2n}) = 0$.*

3.3. We now return to the situation described in the Preliminaries (Section 1). If $e \in \mathscr{E}$ with $e \neq \varnothing$ and if $e = \{j_1 < \ldots < j_d\}$ we define the element v_e in $U(\mathfrak{g}_C)$ by

$$v_e = \frac{(-i)^{d/2}}{(d/2)!} A_d(X_{j_1}, \ldots, X_{j_d}).$$

If $e=\emptyset$ we set $v_e \equiv 1$. Note that according to Proposition 3.2.5 the element v_e corresponds via symmetrization to the polynomial function $l \to (-i)^{d/2} P_e(l)$ on $\mathfrak{g}^*$.

THEOREM 3.3.1. — *If $g \in \Omega_e$ and if π is the irreducible representation of G corresponding to the orbit $O = Gg$, then*

$$d\pi(v_e) = P_e(g) I.$$

Remark 3.3.2. — This was actually proved (in a slightly different form) in [8] (Proposition 2.2.1) in a considerably greater generality. For the convenience of the reader we give here the much simpler proof pertaining to the present special case.

Proof. — The proof is by induction on the dimension of $\mathfrak{g}$. The theorem is clearly valid for $\dim \mathfrak{g} = 1$ (in which case $e=\emptyset$, $P_e \equiv 1$ and $v_e \equiv 1$. Assume then that the theorem has been proved for all dimensions less than or equal to $m-1$ and that $\dim \mathfrak{g} = m$. The case $e=\emptyset$ being trivial we can assume that $e \neq \emptyset$, and write $e = \{j_1 < \ldots < j_d\}$.

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and set $\mathfrak{z}_0 = \ker g \,|\, \mathfrak{z}$. We distinguish two cases: case (*a*): $\dim \mathfrak{z}_0 > 0$ and case (*b*): $\dim \mathfrak{z}_0 = 0$.

Case (*a*). — We use all the notation from the proof of Theorem 2.1.1, and get

$$\begin{aligned} c(v_e) &= \frac{(-i)^{d/2}}{(d/2)!} c(A_d(X_{j_1}, \ldots, X_{j_d})) \\ &= \frac{(-i)^{d/2}}{(d/2)!} A_d(c(X_{i\tilde{j}_1}), \ldots, c(X_{i\tilde{j}_d})) \\ &= \frac{(-i)^{d/2}}{(d/2)!} A_d(\tilde{X}_{\tilde{j}_1}, \ldots, \tilde{X}_{\tilde{j}_d}) = v_{\tilde{e}}, \end{aligned}$$

and therefore also $P_e(\tilde{l} \circ c) = P_{\tilde{e}}(\tilde{l})$ for $\tilde{l} \in \tilde{\mathfrak{g}}^*$. By the induction hypothesis we have $d\tilde{\pi}(v_{\tilde{e}}) = P_{\tilde{e}}(\tilde{g}) I$, and therefore

$$d\pi(v_e) = d\tilde{\pi}(c(v_e)) = d\tilde{\pi}(v_{\tilde{e}}) = P_{\tilde{e}}(\tilde{g}) I = P_e(g) I,$$

and this settles case (*a*).

Case (*b*). – Again we use the notation from the proof of Theorem 2.1.1. We have

$$\hat{X}_{\hat{j}_h}=X_{j_h} \qquad \text{for} \quad 1\leqslant h\leqslant \alpha-1,$$

$$\hat{X}_{\hat{j}_h}=X_{j_{h+1}}+c_{j_{h+1}}X_{j_\alpha} \qquad \text{for} \quad \alpha\leqslant h\leqslant d-1, \qquad \hat{X}_{\hat{j}_d}=X_{j_\alpha},$$

so if we let $C=[c_{rs}]_{1\leqslant r,s\leqslant d}$ be the $d\times d$-matrix:

$$C=\left[\begin{array}{ccc|ccc|c} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & c_{j_{\alpha+1}} & \cdots & c_{j_d} & 1 \\ \hline & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \end{array}\right]$$

where the empty entries are zero, we have $\hat{X}_{\hat{j}_s}=\sum_{r=1}^{d} c_{rs}X_{j_r}$, and therefore $M_{\hat{e}}(l)={}^tCM_e(l)\,C$. Now $\det C=(-1)^\alpha$, so

$$P_{\hat{e}}(l)=Pf(M_{\hat{e}}(l))=\det C\; Pf(M_e(l)) \;=(-1)^\alpha\, P_e(l),$$

and therefore $v_{\hat{e}}=(-1)^\alpha v_e$. The conclusion is then that we can assume that $\mathfrak{g}_{m-1}=\mathfrak{h}$, and this assumption will be in effect from now on.

Now recalling that $j_1=2$, and $j_d=m$, and that X_2 is central in $\mathfrak{h}$ we get using Lemma 3.2.2:

$$\alpha_{d/2}(X_{j_1},\ldots,X_{j_d})=[X_{j_1},X_{j_d}]\,\alpha_{d/2-1}(X_{j_2},\ldots,X_{j_{d-1}})$$
$$=[X_2,X_m]\,\alpha_{d/2-1}(X_{j_1^0},\ldots,X_{j_{d-2}^0}),$$

and therefore $P_e(l)=\langle l,[X_2,X_m]\rangle P_{e^0}(l_0)$, where $l_0=l\,|\,\mathfrak{h}$, and similarly $v_e=-i[X_2,X_m]\,v_{e^0}$ (Corollary 3.2.6). By the induction hypothesis we get that $d\pi_0(v_{e^0})=P_{e^0}(g_0)I$, and therefore

$$d\pi_0(v_e)=-id\pi_0([X_2, X_m])\,d\pi_0(v_{e^0})=-i.i\langle g,[X_2, X_m]\rangle P_{e^0}(g_0)I=P_e(g)I.$$

Now applying the above to the functional sg, $s\in G$, we have $sg\in\Omega_e$ and therefore

$$d(s\pi_0)(v_e)=P_e(sg)I=P_e(g)I \quad \text{(Lemma 1.1.1)},$$

i. e. $d\pi_0(Ad(s^{-1})v_e)=P_e(g)I$ for all $s\in G$, and from this it follows that $d\pi(v_e)=P_e(g)I$. This ends the proof of the theorem.

