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CHARACTERIZATION OF NUCLEAR FRÉCHET SPACES IN WHICH EVERY BOUNDED SET IS POLAR

BY

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In his investigation of control sets in locally convex spaces LELONG [13] gave a sufficient condition for a Fréchet space to have the property that every bounded subset B is polar in some neighbourhood of B (*see* also LELONG [12] and KISELMAN [14]). He remarked that this condition is satisfied by the spaces $H(\mathbb{C}^n)$ of all entire functions on $\mathbb{C}^n$ and asked for a classification of the Fréchet spaces having this property.

In the present article we prove the following two main results concerning this question: *A* nuclear Fréchet space E contains a bounded subset which is not uniformly polar if and only if E has the linear topological invariant $(\tilde{\Omega})$ introduced by VOGT [24]. If, moreover, E has the bounded approximation property then E contains a bounded non-polar subset if and only if E has $(\tilde{\Omega})$. Property $(\tilde{\Omega})$ is of the same type but stronger (resp. weaker) than the linear topological invariant (Ω) (resp. $(\bar{\Omega})$) which has been used to characterize the quotient spaces of power series spaces of infinite (resp. finite) type (*see* VOGT and WAGNER [27] (resp. VOGT [23] and WAGNER [28])).

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In order to obtain these characterizations we first show that a closed bounded absolutely convex subset B of an arbitrary Fréchet space F is uniformly polar and hence polar in F if F does not have the property $(\tilde{\Omega}_B)$. This implies that non-polar compact subsets can only exist in Fréchet-Schwartz spaces and that property $(\tilde{\Omega})$ is necessary for the existence of bounded subsets which are not uniformly polar. In order to show that $(\tilde{\Omega})$ is also sufficient in the class of all nuclear Fréchet spaces we use the following fact: If E has $(\tilde{\Omega})$ then there exists a bounded subset B of E such that for every continuous semi-norm p on E there exists a zero neighbourhood V such that for all $x \in V$ there exists a holomorphic path in the canonical Banach space $\hat{E}_p$ which passes through the canonical image of x and has values in the canonical image of B on a non-empty open subset of $\mathbb{C}$.

To prove our second main result we use that from NOVERRAZ [18] and SCHOTTENLOHER [22] it follows that a dense linear subspace F of a Fréchet space E with the bounded approximation property is non-polar in E if and only if the holomorphic completion $F_{\mathscr{O}}$ of F coincides with E. We get the desired result by then proving that a nuclear Fréchet space E with $(\tilde{\Omega})$ contains a total bounded absolutely convex subset B with the property that every holomorphic function on the linear hull E_B of B has a holomorphic extension to E. This property is established by an interpolation argument from MEISE and VOGT [16], which is based on the fact that nuclearity implies that holomorphic functions locally have an absolutely convergent monomial Taylor expansion.

We also obtain for a certain class of Fréchet spaces, including all Schwartz sequence spaces $\lambda^0(A)$, that the property $(\tilde{\Omega})$ is equivalent to the existence of a bounded subset which is not uniformly polar. The proof uses the fact that for these spaces the bounded sets are essentially weighted l^∞-balls.

It is reasonable to conjecture that for a larger class of Fréchet spaces E property $(\tilde{\Omega})$ characterizes the existence of a bounded subset of E which is not uniformly polar. However, our methods use strongly the nuclearity and the structure of the bounded sets.

Concluding, we want to remark that the present article and [16] influenced each other in various ways. For example the interpolation argument used in Theorem 10 originally appeared in [16] for nuclear Fréchet spaces with basis. For such spaces a first proof of Theorem 10 showed its importance for the present article. This observation motivated the further development which finally led to Theorem 10 and to the results of [16], section 3.

The main results of the present article have been announced in [6]. We wish to indicate that Theorem 2 (resp. Cor. 3) of [6] should be replaced by Theorem 10 (resp. Thm. 11) of the present article since they were based on a result of NOVERRAZ [18] which is not known to hold in the generality stated in [18] (*see* the remark preceding Theorem 10).

1. Preliminaries

We shall use standard notation from the theory of locally convex spaces as presented in the books of PIETSCH [19] or SCHAEFER [20]. A l. c. space always denotes a complex vector space with a locally convex Hausdorff topology.

For a Fréchet space E we always assume that its l. c. structure is generated by an increasing system $(\|.\|_n)_{n\in\mathbb{N}}$ of semi-norms. Then we denote by E_n the canonically normed space $E/\|.\|_n^{-1}(0)$ and by $\hat{E}_n$ its completion. $\pi_n : E \to \hat{E}_n$ denotes the canonical map and U_n denotes the set $\{x\in E \mid \|x\|_n<1\}$. Sometimes it is convenient to assume that $(U_n)_{n\in\mathbb{N}}$ is a neighbourhood basis of zero.

If M is an absolutely convex subset of E, we define:

$$\|.\|_M^*: \quad E' \to [0, \infty] \quad \text{by } \|y\|_M^* := \sup_{x\in M}|y(x)|,$$

where E' denotes the topological dual of E. Obviously $\|.\|_M^*$ is the gauge functional of the polar of M. Instead of $\|.\|_{U_n}^*$ we write $\|.\|_n^*$. We remark that the adjoint π_n^t of π_n gives an isometry between:

$$(E_n)'_b=(\hat{E}_n)'_b \qquad \text{and} \qquad (E'_{U_n^0}, \|.\|_n^*).$$

By E_M we denote the linear hull of M, which becomes a normed space in a canonical way if M is bounded.

(i) SEQUENCE SPACES.

Let $A=(a_{j,k})_{(j,k)\in\mathbb{N}^2}$ be a matrix which satisfies:

(1) $0\leqslant a_{j,k}\leqslant a_{j,k+1}$ for all $j, k\in\mathbb{N}$

(2) for each $j\in\mathbb{N}$ there exists $k\in\mathbb{N}$ with $a_{j,k}>0$.

Then we define for $1 \leqslant s \leqslant \infty$ and $s=0$ the sequence spaces :

$$\lambda^s(A) := \{x \in \mathbb{C}^{\mathbb{N}} \mid \|x\|_k := (\textstyle\sum_{j=1}^{\infty}(|x_j| a_{j,k})^s)^{1/s} < \infty \text{ for all } k \in \mathbb{N}\},$$

$$1 \leqslant s < \infty,$$

$$\lambda^{\infty}(A) := \{x \in \mathbb{C}^{\mathbb{N}} \mid \|x\|_k := \sup_{j \in \mathbb{N}} |x_j| a_{j,k} < \infty \text{ for all } k \in \mathbb{N}\},$$

$$\lambda^0(A) := \{x \in \lambda^{\infty}(A) \mid \lim_{j \to \infty} x_j a_{j,k} = 0 \text{ for all } k \in \mathbb{N}\}.$$

Obviously $\lambda^s(A)$ is a Fréchet space under the natural topology induced by the semi-norms $(\|.\|_k)_{k \in \mathbb{N}}$. We write $\lambda(A)$ instead of $\lambda^1(A)$.

We recall that $\lambda^s(A)$ is Schwartz (resp. nuclear) iff for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ and $\mu \in c_0$ (resp. $\mu \in l^1$) such that:

$$a_{j,k} \leqslant \mu_j a_{j,m} \quad \text{for all } j \in \mathbb{N}.$$

We recall from BIERSTEDT, MEISE and SUMMERS [1], 2.5, that for $s=0$ and $1 \leqslant s \leqslant \infty$ the sets $\{N_a^s \mid a \in \lambda^{\infty}(A), a > 0\}$ form a fundamental system for the bounded subsets of $\lambda^s(A)$, where:

$$N_a^s := \left\{x \in \lambda^s(A) \mid \textstyle\sum_{j=1}^{\infty} \left(\dfrac{|x_j|}{a_j}\right)^s \leqslant 1\right\} \qquad \text{for} \quad 1 \leqslant s < \infty$$

and:

$$N_a^{\infty} := \left\{x \in \lambda^{\infty}(A) \mid \sup \frac{|x_j|}{a_j} \leqslant 1\right\},$$

$$N_a^0 := \left\{x \in \lambda^0(A) \mid \sup \frac{|x_j|}{a_j} \leqslant 1\right\}.$$

If α is an increasing unbounded sequence of positive real numbers (called an exponent sequence) and if $R=1$ or $R=\infty$ then we define for $1 \leqslant s \leqslant \infty$ the power series spaces:

$$\Lambda_R^s(\alpha) := \lambda^s(A(R, \alpha)),$$

where:

$$A(R, \alpha) = \{(r_k^{\alpha_j})_{j \in \mathbb{N}} \mid k \in \mathbb{N}\}, \qquad r_k \nearrow R.$$

$\Lambda_R^s(\alpha)$ is called a power series space of finite (resp. infinite type) if $R=1$ (resp. $R=\infty$). We remark that well-known examples of nuclear power series spaces are:

$$s=C^\infty(S^1)\simeq\Lambda_\infty((\log n+1)_{n\in\mathbb{N}}),$$

$$H(\mathbb{C}^k)\simeq\Lambda_\infty((\sqrt[k]{n})_{n\in\mathbb{N}})\qquad\text{and}\qquad H(\mathbb{D}^k)\simeq\Lambda_1((\sqrt[k]{n})_{n\in\mathbb{N}}),$$

where $\mathbb{D}$ stands for the open unit disk in $\mathbb{C}$ and where $H(\Omega)$ denotes the space of all holomorphic functions on Ω endowed with the compact-open topology.

(ii) Polar sets.

Let E be a l.c. space. $f: E\to[-\infty, \infty[$ is called plurisubharmonic (psh.) if f is not identically $-\infty$, if f is upper semi-continuous (i.e. $f^{-1}([-\infty, c[)$ is open for every $c\in\mathbb{R}$) and if $z\mapsto f(a+zb)$ is subharmonic or identically $-\infty$ for every a, $b\in E$. f is called uniformly plurisubharmonic if there exists a continuous seminorm p on E and a plurisubharmonic function g on the canonical Banach space $\hat{E}_p$ such that $f=g\circ\pi_p$.

A subset B of E is called (uniformly) polar, if there exists a (uniformly) psh. function f on E such that:

$$B\subset\{x\in E\,|\,f(x)=-\infty\}.$$

We shall use that an absolutely convex bounded subset B of E is not (uniformly) polar if the canonical space E_B is not (uniformly) polar. This is an immediate consequence of the fact that for each psh. function f on E its restriction to E_B is $-\infty$ or plurisubharmonic for the canonical norm topology on E_B and that B is a zero neighbourhood for this topology.

For more information on psh. functions on l.c. spaces we refer to Noverraz [17].

(iii) Holomorphic functions.

Let E be a l.c. space and let $\Omega\subset E$ be open, $\Omega\neq\emptyset$. $f:\Omega\to\mathbb{C}$ is called holomorphic if f is continuous and its restriction to each finite dimensional section of Ω is holomorphic as a function of several complex variables. By $H(\Omega)$ (resp. $H^\infty(\Omega)$) we denote the space of all holomorphic (resp. bounded holomorphic) functions on Ω. For details concerning holomorphic functions on l.c. spaces we refer to the books of Dineen [5] and Noverraz [17].

Here we only recall that there exists a unique maximal subspace $E_{\mathscr{O}}$ of the completion $\hat{E}$ of E with the property that every $f \in H(E)$ has a holomorphic extension to $E_{\mathscr{O}}$. The linear space $E_{\mathscr{O}}$ is called the holomorphic completion of E.

2. Definition. — *Let E be a Fréchet space.*

(*a*) *Let $B \subset E$ be closed bounded and absolutely convex. E has property $(\tilde{\Omega}_B)$ if the following is true:*

For every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$, $C>0$ and $d>0$ such that for all $y \in E'$:

$$\|y\|_q^{*1+d} \leqslant C \|y\|_B^* \|y\|_p^{*d}.$$

(*b*) *E has property $(\tilde{\Omega}_b)$ (resp. $(\tilde{\Omega}_c)$) if there exists a closed absolutely convex bounded (resp. compact) subset B of E such that E has $(\tilde{\Omega}_B)$.*

(*c*) *E has property $(\tilde{\Omega})$ if:*

For every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ and $d>0$ such that for every $k \in \mathbb{N}$ there exists $C>0$ such that for every $y \in E'$:

$$\|y\|_q^{*1+d} \leqslant C \|y\|_k^* \|y\|_p^{*d}.$$

Remark. — (*a*) Property $(\tilde{\Omega})$ was introduced by Vogt in [25], sect. 5.

(*b*) It is easy to check that the properties $(\tilde{\Omega}_b)$, $(\tilde{\Omega}_c)$ and $(\tilde{\Omega})$ do not depend on the choice of the semi-norm system and that they are linear topological invariants which are inherited by separated quotients. It is obvious that the implications $(\tilde{\Omega}_c) \Rightarrow (\tilde{\Omega}_b) \Rightarrow (\tilde{\Omega})$ are true.

3. Proposition. — (*a*) *Every Fréchet space with $(\tilde{\Omega}_b)$ (resp. $(\tilde{\Omega}_c)$) is quasi-normable (resp. a. Schwartz space).*

(*b*) *For Fréchet-Schwartz spaces the properties $(\tilde{\Omega}_b)$, $(\tilde{\Omega}_c)$ and $(\tilde{\Omega})$ are equivalent.*

Proof. — (*a*). If E has $(\tilde{\Omega}_b)$ then there exists a closed bounded absolutely convex set B in E such that 2(*a*) is true. From this one obtains, using the idea of proof of Lemma 2.1 and Corollary 2.2 of Vogt and Wagner [27], that for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$, $C>0$ and $d>0$ such that for all $r>0$:

$$U_q \subset C r B + \frac{1}{r^d} U_p. \tag{1}$$

From this it is evident that for every zero-neighbourhood U there exists another one V such that for every $\alpha>0$ there is a bounded set M in E with:

$$V \subset M + \alpha U.$$

By GROTHENDIECK [11], p. 107, this shows that E is quasi-normable.