3.4. Let $\mathscr{D}=\mathscr{D}_m$ designate the set of all subsets of the set $\{1,\ldots,m\}$. We define an irreflexive total ordering $\prec$ on $\mathscr{D}$ in the following way:

(*a*) $\varnothing$ is the maximal element:

(*b*) if $e, e'\neq\varnothing$ and $e=\{j_1<\ldots<j_d\}$, $e'=\{j'_1<\ldots<j'_{d'}\}$, then $e\prec e'$ if either

(1) $d'<d$ and $j_r=j'_r$ for all $r\leqslant d'$

or

(2) there exists $r\leqslant min\{d, d'\}$ such that $j_r\neq j'_r$ and $j_k<j'_k$, where

$$k=\min\{1\leqslant r\leqslant\min\{d,d'\}\,|\,j_r\neq j'_r\}.$$

We let $\mathscr{D}_m^{\text{even}}$ denote the set of elements in $\mathscr{D}_m$ containing an even number of elements. For $e\in\mathscr{D}_m^{\text{even}}$ with $e\neq\varnothing$ and $e=\{j_1<\ldots<j_d\}$ we let $M_e(l)$ designate the $d\times d$-matrix

$$[\langle l,[X_{j_r}, X_{j_s}]\rangle]_{1\leqslant r,s\leqslant d}, \qquad l\in\mathfrak{g}^*,$$

and set $P_e(l)=Pf(M_e(l))$ (*cf.* Section 1). We set

$$v_e=\frac{(-1)^{d/2}}{(d/2)!}A_d(X_{j_1},\ldots,X_{j_d}).$$

If $e=\varnothing$ we set $M_e(l)=1$, $P_e(l)=1$, $v_e\equiv 1$. This is consistent with our earlier notation (Section 1 and 3.3).

LEMMA 3.4.1. — *Let* $e \in \mathscr{D}_m^{\text{even}}$ *and let* $g \in \Omega_{e'}$, *with* $e \prec e'$. *Then* $P_e(g) = 0$.

Proof. — Since $e \prec e'$ we have $e \neq \varnothing$, so we can write $e = \{j_1 < \ldots < j_d\}$. If $e' = \varnothing$ we have that $\mathfrak{g}_g = \mathfrak{g}$, and therefore $M_e(g) = 0$, hence $P_e(g) = 0$. Suppose that $e' \neq \varnothing$, and write $e' = \{j'_1 < \ldots < j'_{d'}\}$. If $j_r = j'_r$ for all $r \leqslant \min\{\mathrm{d}, \mathrm{d}'\}$ and $d > d'$, then $X_{j_1}, \ldots, X_{j_d}$ are linearly dependent $(\operatorname{mod} \mathfrak{g}_g)$, since $X_{j'_1}, \ldots, X_{j'_{d'}}$ is a basis in $\mathfrak{g} (\operatorname{mod} \mathfrak{g}_g)$. But this implies that $M_e(g)$ is singular, hence $P_e(g) = 0$. If $j_k < j'_k$ for $k \leqslant \min\{d, d'\}$ and $r < k \Rightarrow j_r = j'_r$ we have that $X_{j_k} \in \mathbf{R} X_{j_{k-1}} \oplus \ldots \oplus \mathbf{R} X_{j_1} + \mathfrak{g}_g$, so $X_{j_1}, \ldots, X_{j_k}$ are linearly dependent $(\operatorname{mod} \mathfrak{g}_g)$ and again we find that $M_e(g)$ is singular. This proves the lemma.

COROLLARY 3.4.2. — *For all* $e \in \mathscr{E}$ we have:

$$\Omega_e = \left\{ g \in \mathfrak{g}^* \;\middle|\; \begin{array}{l} P_e(g) \neq 0 \text{ and } P_{e'}(g) = 0 \\ \text{for all } e' \in \mathscr{E} \text{ with } e' \prec e \end{array} \right\}.$$

Proof. — This follows from Lemma 1.1.1 and 3.4.1.

Remark 3.4.3. — In [11], p. 525 was introcuced a total ordering $<$ on $\mathscr{E}^*$ and this ordering was used also in [8]. The ordering introduced here is *different* from the one from [11] (and [8]).

THEOREM 3.4.4. — *Let* $e \in \mathscr{D}_m^{\text{even}}$, *and let* π *be an irreducible representation of G corresponding to a coadjoint orbit* 0 *contained in* $\Omega_{e'}$, $e' \in \mathscr{E}$. *If* $e \prec e'$, *then* $d\pi(v_e) = 0$.

Proof. — The proof is by induction on the dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 1$ there is nothing to prove, since $e' = \varnothing$ and $\mathscr{D}_1^{\text{even}} = \{\varnothing\}$.

Assume then that the theorem has been proved for all dimensions less than or equal to $m-1$ and that $\dim \mathfrak{g} = m\ (\geqslant 3)$. Since $e \prec e' \leqq \varnothing$ we have that $e \neq \varnothing$ and we can write $e = \{j_1 < \ldots < j_d\}$. Suppose first that $e' = \varnothing$. Then π is a unitary character, and all $d\pi(X_{j_1}), \ldots, d\pi(X_{j_d})$ commute, so

$$d\pi(v_e) = \frac{(-i)^{d/2}}{(d/2)!} A_d(d\pi(X_{j_1}), \ldots, d\pi(X_{j_d})) = 0$$

(Corollary 3.2.7), and this settles the case $e'=\emptyset$. We can then assume that $e'\neq\emptyset$, and write $e'=\{j'_1<\ldots<j'_{d'}\}$.