If E has $(\tilde{\Omega}_c)$ then we have (1) for some compact set B. This implies that for every $\varepsilon>0$ the set U_q can be covered by finitely many sets of the form $x+\varepsilon U_p$. Hence E is a Schwartz space.

(b) It suffices to show that $(\tilde{\Omega})$ implies $(\tilde{\Omega}_c)$. This is obtained by a suitable modification of the proof of Lemma 1.4 of VOGT [23].

4. PROPOSITION. — *Assume that* $1\leqslant s<\infty$ *or* $s=0$.

(a) $\lambda^s(A)$ *has* $(\tilde{\Omega})$ *if and only if A satisfies:*

For each $p\in\mathbb{N}$ *there exists* $q\in\mathbb{N}$ *and* $d>0$ *such that for each* $k\in\mathbb{N}$ *there exists* $C>0$ *such that for all* $j\in\mathbb{N}$:

$$a_{j,k}\,a_{j,p}^d\leqslant C\,a_{j,q}^{1+d}. \tag{1}$$

(b) $\lambda^s(A)$ *has* $(\tilde{\Omega}_{N_a^*})$ *for* $a\in\lambda^\infty(A)$, $a>0$ *iff:*

For each $p\in\mathbb{N}$ *there exists* $q\in\mathbb{N}$, $C>0$ *and* $d>0$ *such that for all* $j\in\mathbb{N}$:

$$a_{j,p}^d\leqslant C\,a_j\,a_{j,q}^{1+d}. \tag{2}$$

(c) *The following are equivalent:*

(α) $\lambda^s(A)$ *has* $(\tilde{\Omega})$;

(β) *there exists* $a\in\lambda^\infty(A)$, $a>0$, *satisfying* (2) *of* (b);

(γ) $\lambda^s(A)$ *has* $(\tilde{\Omega}_b)$.

Proof. — We first remark that for $1<s<\infty$ and $s=0$:

$$\lambda^s(A)'=\left\{y\in\mathbb{C}^{\mathbb{N}}\mid \text{there exists } k\in\mathbb{N}\colon \|y\|_k^*=\left(\sum_{j=1}^\infty\left(\frac{|y_j|}{a_{j,k}}\right)^{s'}\right)^{1/s'}<\infty\right\},$$

where:

$$\frac{1}{s}+\frac{1}{s'}=1 \quad\text{and}\quad s'=1 \quad\text{for } s=0.$$

Here and in the sequel we let:

$$\frac{a}{0}:=\infty \quad\text{for } a>0 \quad\text{and}\quad \frac{0}{0}:=0.$$

For $s=1$ we have:

$$\lambda^1(A)'=\left\{y\in\mathbb{C}^{\mathbb{N}}\mid \text{there exists } k\in\mathbb{N}: \|y\|_k^*=\sup_{j\in\mathbb{N}}\frac{|y_j|}{a_{j,k}}<\infty\right\}.$$

(*a*) If $\lambda^s(A)$ has $(\tilde{\Omega})$ we apply the definition to:

$$e_j' = (\delta_{nj})_{n \in \mathbb{N}} \in \lambda^s(A)'$$

and get:

$$a_{j,q}^{-(1+d)} \leqslant C\, a_{j,k}^{-1} a_{j,p}^{-d},$$

which implies (1). If (1) is satisfied we get for $1<s<\infty$, and $s=0$ by Hölder's inequality, for every $y \in \lambda^s(A)'$ with $\|y\|_q^* < \infty$:

$$\|y\|_q^{*s'} = \sum_{j=1}^{\infty} \left(\frac{|y_j|}{a_{j,q}}\right)^{s'} \leqslant C^{s'/(1+d)} \sum_{j=1}^{\infty} \left(\frac{|y_j|}{a_{j,k}}\right)^{s'/(1+d)} \left(\frac{|y_j|}{a_{j,p}}\right)^{s'd/(1+d)}$$
$$\leqslant C^{s'/(1+d)} \left(\sum_{j=1}^{\infty} \left(\frac{|y_j|}{a_{j,k}}\right)^{s'}\right)^{1/(1+d)} \left(\sum_{j=1}^{\infty} \left(\frac{|y_j|}{a_{j,p}}\right)^{s'}\right)^{d/(1+d)}$$
$$= C^{s'/(1+d)} \|y\|_k^{*\,s'/(1+d)} \|y\|_p^{*\,s'/(1+d)},$$

which implies $(\tilde{\Omega})$. The case $s=1$ is treated similarly.

(*b*) We remark that for $1<s<\infty$ and $s=0$ we have:

$$\|y\|_{N_a^s}^* = \left(\sum_{j=1}^{\infty} (|y_j|\, a_j)^{s'}\right)^{1/s'} \qquad \text{for} \quad \frac{1}{s} + \frac{1}{s'} = 1$$

and:

$$s' = 1 \qquad \text{for} \quad s=0,$$

while $\|y\|_{N_a^1}^* = \sup_{j \in \mathbb{N}} |y_j|\, a_j$. From this it follows as in part (*a*) that $(\tilde{\Omega}_{N_a^s})$ is equivalent to (2).

(*c*) $(\alpha) \Rightarrow (\beta)$: For $p \in \mathbb{N}$ we let:

$$b_j^{(p)} := \frac{a_{j,p}^d}{a_{j,q}^{1+d}} \qquad \text{for} \quad a_{j,q} > 0,$$

and zero otherwise. Then we get from (1) that $b^{(p)}$ is in $\lambda^\infty(A)$. Hence there exists $a \in \lambda^\infty(A)$ such that $b^{(p)} \leqslant C_p a$ for all $p \in \mathbb{N}$. This implies (β).

$(\beta) \Rightarrow (\gamma)$: This is an immediate consequence of Bierstedt, Meise and Summers [1], 2.5 and (*b*).

$(\gamma) \Rightarrow (\alpha)$: obvious.

5. EXAMPLES. — (1) For every exponent sequence α and $1 \leqslant s \leqslant \infty$ the power series space $\Lambda_1^s(\alpha)$ together with all its separated quotient spaces has $(\tilde{\Omega}_c)$. Since $\Lambda_1^s(\alpha)$ is a Schwartz space (and hence $\Lambda_1^\infty(\alpha) = \Lambda_1^0(\alpha)$) this follows immediately from 4(b) with:

$$a = (1)_{j \in \mathbb{N}} \quad \text{and} \quad d = C = 1$$

since for every $p \in \mathbb{N}$ there exists $q > p$ with $r_p \leqslant r_q^2$.

(2) For every exponent sequence α and $1 \leqslant s \leqslant \infty$ the power series space $\Lambda_\infty^s(\alpha)$ does not have $(\tilde{\Omega})$. In order to see this, choose $R = 1$, and take $S > 1$ and $d > 0$ arbitrarily. Then we get for $T > S^{1+d}$ that:

$$\sup_{j \in \mathbb{N}} T^{\alpha_j} S^{-(1+d)\alpha_j} = \sup_{j \in \mathbb{N}} \left(\frac{T}{S^{1+d}}\right)^{\alpha_j} = \infty,$$

since $\lim_{j \to \infty} \alpha_j = \infty$.