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and let $g\in O$. Set $\mathfrak{z}_0=\ker g|\mathfrak{z}$. We consider two cases: case (*a*): $\dim\mathfrak{z}_0>0$ and case (*b*): $\dim\mathfrak{z}_0=0$.

Case (*a*). — We use all the notation from the proof of Theorem 2.1.1. We first reduce to the case where $e\subset I$: We can write

$$X_{j_k}=\sum_{r=1}^{n} a_{rk}X_{i_r}+Z_k,$$

where $Z_k\in\mathfrak{z}_0$, and where $a_{rk}=0$ if $i_r>j_k$. Since the Z_k are central in $\mathfrak{g}$ we have

$$v_e=\frac{(-i)^{d/2}}{(d/2)!}A_d(X_{j_1},\ldots,X_{j_d})$$
$$=\frac{(-i)^{d/2}}{(d/2)!}\sum_{r_1=1,\ldots,r_d=1}^{n} a_{r_1 1}\ldots a_{r_d d}A_d(X_{i_{r_1}},\ldots,X_{i_{r_d}}).$$

Now a necessary condition for the non-vanishing of the term in this sum corresponding to the multi-index $(r_1,\ldots,r_d)$ is: $i_{r_1}\leqslant j_1,\ldots,i_{r_d}\leqslant j_d$ and the set $\{i_{r_1},\ldots,i_{r_d}\}$ contains d elements. Suppose then that $(r_1,\ldots,r_d)$ is such a multi-index, and write $\{i_{r_1},\ldots,i_{r_d}\}=\{\bar{j}_1<\ldots<\bar{j}_d\}=\bar{e}$. It is then immediate that $\bar{e}\precneqq e$. The conclusion is that we can write v_e as a linear combination of elements $v_{\bar{e}}$ where $\bar{e}\precneqq e\prec e'$, and where $\bar{e}\subset I$. So we just have to show that if $e\prec e'$ and if $e\subset I$ then $d\pi(v_e)=0$. So assume that $e\prec e'$ and $e\subset I$, write $e=\{j_1<\ldots<j_d\}=\{i_{\tilde{j}_1}<\ldots<i_{\tilde{j}_d}\}$, and set $\tilde{e}=\{\tilde{j}_1<\ldots<\tilde{j}_d\}\in\mathcal{D}_n^{\text{even}}$. We have $e'=\{i_{\tilde{j}'_1}<\ldots<i_{\tilde{j}'_{d'}}\}$, where $\{\tilde{j}'_1<\ldots<\tilde{j}'_{d'}\}=J_{\tilde{g}}=\tilde{e}'$, and clearly $\tilde{e}\prec\tilde{e}'$. As in the proof of Theorem 2.1.1 we see that $c(v_e)=v_{\tilde{e}}$, and therefore, using the induction hypothesis, $0=d\tilde{\pi}(v_{\tilde{e}})=d\pi(v_e)$. This settles case (*a*).

Case (*b*). — Again we use the notation from the proof of Theorem 2.1.1. We have that $j'_1=2$, so, since $e\prec e'$, either $j_2=1$ or $j_1=2$. If $j_1=1$, then $v_e=0$, since X_1 is central. We can therefore assume that $j_1=2$.

Set $p=\min\{1\leqslant j\leqslant m \mid X_j\notin \mathfrak{h}\}$. We then construct the Jordan-Hölder basis $\hat{X}_1, \ldots, \hat{X}_m$, and we see that $p\in e'$, so we can write $p=j'_{\alpha'}$ with $2\leqslant \alpha'\leqslant d'$. We then distinguish two subcases: case (*b* 1): $p\in e$ and case (*b* 2): $p\notin e$.

Case (*b* 1). – Write $p=j_\alpha$, $2\leqslant\alpha\leqslant d$. As in the proof of Theorem 3.3.1 we have $v_{\hat{e}=(-1)^{\alpha} ve, \text{ where } e=\{j1}<\ldots<\hat{j}_d\}$, $\hat{j}_h$ being defined by $\hat{j}_h=j_h$ for $1\leqslant h\leqslant\alpha-1$, $\hat{j}_h=j_{h+1}-1$ for $\alpha\leqslant h\leqslant d-1$ and $\hat{j}_d=m$. Setting $\hat{e}'=\hat{J}_g=\{\hat{j}'_1<\ldots<\hat{j}'_{d'}\}$, we see as in the proof of Theorem 3.3.1 that $\hat{j}'_h=j'_h$ for $1\leqslant h\leqslant\alpha'-1$, $\hat{j}'_h=j'_{h+1}-1$ for $\alpha'\leqslant h\leqslant d'-1$ and $\hat{j}'_{d'}=m$. It is easily seen that $\hat{e}\prec\hat{e}'$. (In fact, suppose first that $d>d'$ and $j_r=j'_r$ for all $r\leqslant d'$; then $\alpha=\alpha'$, and $\hat{j}_r=\hat{j}'_r$ for all $r\leqslant d'-1$, while

$$\hat{j}_{d'}=j_{d'1}-1\leqslant m-1<m=\hat{j}'_{d'}, \quad \text{so } \hat{e}\prec\hat{e}'.$$