(3) The following example of VOGT [25], 5.6, shows the existence of nuclear Fréchet spaces $\lambda^s(\mathbb{N} \times \mathbb{N}, A)$ which have $(\tilde{\Omega})$ and which are not quotient spaces of finite type power series spaces.

The matrix:

$$A = \{ (a_{i,\, j;\, k})_{(i,\, j) \in \mathbb{N} \times \mathbb{N}} \mid k \in \mathbb{N} \}$$

is defined as follows:

Let α and β be exponent sequences, choose a strictly increasing sequence σ of positive real numbers with $\lim_{k \to \infty} \sigma_k = 1$ and put:

$$\rho_{j,\, k} := \begin{cases} 1 + \sigma_k & \text{for } j < k, \\ \sigma_k & \text{for } j \geqslant k. \end{cases}$$

Finally define $a_{i,\, j,\, k} := \exp(\rho_{j,\, k} \alpha_i \beta_j)$.

(4) An example of a non-normable Fréchet space with $(\tilde{\Omega}_b)$ which is not a Schwartz space is given in [13].

In order to give a sufficient condition for a closed bounded absolutely convex set B to be polar, we introduce the following notation which is used to derive a technical lemma.

NOTATION. — Let E be a Fréchet space, B a closed bounded absolutely convex subset of E and $p \in \mathbb{N}$. A projection π on E is called $B-p$-admissible if there exist $n \in \mathbb{N}$, $x_j \in E_B$ and $y_j \in E'_p$ for $1 \leqslant j \leqslant n$, such that:

$$\pi(x) = \sum_{j=1}^{n} y_j(x) x_j.$$

6. Corollary. — *Let E be a Fréchet space and B a closed, bounded, absolutely convex subset with dense linear hull. E has $(\tilde{\Omega}_B)$ if the following holds:*

For each $p\in\mathbb{N}$ there exist $q\in\mathbb{N}$, $C>0$, $d>0$ and a $B-p$-admissible projection π on E such that for all $y\in E'_p\cap\ker\pi^t$:

$$(1)\qquad \|y\|_{q,\pi}^{*1+d}\leqslant C\|y\|_B^*\|y\|_p^{*d},$$

where:

$$\|y\|_{q,\pi}^* := \sup\{|y(x)|\ |x\in\ker\pi,\ \|x\|_q\leqslant 1\}.$$

Proof. — If $p\in\mathbb{N}$ is given we choose $q\in\mathbb{N}$, $C>1$, $d>0$ and π such that (1) holds. Since $\|\ \|_B^*$, $\|\ \|_p^*$ and $\|\ \|_q^*$ are norms on E'_p we assume that C is chosen so large that (2) and (3) are satisfied:

$$(2)\qquad \begin{gathered}\|\pi x\|_q\leqslant C\|x\|_q,\qquad \|\pi x\|_p\leqslant C\|x\|_p,\\ \|\pi x\|_B\leqq C\|x\|_B\quad\text{for all } x\in E,\end{gathered}$$

$$(3)\qquad \|y\|_q^{*1+d}\leqslant C\|y\|_B^*\|y\|_p^{*d}\quad\text{for all } y\in\operatorname{im}\pi^t\subset\operatorname{span}(y_1,\ldots,y_n)\subset E'_p.$$

Next we let $\pi_0 := \mathrm{id}_E-\pi$ and note that for $y\in E'_p$ we have:

$$\pi_0^t y\in E'_p\cap\ker\pi^t\qquad\text{and}\qquad y=\pi^t y+\pi_0^t y.$$

Then we get for all $x\in E$, because of (2), $\pi_0 x\in\ker\pi$,

$$\|\pi_0 x\|_q\leqslant(1+C)\|x\|_q\qquad\text{and}\qquad \pi_0^t y(x)=\pi_0^t y(\pi_0 x).$$

This implies:

$$(4)\qquad \|\pi_0^t y\|_q^*\leqslant(1+C)\|\pi_0^t y\|_{q,\pi}^*.$$

Hence we get from (3), (4), (1) and (2) that for all $y\in E'_p$:

$$\begin{aligned}\|y\|_q^{*1+d}&\leqslant(2\max(\|\pi^t y\|_q^*,\|\pi_0^t y\|_q^*))^{1+d}\\ &\leqslant 2^{1+d}\max(C\|\pi^t y\|_B^*\|\pi^t y\|_p^{*d},(1+C)^{1+d}C\|\pi_0^t y\|_B^*\|\pi_0^t y\|_p^{*d})\\ &\leqslant C2^{1+d}(1+C)^{2+2d}\|y\|_B^*\|y\|_p^{*d}.\end{aligned}$$

Since this estimate holds trivially for all $y\in E'\setminus E'_p$, it follows that E has $(\tilde{\Omega}_B)$.

7. Theorem. — *Let E be a Fréchet space and let B be a closed bounded and absolutely convex subset of E. If E does not have $(\tilde{\Omega}_B)$ then B is uniformly polar in E.*

Proof. — If E_B is not dense in E, then there exists $y \in E'$ with $y \neq 0$ and $y \mid E_B = 0$. Then $f: x \mapsto \log |y(x)|$ is uniformly psh. and $B \subset \{x \in E \mid f(x) = -\infty\}$. Hence we can assume for the rest of the proof that E_B is dense in E.

Since E does not have $(\tilde{\Omega}_B)$ the hypotheses of Lemma 6 are not satisfied. Hence we may assume without loss of generality that the semi-norm system $(\|\cdot\|_n)_{n \in \mathbb{N}_0}$ of E satisfies:

$$2\| \quad \|_k \leqslant \| \quad \|_{k+1} \text{ for all } k \in \mathbb{N}_0$$

and that we have for $p = 0$:

for every $q \in \mathbb{N}$, $C > 0$, $d > 0$ and every $B-0$-admissible projection π there exists $y \in E'_0 \cap \ker \pi^t$ with:

$$\|y\|_{q,\pi}^{*\,1+d} > C \|y\|_B^* \|y\|_0^{*\,d}. \tag{1}$$

Since the inequality in (1) is $(1+d)$-homogeneous it follows that for $C = q^q$ and $d = q$ there exists:

$$y \in E'_0 \cap \ker \pi^t \qquad \text{with} \quad \|y\|_0^* = \frac{1}{q},$$

which satisfies (1). Since:

$$\| \quad \|_{q,\pi}^* \leqslant \| \quad \|_q^* \leqslant \| \quad \|_0^*,$$

we get for all $q \in \mathbb{N}$ and every $B-0$-admissible projection π there exists:

$$y \in E'_0 \cap \ker \pi^t \qquad \text{with} \qquad \|y\|_0^* = \frac{1}{q}$$

such that:

$$\|y\|_{q,\pi}^{*\,q} \geqslant \|y\|_{q,\pi}^{*\,1+q} \geqslant \|y\|_B^*. \tag{2}$$