Suppose next that $k\leqslant\min\{d, d'\}$, that $j_k<j'_k$ and that $r<k\Rightarrow j_r=j'_r$. If $k<\alpha$, and if also $k<\alpha'$ we clearly have $\hat{e}'\prec e'$, and if $k\geqslant\alpha'$ we actually have $k=\alpha'$, since $k>\alpha'$ implies that $p=j'_{\alpha'}=j_{\alpha'}<j_k<j_\alpha=p$ which is a contradiction so, $r<\alpha'\Rightarrow\hat{j}_r=j_r=j'_r=\hat{j}'_r$ while $\hat{j}'_{\alpha'}=j'_{\alpha'+1}-1\geqslant j'_{\alpha'}=p=j_\alpha>j_{\alpha'}$ so again $\hat{e}\prec\hat{e}'$. If $k\geqslant\alpha$, then $j_1=j'_1,\ldots,j_{\alpha-1}=j'_{\alpha-1}<p$ and $p=j\alpha\leqslant j'_\alpha$, implying that $j'_\alpha=p$, and therefore that $k>\alpha$, and that $\alpha=\alpha'$. But then we clearly have $\hat{j}'_r=\hat{j}_r$ for $r\leqslant k-2$, $\hat{j}'_{k-1}>j_{k-1}$, so again $\hat{e}\prec\hat{e}'$.) We have thus reduced to the case where $\mathfrak{g}_{m-1}=\mathfrak{h}$ and $j_d=m$. We shall then assume that this is the case from now on. We get as in the proof of Theorem 3.3.1 that $v_e=-i[X_2, X_m]v_{e^0}$, where

$$e^0=\{j_1^0<\ldots<j_{d-2}^0\}=\{j_2<\ldots<j_{d-1}\}.$$

Now clearly $e^0\prec e'^0$, where

$$e'^0=J_{g0}=\{j_1'^0<\ldots<j_{d'-2}'^0\}=\{j'_2<\ldots<j'_{d'-1}\}$$

(*v.* proof of Theorem 3.3.1). (In fact we cannot have that $d>d'$ and $j_r=j'_r$ for all $r\leqslant d'$, since $j'_{d'}=m$. Therefore there exists k such that $j_k<j'_k$ and $r<k \Rightarrow j_r=j'_r$. Clearly $2\leqslant k\leqslant d'-1$, and therefore $j_r^0=j_r'^0$ for all $r\leqslant k-1$ and $j_{k-1}^0<j_{k-1}'^0$, so $e^0\prec e'^0$.) By the induction hypothesis we then get that $d\pi_0(v_{e^0})=0$ and therefore that $d\pi_0(v_e)=0$.

Applying this to the functional sg, $s\in G$, we get similarly that $d(s\pi_0)(v_e)=0$, i. e. that $d\pi_0(\mathrm{Ad}(s^{-1})v_e)=0$ for all $s\in G$, and therefore, as in the proof of Theorem 3.1.1, we get that $d\pi(v_e)=0$. This settles case (*b* 1).

Case (*b* 2). – Here $p \notin e$. Suppose that $d > d'$ and that $j'_r = j_r$ for all $r \leqslant d'$. This would imply that $p \in e$ which is a contradiction. So there exists $k \leqslant \min\{d, d'\}$ such that $r < k \Rightarrow j_r = j'_r$ and $j_k < j'_k$. Suppose that $k > \alpha'$. Then $\alpha' \leqslant \min\{d, d'\}$ and $j_{\alpha'} = j'_{\alpha'} = p$ which is again a contradiction. So $k \leqslant \alpha'$. Therefore $j_k < j'_k \leqslant j'_{\alpha'} = p$. Set

$$\alpha = \min\{1 \leqslant r \leqslant d \mid X_{j_r} \notin \mathfrak{h}\}.$$

Then $j_{\alpha-1} < p < j_\alpha$, and $k < \alpha$.

Define $\hat{e} = \{\hat{j}_1 < \ldots < \hat{j}_d\}$, where $\hat{j}_h = j_h$ for $1 \leqslant h \leqslant \alpha - 1$, $\hat{j}_h = j_h - 1$ for $\alpha \leqslant h \leqslant d$.

Then

$$X_{j_h} = \hat{X}_{j_h} \quad \text{for} \quad 1 \leqslant h \leqslant \alpha - 1,$$
$$X_{j_h} = \hat{X}_{\hat{j}_h} - c_{j_h} \hat{X}_m \quad \text{for} \quad \alpha \leqslant h \leqslant d$$

(in fact, $\hat{X}_j = X_j$ for $1 \leqslant j \leqslant p-1$ and $\hat{X}_j = X_{j+1} + c_{j+1} X_p$ for $p \leqslant j \leqslant m-1$, so $\hat{X}_j = X_j$ for $1 \leqslant j \leqslant p-1$ and $\hat{X}_j = X_{j+1} + c_{j+1} X_p$ for $p \leqslant j \leqslant m-1$, so $X_j = \hat{X}_{j-1} - c_j \hat{X}_m$ for $p+1 \leqslant j \leqslant m$, and from this the relations follow), and therefore

$$v_e = \frac{(-i)^{d/2}}{(d/2)!} A_d(X_{j_1}, \ldots, X_{j_{\alpha-1}}, X_{j_\alpha}, \ldots, X_{j_d})$$
$$= \frac{(-i)^{d/2}}{(d/2)!} A_d(\hat{X}_{\hat{j}_1}, \ldots, \hat{X}_{\hat{j}_{\alpha-1}}, \hat{X}_{\hat{j}_\alpha} - c_{j_\alpha} \hat{X}_m, \ldots, \hat{X}_{\hat{j}_d} - c_{j_d} \hat{X}_m).$$

For $\alpha \leqslant \tau \leqslant d$, define the element $\hat{e}_\tau \in \mathscr{D}_m^{\text{even}}$ by

$$\hat{e}_\tau = \{\hat{j}_1 < \ldots < \hat{j}_{\alpha-1} < \ldots < \hat{j}\!\!/_\tau < \ldots < \hat{j}_d < m\}$$