Now we determine inductively $(y_k)_{k \in \mathbb{N}}$ in E'_0, $(x_k)_{k \in \mathbb{N}}$ in E_B and a zero-sequence $(a_k)_{k \in \mathbb{N}}$ of strictly positive real numbers with the following properties:

$$\left\{\begin{array}{l} \|y_k\|_0^* = \dfrac{1}{k}, \quad \|x_k\|_k = 1 \quad \text{for all } k \in \mathbb{N}, \\ y_j(x_k) = 0 \quad \text{for all } j, k \in \mathbb{N} \qquad \text{with } j \neq k, \\ y_k(x_k) \geqslant a_k^2, \qquad a_k^k \geqslant \|y_k\|_B^* \quad \text{for all } k \in \mathbb{N}. \end{array}\right. \tag{3}$$

To begin the induction we apply (2) with $q=1$ and $\pi_0 := 0$. Then we get:

$$y_1 \in E_0' \qquad \text{with} \quad \|y_1\|_0^* = 1,$$

such that:

$$a_1 := \|y_1\|_{1,\pi_0}^* \geqslant \|y_1\|_B^*.$$

Since E_B is dense we have:

$$0 < a_1 = \|y_1\|_1^* \leqslant \frac{1}{2}\|y_1\|_0^* = \frac{1}{2},$$

which implies $a_1^2 < a_1$. Hence there exists $x_1 \in E_B$ with:

$$\|x_1\|_1 = 1 \qquad \text{and} \qquad y_1(x_1) \geqslant \|y_1\|_{1,\pi_0}^{*\,2} = a_1^2.$$

For the induction step we assume that for $1 \leqslant k \leqslant n$ y_k, x_k and a_k have been chosen in such a way that (3) holds for these k.

Then we put $\lambda_j := (y_j(x_j))^{-1}$ and note that:

$$\pi_n : \quad x \mapsto \textstyle\sum_{j=1}^n y_j(x)\lambda_j x_j,$$

is a $B-0$-admissible projection on E. Hence we may apply (2) with:

$$q = n+1 \qquad \text{and} \qquad \pi = \pi_n \quad \text{to get } y_{n+1} \in E_0' \cap \ker \pi_n^t$$

with:

$$\|y_{n+1}\|_0^* = \frac{1}{n+1} \qquad \text{and} \qquad \|y_{n+1}\|_{n+1,\pi_n}^{*\,n+1} \geqslant \|y_{n+1}\|_B^*.$$

We let $a_{n+1} := \|y_{n+1}\|_{n+1,\pi_n}^*$ and remark that:

$$0 < a_{n+1} = \|y_{n+1}\|_{n+1,\pi_n}^* \leqslant \|y_{n+1}\|_{n+1}^* \leqslant \|y_{n+1}\|_0^* = \frac{1}{n+1}$$

and hence $a_{n+1}^2 < a_{n+1}$. Note that $E_B \cap \ker \pi_n$ is dense in $\ker \pi_n$ since E_B is dense in E and π_n maps E_B into itself. Hence there exists:

$$x_{n+1} \in E_B \cap \ker \pi_n \qquad \text{with} \quad \|x_{n+1}\|_{n+1} = 1$$

and:

$$y_{n+1}(x_{n+1}) \geqslant a_{n+1}^2.$$

Since:

$$\ker \pi_n = \{ x \in E \,|\, y_j(x) = 0 \text{ for } 1 \leqslant j \leqslant n \}$$

and:

$$\ker \pi_n^{\tau} = (\operatorname{span}(x_1, \ldots, x_n))^{\perp},$$

y_{n+1} satisfies all the conditions in (3). Since $a_{n+1} \leqslant 1/(n+1)$, the existence of the sequences $(y_k)_{k \in \mathbb{N}}$, $(x_k)_{k \in \mathbb{N}}$ and $(a_k)_{k \in \mathbb{N}}$ satisfying (3) follows by induction.

Now we let:

$$\alpha_k := -(k^2 \log a_k)^{-1} \quad \text{for} \quad k \in \mathbb{N}$$

and define $g : \hat{E}_0 \to [-\infty, \infty[$ *by:*

$$g(z) = \sum_{k=1}^{\infty} \alpha_k \log |y_k(z)|,$$

where we have identified $E_0' = \hat{E}_0'$ with $(E')_{U_0^0}$. We remark that under this identification $\| \ \|_0^*$ is the dual norm on $\hat{E}_0'$. Since $\| y_k \|_0^* = 1/k$, for every $R > 0$ there exists $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k > m$ and all $z \in \hat{E}_0$ with $\| z \|_0 < R$ we have $|y_k(z)| < 1$ and hence $\log |y_k(z)| < 0$. Since $\alpha_k > 0$ for all $k \in \mathbb{N}$ this shows that locally g is the decreasing limit of psh. functions and hence is psh. on $\hat{E}_0$ provided that it is not identically $-\infty$.

In order to show that $g \circ \pi_0$ and hence g is not identically $-\infty$ we remark that the series $\sum_{j=1}^{\infty} x_j$ converges in E and defines an element $x_0 \in E$ since:

$$2 \| \ \|_k \leqslant \| \ \|_{k+1} \qquad \text{and} \qquad \| x_k \|_k = 1.$$

Because of (3), the identification mentioned above, and the choice of α_k we have:

$$g(\pi_0(x_0)) = \sum_{k=1}^{\infty} \alpha_k \log |y_k(x_k)| \geqslant \sum_{k=1}^{\infty} \frac{-2 \log a_k}{k^2 \log a_k} = -\sum_{k=1}^{\infty} \frac{2}{k^2} > -\infty.$$

This shows that g is psh. on $\hat{E}_0$. Hence $f := g \circ \pi_0$ is uniformly psh. on E. The proof is now completed by noting that for every $x \in B$ we have, by (3),

$$f(x) = g(\pi_0(x)) = \sum_{k=1}^{\infty} \alpha_k \log |y_k(x)| \leqslant \sum_{k=1}^{\infty} \alpha_k \log \| y_k \|_B^*$$

$$\leqslant \sum_{k=1}^{\infty} \alpha_k k \log a_k = \sum_{k=1}^{\infty} \frac{-k \log a_k}{k^2 \log a_k} = -\sum_{k=1}^{\infty} \frac{1}{k} = -\infty.$$

From Proposition 3 (a) and Theorem 7, we obtain the following Corollary.

8. COROLLARY. — *Let E be a Fréchet space.*

(a) *If E contains a bounded (resp. compact) subset which is not uniformly polar then E has $(\tilde{\Omega}_b)$ (resp. $(\tilde{\Omega}_c)$).*

(b) *If E is not Schwartz then every compact subset of E is uniformly polar and hence polar in E.*

Remark. — An alternative proof of 8 (b) is possible using limiting sets and the fact that every non-Schwartz Fréchet space has a non-Montel quotient.

Now we show that in nuclear Fréchet spaces the converse of Corollary 8(a) is also true.

9. THEOREM. — *Let E be a nuclear Fréchet space. The following are equivalent:*

(i) *E contains a bounded subset which is not uniformly polar;*

(ii) *E has $(\tilde{\Omega})=(\tilde{\Omega}_b)=(\tilde{\Omega}_c)$.*

Proof. — By Corollary 8 and Proposition 3(b), (i) implies (ii). In order to show the converse implication we first remark that by the nuclearity of E we can assume that $(\|\ \|_n)_{n\in\mathbb{N}}$ is chosen in such a way that the corresponding canonical spaces are Hilbert spaces for each $n\in\mathbb{N}$. Moreover we can assume that E has $(\tilde{\Omega}_B)$ for a closed bounded absolutely convex subset B of E which is a Hilbert ball, i.e. for which the canonically normed space E_B is a Hilbert space.

By a remark in 1.(ii) it suffices to show that E_B is not uniformly polar, i.e. that for every uniformly psh. function f on E there exists $b\in E_B$ with $f(b)>-\infty$. In order to show this, let f be given. Then there exists $p\in\mathbb{N}$ and a psh. function g on $\hat{E}_p$ with $f=g\circ\pi_p$.

Next we let $A:=\pi_p|E_B: E_B\to\hat{E}_p$ and remark that by replacing the norm on E_B by a suitable multiple, we may assume $\|A\|\leqslant 1/2$. It is easy to see that E_B is dense in E since E has $(\tilde{\Omega}_B)$. Consequently $\operatorname{im}(A)=\pi_p(E_B)$ is dense in $\hat{E}_p$. Since A factors through the nuclear space E it follows that A is strongly nuclear. Hence the spectral representation theorem (*see* PIETSCH [19], 8.3) implies that A can be represented by:

$$A\,x=\sum_{j=1}^{\infty}\lambda_j(x\,|\,e_j)_B\,f_j,$$

where $\{e_j\,|\,j\in\mathbb{N}\}$ is an orthonormal system in E_B, $\{f_j\,|\,j\in\mathbb{N}\}$ is a complete orthonormal system in $\hat{E}_p$ and $(\lambda_j)_{j\in\mathbb{N}}$ is a decreasing sequence in s with:

$$0<\lambda_j\leqslant\frac{1}{2}\quad\text{for all } j\in\mathbb{N}.$$

Now we define $\varphi_k\in E'$ by $\varphi_k : x\mapsto(\pi_p(x)\,|\,f_k)_p$ and remark that:

(1) $\|\varphi_k\|_p^*=\sup_{x\in U_p}|\varphi_k(x)|$

$$\leqslant\sup_{x\in U_p}\|\pi_p(x)\|_p\,\|f_k\|_p\leqslant 1\qquad\text{for all } k\in\mathbb{N}.$$

Furthermore we have for all $x\in E_B$:

(2) $$\varphi_k(x)=(\pi_p(x)\,|\,f_k)_p=(A\,x\,|\,f_k)_p=\lambda_k(x\,|\,e_k)_B$$

and hence:

(3) $$\|\varphi_k\|_B^*=\sup_{x\in B}|\varphi_k(x)|=\lambda_k\sup_{x\in B}|(x\,|\,e_k)|=\lambda_k.$$

Now we choose $q\in\mathbb{N}$, $q>p$, $C>1$ and d according to $(\tilde{\Omega}_B)$ and get from (1) and (3) that:

(4) $$\|\varphi_k\|_q^{*\,1+d}\leqslant C\,\|\varphi_k\|_B^*\,\|\varphi_k\|_p^{*\,d}\leqslant C\lambda_k\quad\text{for all } k\in\mathbb{N}.$$

Since U_q is open in E there exists $x\in U_q$ with $g(\pi_p(x))=f(x)>-\infty$.

We let $\xi:=\pi_p(x)\in\hat{E}_p$ and see from (4) that:

(5) $$|(\xi\,|\,f_k)_p|=|(\pi_p(x)\,|\,f_k)_p|=|\varphi_k(x)|\leqslant\|\varphi_k\|_q^*\leqslant D\lambda_k^a,$$

where $a:=1/(1+d)$ and $D=C^a$.

Next we let $G:=\{z\in\mathbb{C}\,|\,\operatorname{Re}z>1/2\}$ and we define for $k\in\mathbb{N}$ the functions $h_k : G\to\mathbb{C}$ in the following way:

$$h_k(z):=\begin{cases}\dfrac{\xi_k}{|\xi_k|}e^{z\log|\xi_k|} & \text{if } |\xi_k|>\lambda_k^2,\\[2mm] \xi_k & \text{if } |\xi_k|\leqslant\lambda_k^2,\end{cases}$$

where $\xi_k = (\xi \mid f_k)_p$. Because of (5) we have for each $z \in G$:

$$\sum_{k=1}^{\infty} |h_k(z)|^2 \leqslant \sum_{k=1}^{\infty} |\xi_k| \leqslant D \sum_{k=1}^{\infty} \lambda_k^a < \infty,$$

since $\lambda \in s$. Hence the function $h : G \to \hat{E}_p$ defined by:

$$h(z) := \sum_{k=1}^{\infty} h_k(z) f_k,$$

is holomorphic. Consequently $g \circ h$ is subharmonic on G since:

$$g(h(1)) = g(\xi) = g(\pi_p(x)) = f(x) > -\infty.$$

Hence there exists w with $\operatorname{Re} w > 2/a = 2(1+d)$ such that $g(h(w)) > -\infty$. Because of (5) and the definition of the functions h_k we have:

$$(6) \qquad |h_k(w)| \leqslant C^2 \lambda_k^2 \quad \text{for all } k \in \mathbb{N}.$$

Since $\lambda \in s$ it follows from (6) that:

$$b := \sum_{k=1}^{\infty} \frac{h_k(w)}{\lambda_k} e_k,$$

defines an element of E_B for which:

$$f(b) = g(\pi_p(b)) = g(A\,b) = g\left(\sum_{k=1}^{\infty} h_k(w) f_k\right) = g(h(w)) > -\infty.$$

This completes the proof.

Remark. — In general it is an open question whether the existence of a bounded set which is not uniformly polar is equivalent to the existence of a bounded non-polar set. Under additional hypotheses on E, this can be shown by an application of a result of Noverraz [18] (which for Banach spaces with basis is also due to Coeuré [4], Thm. 2). However, at present it is not known whether this result holds in the generality stated in [18], since the proof of [18], Prop. 10, is incomplete. Therefore we recall here what is known to hold:

Let X be a dense linear subspace of the Fréchet space E and let:

$$X_{\mathscr{P}} := \bigcap \{ U \mid X \subset U \subset E,\ U \text{ open pseudoconvex} \}.$$

The result of Noverraz [18] which we are going to use reads as follows:

If $X_{\mathscr{O}} = X_{\mathscr{P}}$ then X is non-polar in E iff $X_{\mathscr{O}} = E$.

Since $X_{\mathscr{C}} = \bigcap \{U \mid X \subset U \subset E,\ U \text{ a domain of existence}\}$, $X_{\mathscr{C}} = X_{\mathscr{P}}$ obviously holds if the Levi problem for E has a positive solution, i. e. if every pseudoconvex domain in E is a domain of existence. From [18] in connection with SCHOTTENLOHER [21], Thm. 1, it follows that $X_{\mathscr{C}} = X_{\mathscr{P}}$ also holds if E has the approximation property and if every holomorphically convex open subset of E is a domain of existence.