($= \{\hat{j}_1^\tau < \ldots < \hat{j}_d^\tau\}$). Since A_d is alternating we then get

$$v_e = \frac{(-i)^{d/2}}{(d/2)!} A_d(\hat{X}_{\hat{j}_1}, \ldots, \hat{X}_{\hat{j}_{\alpha-1}}, \hat{X}_{\hat{j}_\alpha}, \ldots, \hat{X}_{\hat{j}_d})$$
$$+ \frac{(-i)^{d/2}}{(d/2)!} \sum_{\tau=\alpha}^{d} -c_{j_\tau} A_d(\underset{\substack{\uparrow \\ 1}}{\hat{X}_{\hat{j}_1}}, \ldots, \underset{\substack{\uparrow \\ \alpha-1}}{\hat{X}_{\hat{j}_{\alpha-1}}}, \underset{\substack{\uparrow \\ \alpha}}{\hat{X}_{\hat{j}_\alpha}}, \ldots, \underset{\substack{\uparrow \\ \tau}}{\hat{X}_m}, \ldots, \underset{\substack{\uparrow \\ d}}{\hat{X}_{\hat{j}_d}})$$
$$= v_{\hat{e}} + \sum_{\tau=\alpha}^{d} (-1)^{\tau+1} c_{j_\tau} v_{\hat{e}_\tau}.$$

Now since $\hat{e} \subset \{1, \ldots, m-1\}$ and since $\hat{X}_{j_1} = X_2$ is central in $\mathfrak{h}$ we get that $v_{\hat{e}} = 0$ (Corollary 3.2.7). We then claim that $\hat{e}_\tau \prec \hat{e}'$ for all $\alpha \leqslant \tau \leqslant d$. In fact, for $r < k$ we have $\hat{j}_r^\tau = \hat{j}_r$ (since $k \leqslant \alpha - 1) = j_r = j'_r = \hat{j}'_r$ (since $k \leqslant \alpha'$), and $\hat{j}_k^\tau = \hat{j}_k$ (since $k \leqslant \alpha - 1) = j_k < j'_k \leqslant \hat{j}'_k$ ("=" if $k < \alpha'$, and if $k = \alpha'$, then $j'_k = p \leqslant j'_{k+1} - 1 = \hat{j}'_k$). This shows our claim.

It now follows from case (b 1) that $d\pi(v_{\hat{e}_\tau}) = 0$, for all $\alpha \leqslant \tau \leqslant d$, and therefore we finally get that $d\pi(v_e) = 0$. This settles case (b 2), and ends the proof of the theorem.

Corollary 3.4.5. — *If $g \in \Omega_{e'}$, if π is the irreducible representation of G corresponding to the orbit $O = Gg$ and if $e \in \mathcal{D}_m^{\text{even}}$ with $e \leqq e'$, then $d\pi(v_e) = P_e(g) I$.*

Proof. — This follows from Corollary 3.4.2, Theorem 3.3.1 and 3.4.4.

Let, for $e \in \mathcal{E}$, Ξ_e denote the set of irreducible representations π of G whose associated coadjoint orbit is contained in Ω_{e}. Using Corollary 3.4.5 and 3.4.2 we get.

Theorem 3.4.6. — *For all $e \in \mathcal{E}$ we have*

$$\Xi_e = \left\{ \pi \in \hat{G} \,\middle|\, \begin{array}{l} d\pi(v_e) \neq 0 \text{ and } d\pi(v_{e'}) = 0 \\ \textit{for all } e' \in \mathcal{E} \textit{ with } e' \prec e \end{array} \right\}.$$

We can now give a satisfactory answer to the question posed in the beginning of this section: Given an irreducible representation π of G we use Theorem 3.4.6 to find the $e \in \mathcal{E}$ such that the coadjoint orbit O associated with π is contained in Ω_e, and then proceed using Theorem 3.1.1 to find the orbit O itself. In an obvious way we also get an algebraic way of chechking whether a given representation of G is factorial, and, if so, of finding the orbit associated with it.

4. An application concerning the continuity of the trace

Let A be a C^*-algebra.

4.1. First we recall what it means that A is with generalized continuous trace: Set $\mathfrak{n}=\mathfrak{n}(A)$ to be the set of elements x in A such that the map $\pi \to \operatorname{Tr}(\pi(x^* x))$ is finite and continuous on $\hat{A}$. $\mathfrak{n}(A)$ is a selfadjoint ideal in A. Furthermore set $\mathfrak{m}=\mathfrak{m}(A)=\mathfrak{n}^2$. $\mathfrak{m}$ is a heriditary ideal in A contained in $\mathfrak{n}$, and it has the same closure in A as $\mathfrak{n}$. Set $J(A)=\overline{\mathfrak{m}(A)}=\overline{\mathfrak{n}(A)}$ which is a closed ideal in A (*cf.* [2], p. 240).

There exists an ordinal $\alpha=\alpha(A)$ and an increasing family of closed ideals $(J_\beta)_{0\leqslant\beta\leqslant\alpha}$ such that (*a*) $J_0=\{0\}$, $J(A/J_\alpha)=\{0\}$, (*b*) if $\beta\leqslant\alpha$ is a limit ordinal, then J_β is the closure of $\bigcup_{\beta'<\beta} J_{\beta'}$, (*c*) if $\beta<\alpha$, then $J_{\beta+1}/J_\beta=J(A/J_\beta)\neq\{0\}$. Furthermore α and the family $(J_\beta)_{0\leqslant\beta\leqslant\alpha}$ are uniquely determined by these properties ([2], p. 242).

The C^*-algebra A is said to be with generalized continuous trace (GCT) if $J_\alpha=A$ ([2], Définition 4, p. 243).

4.2. Suppose that B is a dense $*$-subalgebra in A. We now define what it means that A is GCT with respect to B: We set $\mathfrak{n}_A(B)=\mathfrak{n}(A)\cap B$, and we set $\mathfrak{m}_A(B)=\mathfrak{n}_A(B)^2$. Then $\mathfrak{n}_A(B)$ and $\mathfrak{m}_A(B)$ are twosided $*$-ideals in B. We set $J_A(B)$ to be the closure of $\mathfrak{m}_A(B)$ in A. Then $J_A(B)$ is a closed ideal in A.