10. THEOREM. — *Let E be a nuclear Fréchet space in which every holomorphically convex open subset is a domain of existence. Then the following are equivalent:*

(i) *E contains a bounded non-polar subset;*

(ii) *E contains a bounded subset which is not uniformly polar;*

(iii) *E has $(\tilde{\Omega}) = (\tilde{\Omega}_b) = (\tilde{\Omega}_c)$.*

Proof. — It is trivial that (i) implies (ii). The equivalence of (ii) and (iii) has been shown in Theorem 9. Hence it remains to show that (iii) implies (i). In order to do this we choose $(\|\ \|_n)_{n\in\mathbb{N}}$ and B as in the proof of Theorem 9. Then E_B is dense in E, since E has $(\tilde{\Omega}_B)$. Hence it follows from a remark in 1. (ii) and the result of NOVERRAZ stated in the preceding remark that B is non-polar in E, iff $(E_B, \tau_E)_{\mathscr{C}} = E$, where E_B carries the topology τ_E induced by E. Thus it suffices to show that every $f \in H(E_B, \tau_E)$ has a holomorphic extension to E.

If $f \in H(E_B, \tau_E)$ is given then there exists $p \in \mathbb{N}$ such that f is bounded on $E_B \cap U_p$. Hence we have for all $x \in E_B \cap U_p$:

$$f(x) = \sum_{n=0}^{\infty} p_n(x) \qquad \text{where} \quad p_n(x) = \frac{1}{2\pi i}\int_{|t|=1} \frac{f(tx)}{t^{n+1}}\,dt.$$

Moreover we have for all $n \in \mathbb{N}_0$:

$$\sup\{|p_n(x)| \mid x \in E_B \cap U_p\} \leqslant \sup\{|f(x)| \mid x \in E_B \cap U_p\} =: M.$$

Since $E_B \cap U_p$ is dense in U_p this implies that there exist n-homogeneous polynomials $\hat{p}_n$ on $\hat{E}_p$ satisfying $p_n = \hat{p}_n \circ \pi_p$ and:

$$\sup\{|\hat{p}_n(y)| \mid \|y\|_{\hat{p}} < 1\} \leqslant M.$$

Hence $\sum_{n=0}^{\infty} \hat{p}_n(y)$ converges absolutely and locally uniformly for all $y \in \hat{U}_p$, the open unit ball of $\hat{E}_p$. Since $\pi_p(E_B)$ is dense in $\hat{E}_p$, this implies that:

$$\hat{f}: \quad y \mapsto \sum_{n=0}^{\infty} \hat{p}_n(y),$$

is in $H^\infty(\hat{U}_p)$ and satisfies $f \mid (E_B \cap U_p) = \hat{f} \circ \pi_p$.

Since we have the same hypotheses as in the proof of Theorem 9 we continue as in that proof by representing the map $A := \pi_p \mid E_B$ and by defining the functionals $\varphi_k \in E'$ till we get (4) of Theorem 9. We shall refer to (1)-(4) of Theorem 9 in the sequel without mentioning Theorem 9. Moreover, we let:

$$F_B := \overline{\operatorname{span}\{e_j \mid j \in \mathbb{N}\}}^{E_B}$$

and denote by $\pi : E_B \to E_B$ the orthogonal projection of E_B onto F_B. We remark that $\ker \pi = E_B \cap \ker \pi_p$. Since f is bounded on $E_B \cap U_p$, an application of Liouville's theorem shows that:

$$f(x) = f \circ \pi(x) \quad \text{for all } x \in E_B \cap U_p.$$

Since $E_B \cap U_p$ is open in (E_B, τ_E) this implies by analytic continuation that $f = f \circ \pi$.

Next choose $\delta > 0$ such that:

$$\sum_{k=1}^{\infty} \left(\frac{\delta}{k}\right)^2 < 1.$$

For each $z \in \mathbb{C}^{\mathbb{N}}$ with $|z_j| < \delta/j$ we have:

$$\left\|\sum_{j=1}^{\infty} z_j f_j\right\|_p^2 = \sum_{j=1}^{\infty} |z_j|^2 \leqslant \sum_{j=1}^{\infty} \left(\frac{\delta}{j}\right)^2 < 1,$$

and hence:

$$\sum_{j=1}^{\infty} z_j f_j \in \hat{U}_p.$$

We let:

$$\mathbb{M} := \{m \in \mathbb{N}_0^{\mathbb{N}} \mid m_j \neq 0 \text{ only for finitely many } j \in \mathbb{N}\}$$

and define, in the same way as BOLAND and DINEEN [2], for $m=(m_1, \ldots, m_n, 0 \ldots) \in \mathbb{M}$ the m-th Taylor coefficient of $\hat{f}$ by:

$$a_m := \left(\frac{1}{2\pi i}\right)^n \int_{|z_1|=\mu_1} \cdots \int_{|z_n|=\mu_n} \frac{\hat{f}(z_1 f_1 + \ldots + z_n f_n)}{z_1^{m_1+1} \ldots z_n^{m_n+1}} dz_1 \ldots dz_n,$$

where $\mu_j := \delta/j$ for $j \in \mathbb{N}$. Then we have the estimate:

$$(5) \qquad |a_m| \leqslant \frac{M}{\mu^m} \quad \text{for all } m \in \mathbb{M}.$$

Since $\pi_p(e_j) = A e_j = \lambda_j f_j$ for all $j \in \mathbb{N}$ and $f = \hat{f} \circ \pi_p$ we also have the following representation of a_m:

$$(6) \quad a_m = \left(\frac{1}{2\pi i}\right)^n$$

$$\int_{|z_1|=\mu_1} \cdots \int_{|z_n|=\mu_n} \frac{\hat{f}(\pi_p((z_1/\lambda_1) e_1 + \ldots + (z_n/\lambda_n) e_n))}{z_1^{m_1+1} \ldots z_n^{m_n+1}} dz_1 \ldots dz_n$$

$$= \frac{1}{\lambda^m} \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_1|=r_1} \cdots \int_{|\zeta_n|=r_n} \frac{f(\zeta_1 e_1 + \ldots + \zeta_n e_n)}{\zeta_1^{m_1+1} \ldots \zeta_n^{m_n+1}} d\zeta_1 \ldots d\zeta_n.$$

In order to obtain a further estimate from this representation we remark that for $t>0$ the set:

$$K(t) := \{\textstyle\sum_{j=1}^{\infty} z_j e_j \mid |z_j| \leqslant t\mu_j \quad \text{for all } j \in \mathbb{N}\},$$

is compact in the Hilbert space E_B. Hence it is compact in the topology induced by E on E_B, which implies:

$$\sup\{|f(x)| \mid x \in K(t)\} =: N(t) < \infty \quad \text{for all } t>0.$$

Since f is holomorphic on E_B with respect to the topology induced by E we get from (6):

$$(7) \qquad |a_m| \leqslant \frac{N(t)}{\lambda^m \mu^m t^{|m|}} \quad \text{for all } m \in \mathbb{M} \text{ and all } t>0.$$

Now let $r>0$ be given. Then we let:

$$\eta:=\frac{1}{1+d},\qquad \gamma:=\frac{\eta}{2},\qquad \beta:=1-\gamma$$

and remark that:

$$\left(\frac{\lambda_k^\gamma}{\mu_k}\right)_{k\in\mathbf{N}}=\left(\frac{k}{\delta}\lambda_k^\gamma\right)_{k\in\mathbf{N}}$$

is in l^1 since λ is in s.