Using transfinite induction we get a result analogous to the one above: There exists an ordinal $\alpha=\alpha_A(B)$ and an increasing family $(J_\beta)_{0\leqslant\beta\leqslant\alpha}$ of closed ideals in A such that:

(*a*) $J_0=\{0\}, J_{A/J_\alpha}(B+J_\alpha/J_\alpha)=\{0\}$, (*b*) if $\beta\leqslant\alpha$ is a limit ordinal, then J_β is the closure of $\bigcup_{\beta'<\beta} J_{\beta'}$, (*c*) if $\beta<\alpha$, then

$$J_{\beta+1}/J_\beta=J_{A/J_\beta}(B+J_\beta/J_\beta)\neq\{0\}.$$

Furthermore α and the family $(J_\beta)_{0\leqslant\beta\leqslant\alpha}$ are uniquely determined by these properties.

We say that A is with generalized continuous trace with respect to B if $J_\alpha=A$. Clearly, if A is with generalized continuous trace with respect to B, then A is with generalized continuous trace, and $\alpha(A)=\alpha_A(A)\leqslant\alpha_A(B)$.

For $0<\beta\leqslant\alpha$ not a limit ordinal we set $\mathcal{J}_\beta(B)$ to be the inverse image in A by the quotient map of $\mathfrak{m}_{A/J_{\beta-}}(B+J_{\beta-}/J_{\beta-})$, where $\beta-$ is the immediate predecessor of β, and we set $\mathcal{J}_0(B)=0$.

4.3. Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and set $A=C^*(G)$. In [4] DIXMIER showed ([4], 8. Théorème, p. 117):

THEOREM 4.3.1. (Dixmier). – *A is GCT, and $\alpha=\alpha(A)$ is finite.*

Set $B=C_c^\infty(G)$ which is a dense $*$-subalgebra of A. In the next section we use the results of Section 3 to prove the following.

THEOREM 4.3.2. – *A is GCT with respect to B, $\alpha=\alpha_A(B)$ is finite and $\mathcal{J}_\alpha(B)=B$.*

4.4. Let $\mathfrak{g}=\mathfrak{g}_m\supset\mathfrak{g}_{m-1}\supset\ldots\supset\mathfrak{g}_1\supset\mathfrak{g}_0=\{0\}$ be a Jordan-Hölder sequence for $\mathfrak{g}$, and retain the notation from the Preliminaries (Section 1). Write $\mathcal{E}=\{e_1\prec\ldots\prec e_n=\varnothing\}$, set $\mathcal{J}_0=\{0\}$ and set for $1\leqslant j\leqslant n$

$$\mathcal{J}_j=\sum\nolimits_{j'\leqslant j}C_c^\infty(G)*v_{e_{j'}}^**v_{e_{j'}}*C_c^\infty(G).$$

Then $\mathcal{J}_j$, $0\leqslant j\leqslant m$, is a two-sided $*$-ideal in $C_c^\infty(G)$, and since $v_{e_n}\equiv 1$ we have a finite composition series

$$C_c^\infty(G)=\mathcal{J}_n\supset\mathcal{J}_{n-1}\supset\ldots\supset\mathcal{J}_1\supset\mathcal{J}_0=\{0\}.$$

Set $\overline{\mathcal{J}}_j$, $0\leqslant j\leqslant n$, to be the norm closure of $\mathcal{J}_j$ in $C^*(G)$. Each $\overline{\mathcal{J}}_j$ is a closed ideal in $C^*(G)$ and gives rise to an open subset $\hat{\bar{\mathcal{J}}}_j$ of $\hat{G}$ (namely: $\hat{\bar{\mathcal{J}}}_j=\{\pi\in\hat{G}\mid\pi|\mathcal{J}_j\not\equiv 0\}$). Set $\hat{\bar{\mathcal{J}}}_j=V_j$. We then have a finite composition series

$$\hat{G}=V_n\supset V_{n-1}\supset\ldots\supset V_1\supset V_0=\varnothing,$$

into open subsets.

Let us then note that the restriction of π to $v_e * C_c^\infty(G)$ is zero if and only if $d\pi(v_e)=0$. Therefore we get from theorem 3.4.6:

LEMMA 4.4.1:

$$\Xi_{e_j}=\{\pi\in\hat{G}\,|\,\pi\,|\,\mathscr{I}_j\neq 0 \textit{ and } \pi\,|\,\mathscr{I}_{j'}=0 \textit{ for all } j'<j\}.$$

COROLLARY 4.4.2:

$$\Xi_{e_j}=V_j\setminus V_{j-1} \qquad \textit{and} \qquad V_j=\cup_{j'\leqslant j}\Xi_{e_{j'}}.$$

PROPOSITION 4.4.3. – *If* $\varphi\in\mathscr{I}_j$, $1\leqslant j\leqslant n$, *then* $\pi\to\operatorname{Tr}(\pi(\varphi))$ *is continuous on* $\complement V_{j-1}=\bigcup_{j\leqslant j'}\Xi_{e_{j'}}$.

Proof. – Let π_n be a sequence in $\complement V_{j-1}$ such that $\pi_n\to\pi\in\hat{G}$. We have to prove that $\operatorname{Tr}(\pi_n(\varphi))\to\operatorname{Tr}(\pi(\varphi))$. We can clearly assume that all the π_n belong to one $\Xi_{e_{j'}}$ for $j'\geqslant j$, and since each $\complement V_j$ is closed we have that $\pi\in\Xi_{e_{j''}}$ for $j''\geqslant j'$. Now if $j''\geqslant j'>j$ we have that $\pi(\varphi)=0$ and $\pi_n(\varphi)=0$ (Lemma 4.4.1) for all n, so this situation is trivial. Suppose then that all π_n are in Ξ_{e_j} so that $\pi\in\Xi_{e_{j'}}$ with $j'\geqslant j$. It is no loss of generality to assume that $\varphi=\varphi_1 * v_{e_j}^* * v_{e_j} * \varphi_2$ where $\varphi_1,\varphi_2\in C_c^\infty(G)$. But $\pi_n(\varphi)=|P_{e_j}(g_n)|^2\pi_n(\varphi_1 * \varphi_2)$, where g_n is a functional in the orbit O_n of π_n (Theorem 3.3.1), and similarly $\pi(\varphi)=|P_{e_j}(g)|^2\pi(\varphi_1 * \varphi_2)$, where g is a functional in the orbit O of π (Corollary 3.4.5). We can assume that g_n and g have been selected such that $g_n\to g$ [1]. Suppose first that $j'>j$. Then $P_{e_j}(g)=0$, since $g\in\Omega_{e_{j'}}$ (Lemma 3.4.1), and we therefore have to prove that $\operatorname{Tr}(\pi_n(\varphi))\to 0$ for $n\to\infty$. Now using e. g. the formula on p. 12 in [8] specialized to the nilpotent case we find

$$|P_{e_j}(g_n)|\int_{O_n}\frac{1}{(1+\|l\|^2)^{(d+1)/2}}\,d\beta_{O_n}(l)$$

$$\leqslant(2\pi)^{-d/2}M(d+1)\ldots M(2)\left(=\frac{1}{1.3\ldots(d-1)}\right)<+\infty,$$

where $\|l\|^2=\sum_{j=1}^{m}|\langle l, X_j\rangle|^2$, $l\in\mathfrak{g}^*$, β_{O_n} is the canonical measure of the orbit O_n and where

$$M(k)=\int_{-\infty}^{+\infty}\frac{1}{(1+x^2)^{k/2}}dx \qquad \text{for} \quad k>0.$$

Since $\varphi_1*\varphi_2\circ\exp$ is a C^∞-function on $\mathfrak{g}$ with compact support, its Fourier transform

$$(\varphi_1*\varphi_2\circ\exp)\hat{}(l)=\int_{\mathfrak{g}}\varphi_1*\varphi_2(\exp X)e^{i\langle l, X\rangle}dX$$

is a Schwartz function on $\mathfrak{g}^*$, hence there is a constant K such that

$$(1+\|l\|^2)^{(d+1)/2}|(\varphi_1*\varphi_2\circ\exp)\hat{}(l)|\leqslant K,$$

for all $l\in\mathfrak{g}^*$. But then using the Kirillov character formula and the result from above we get

$$\begin{aligned}|\mathrm{Tr}(\pi_n(\varphi))| &= |P_{e_j}(g_n)|^2|\mathrm{Tr}(\pi_n(\varphi_1*\varphi_2))|\\ &=|P_{e_j}(g_n)^2|\left|\int_{O_n}(\varphi_1*\varphi_2\circ\exp)\hat{}(l)\,d\beta_{O_n}(l)\right.\\ &\leqslant|P_{e_j}(g_n)|^2\int_{O_n}\frac{K}{(1+\|l\|^2)^{(d+1)/2}}d\beta_{O_n}(l)\\ &\leqslant|P_{e_j}(g_n)|K(2\pi)^{-d/2}M(d+1)\dots M(2)\to 0\end{aligned}$$

for $n\to\infty$, since $P_{e_j}(g_n)\to P_{e_j}(g)=0$. This settles the case $j'>j$. The case $j'=j$ is handled by the following lemma:

LEMMA 4.4.4. — *The function* $\pi\to\mathrm{Tr}(\pi(\varphi))$, $\varphi\in C_c^\infty(G)$,*is continuous on each of the subsets* Ξ_e, $e\in\mathscr{E}$.

Proof. — First, find a constant $K>0$ such that

$$(1+\|l\|^2)^{(d+1)/2}(\varphi\circ\exp)\hat{}(l)<K \text{ for all } l\in\mathfrak{g}^*.$$

Let $g_n, g_0 \in \Omega_e$ with $g_n \to g_0$ and let π_n, π_0 be the associated irreducible representations. Set

$$\psi_n(x) = (\varphi \circ \exp)\hat{}\,(\textstyle\sum_{j=1}^{m} R_j^e(g_n, x)\, l_j), \qquad x \in \mathbf{R}^d, \quad n \geqslant 0.$$

Then ψ_n is a Schwartz function on $\mathbf{R}^d$ and ψ_n converges to ψ, uniformly on compact subsets. Now

$$\begin{aligned}
&|\operatorname{Tr}(\pi_n(\varphi)) - \operatorname{Tr}(\pi(\varphi))| \\
&\quad = \left| \frac{1}{P_e(g_n)} \int_{\mathbf{R}^d} (\varphi \circ \exp)\hat{}\,(\textstyle\sum_{j=1}^{m} R_j^e(g_n, x)\, l_j)\, dx \right. \\
&\qquad \left. - \frac{1}{P_e(g_0)} \int_{\mathbf{R}^d} (\varphi \circ \exp)\hat{}\,(\textstyle\sum_{j=1}^{m} R_j^e(g_0, x)\, l_j)\, dx \right| \\
&\quad = \left| \int_{\mathbf{R}^d} \left(\frac{1}{P_e(g_n)} \psi_n(x) - \frac{1}{P_e(g_0)} \psi_0(x) \right) dx \right| \\
&\quad \leqslant \int_{[-C, C]^d} \left| \frac{1}{P_e(g_n)} \psi_n(x) - \frac{1}{P_e(g_0)} \psi_0(x) \right| dx \\
&\qquad\qquad + \int_{\mathbf{R}^d \setminus [-C, C]^d} \left| \frac{1}{P_e(g_n)} \psi_n(x) - \frac{1}{P_e(g_0)} \psi_0(x) \right| dx,
\end{aligned}$$

where $C > 0$. But the last integral is smaller than

$$\begin{aligned}
&\frac{K}{P_e(g_n)} \int_{\mathbf{R}^d \setminus [-C, C]^d} \frac{1}{(1 + \sum_{j=1}^{m} |R_j^e(g_n, x)|^2)^{(d+1)/2}}\, dx \\
&\qquad + \frac{K}{P_e(g_0)} \int_{\mathbf{R}^d \setminus [-C, C]^d} \frac{1}{(1 + \sum_{j=1}^{m} |R_j^e(g, x)|^2)^{(d+1)/2}}\, dx \\
&\quad \leqslant \frac{K}{P_e(g_n)} \int_{\mathbf{R}^d \setminus [-C, C]^d} \frac{1}{(1 + x_1^2 + \ldots + x_d^2)^{(d+1)/2}}\, dx_1 \ldots dx_d \\
&\qquad + \frac{K}{P_e(g_0)} \int_{\mathbf{R}^d \setminus [-C, C]^d} \frac{1}{(1 + x_1^2 + \ldots + x_d^2)^{(d+1)/2}}\, dx_1 \ldots dx_d.
\end{aligned}$$

Now chosing for a given $\varepsilon>0$ the number $C>0$ such that the last expression is smaller than $\varepsilon/2$ for all n (which is clearly possible) we get that

$$|\operatorname{Tr}(\pi_n(\varphi))-\operatorname{Tr}(\pi(\varphi))| \leqslant \int_{[-C, C]^d} \left| \frac{1}{P_e(g_n)} \psi_n(x) - \frac{1}{P_e(g_0)} \psi_0(x) \right| dx + \frac{\varepsilon}{2}$$

for all n. But this shows that $\operatorname{Tr}(\pi_n(\varphi)) \rightarrow \operatorname{Tr}(\pi(\varphi))$ since ψ_n converges to ψ_0 uniformly on compact subsets.

COROLLARY 4.4.5. — *Theorem* 4.3.2 *is true.*

Proof. — Setting

$$\mathfrak{K}_1 = \{ \varphi \in C_c^\infty(G) \,|\, \pi \rightarrow \operatorname{Tr}(\pi(\varphi^* * \varphi)) \text{ is continuous} \}$$

we have that $\mathscr{I}_1(B) = \mathfrak{K}_1^2$, and since $v_{e_1} * C_c^\infty(G) \subset \mathfrak{K}_1$ (Proposition 4.4.3) we have that $\mathscr{I}_1 \subset \mathscr{I}_1(B)$. But this shows that $\hat{\mathscr{I}}_1 \subset J_1(B)\hat{}$, hence $\complement J_1(B)\hat{} \subset \complement V_1$. Set

$$\mathfrak{K}_2 = \{ \varphi \in C_c^\infty(G) \,|\, \pi \rightarrow \operatorname{Tr}(\pi(\varphi^* * \varphi)) \text{ is continuous on } \complement J_1(B)\hat{} \}.$$

Then, since $\complement J_1(B)\hat{} \subset \complement V_1$ we have by Proposition 4.4.3 that $v_{e_1} * C_c^\infty(G)$ and $v_{e_2} * C_c^\infty(G)$ are contained in $\mathfrak{K}_2$, hence $\mathscr{I}_2 \subset \mathfrak{K}_2^2 = \mathscr{I}_2(B)$. Continuing like this we see that the sequence $\mathscr{I}_1(B)$, $\mathscr{I}_2(B), \ldots$ stops at $C_c^\infty(G)$ in finitely many steps. This ends the proof of the corollary.

Remark 4.4.6. — By Dixmier's result (Theorem 4.3.1) we have a canonical composition series of $A = C^*(G)$:

$$C^*(G) = J_\alpha \supset J_{\alpha-1} \supset \ldots \supset J_1 \supset J_0 = \{0\}$$

by a finite sequence of closed two-sided ideals. By our result (Theorem 4.3.2) we have a canonical composition series of $B = C_c^\infty(G)$:

$$C_c^\infty(G) = \mathscr{I}_\beta \supset \mathscr{I}_{\beta-1} \supset \ldots \supset \mathscr{I}_1 \supset \mathscr{I}_0 = \{0\}$$

by a finite sequence of two-sided $*$-ideals in $C_c^\infty(G)$: In connexion with these two composition series we would like to raise the following problems:

(1) is $\alpha=\beta$ (clearly $\alpha\leqslant\beta$, *cf.* above)?

(2) if so, is $\mathscr{J}_j$ dense in J_j (clearly $\mathscr{J}_j\subset J_j$)?

Let I_j be the two-sided $*$-ideal in $U(\mathfrak{g}_\mathbf{C})$ defined by

$$I_j=\{u\in U(\mathfrak{g}_\mathbf{C})\,|\,C_c^\infty(G)*u*C_c^\infty(G)\in\mathscr{J}_j\}.$$

We then have a canonical composition series of $U(\mathfrak{g}_\mathbf{C})$:

$$U(\mathfrak{g}_\mathbf{C})=I_\beta\supset I_{\beta-1}\supset\ldots\supset I_1\supset I_0=\{0\}$$

by finitely many two-sided $*$-ideals.

(3) is $C_c^\infty(G)*I_j*C_c^\infty(G)$ dense in $\mathscr{J}_j$ (clearly

$$C_c^\infty(G)*I_j*C_c^\infty(G)\subset\mathscr{J}_j)?$$

Of course the answer to the questions posed above will be affirmative if it is true that whenever π is an irreducible representation of G such that $d\pi$ vanishes on I_j, then π [considered as a representation of $C^*_*(G)$] vanishes on J_j (it is clear that if π vanishes on J_j then $d\pi$ vanishes on I_j).

(4) is there an algebraic characterisation of the ideals I_j?

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