Hence $R/2:=\sup_{k\in\mathbf{N}}\lambda_k^\gamma\mu_k^{-1}<\infty$. We let $t=(Rr)^{1/\gamma}$ and $D=C^\eta$.

From (4), (5) and (7) we have the following estimate:

$$\begin{aligned}
\textstyle\sum_{m\in\mathbf{M}} r^{|m|}|a_m|\prod_{j=1}^{\infty}\|\varphi_j\|_q^{*\,m_j} &\leqslant D\textstyle\sum_{m\in\mathbf{M}} r^{|m|}|a_m|(\lambda^m)^\eta\\
&\leqslant D\textstyle\sum_{m\in\mathbf{M}} r^{|m|}|a_m|^\beta(|a_m|\lambda^m)^\gamma(\lambda^m)^\gamma\\
&\leqslant D\textstyle\sum_{m\in\mathbf{M}} r^{|m|}\left(\frac{M}{\mu^m}\right)^\beta\left(\frac{N(t)}{\mu^m t^{|m|}}\right)^\gamma(\lambda^m)^\gamma\\
&\leqslant DM^\beta N(t)^\gamma\textstyle\sum_{m\in\mathbf{M}}\left(\frac{r}{t^\gamma}\right)^{|m|}\frac{(\lambda^m)^\gamma}{\mu^m}\\
&=DM^\beta N(t)^\gamma\textstyle\sum_{m\in\mathbf{M}}\left(\frac{\lambda^\gamma}{R\mu}\right)^m=DM^\beta N(t)\prod_{k=1}^{\infty}\left(1-\frac{\lambda_k^\gamma}{R\mu_k}\right)^{-1}<\infty.
\end{aligned}$$

This implies that the series:

$$\textstyle\sum_{m\in\mathbf{M}} a_m\prod_{j=1}^{\infty}(\varphi_j(x))^{m_j},$$

converges absolutely and uniformly on rU_q for each $r>0$. Hence it defines a holomorphic function $g:E\to\mathbb{C}$. In order to see that g is the desired extension of f it suffices to show $f|F_B=g|F_B$, since:

$$f=f\circ\pi\qquad\text{and}\qquad g|F_B=(g|E_B)\circ\pi\quad\text{by (2).}$$

To show that $g|F_B$ we remark that for:

$$z=\textstyle\sum_{j=1}^{n} z_j e_j\in F_B$$

we have with $z_j=0$, for $j>n$:

$$f(z)=\textstyle\sum_{m\in\mathbf{M}} b_m z^m,$$

where:

$$b_m = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_1|=r_1} \dots \int_{|\zeta_n|=r_n} \frac{f(\zeta_1 e_1 + \dots + \zeta_n e_n)}{\zeta_1^{m_1+1} \dots \zeta_n^{m_n+1}} d\zeta_1 \dots d\zeta_n = \lambda^m a_m,$$

because of (6). From (2) we get that $\varphi_k(z) = \lambda_k z_k$ and hence:

$$g(z) = \sum_{m \in M} a_m \lambda^m z^m = \sum_{m \in M} b_m z^m = f(z).$$

Obviously, this implies $f \mid F_B = g \mid F_B$ and consequently $f = g \mid E_B$.

Remark. — (*a*) The hypothesis of Theorem 10 is, in particular, satisfied if E is a nuclear Fréchet space with the bounded approximation property, i. e. there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of continuous linear operators on E with finite rank such that:

$$\lim_{n \to \infty} A_n x = x \quad \text{for all } x \in E.$$

By the remark preceding Theorem 10 this follows from SCHOTTENLOHER [21], Cor. 3.4, where the Levi problem for E is shown to have a positive solution.

DUBINSKY [9] has shown the existence of nuclear Fréchet spaces which do not have the bounded approximation property. For a simple example of such a space we refer to VOGT [26].

(*b*) Examples of nuclear Fréchet spaces without the bounded approximation property which do contain bounded non-polar subsets and which have $(\tilde{\Omega})$ are obtained by the following arguments: If B is a bounded non-polar subset of the Fréchet space E and if F is a (separated) quotient space of E, then it is easy to see that $q(B)$ is a bounded non-polar subset of F ($q : E \to F$ denotes the quotient map). Hence it follows from Theorem 10 and part (*a*) of this remark that for every nuclear space $\lambda(A)$ with $(\tilde{\Omega})$ each quotient space contains a bounded non-polar subset and has $(\tilde{\Omega})$. However, DUBINSKY and VOGT [10] have shown that there exist quotient spaces of $\lambda(A)$ which do not have the bounded approximation property.

(*c*) From part (*a*) of this remark and the proof of Theorem 10 it follows that every nuclear Fréchet space E which has $(\tilde{\Omega})$ and the bounded approximation property contains dense non-complete linear subspaces F for which the holomorphic completion F_θ coincides with the topological completion $\hat{F}=E$. The only previous examples for this phenomenon were certain algebraic hyperplanes given in DINEEN and NOVERRAZ [8].

From Theorem 10 and the preceding remark (*a*) we get the characterization mentioned in the title of the present article which gives a reasonable answer to the question of LELONG [13]:

11. THEOREM. — *Let E be a nuclear Fréchet space with the bounded approximation property. Then every bounded subset of E is polar if and only if E does not have* $(\tilde{\Omega})$.

As a further corollary of Theorem 10 we get the following characterization of the compact non-polar polydiscs in the nuclear power series spaces of finite type.

12. COROLLARY. — *Let* $\Lambda_1(\alpha)$ *be nuclear and let* $a \in \Lambda_1(\alpha)$ *satisfy* $a>0$. *The set:*

$$N_a := \{ x \in \Lambda_1(\alpha) \mid |x_j| \leqslant a_j \text{ for all } j \in \mathbb{N} \}$$

is non-polar if and only if $\liminf_{j\to\infty} a_j^{1/\alpha_j}>0$.

Proof. — If $a>0$ and $\liminf_{j\to\infty} a_j^{1/\alpha_j}>0$ then there exists $\varepsilon>0$ and $\delta>0$ such that $a_j \geqslant \varepsilon\,\delta^{\alpha_j}$ for all j. Let $(r_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers with $\lim_{k\to\infty} r_k=1$. If $p\in\mathbb{N}$ is given choose $d>0$ such that:

$$\left(\frac{r_p}{r_{p+1}}\right)^d \frac{1}{r_{p+1}\,\delta} \leqslant 1.$$

Then we get for all $j\in\mathbb{N}$:

$$r_p^{\alpha_j d} \leqslant \delta^{\alpha_j} r_{p+1}^{\alpha_j(1+d)} \leqslant \frac{1}{\varepsilon} a_j r_{p+1}^{\alpha_j(1+d)}.$$

By Proposition 4 (*b*) this shows that $\Lambda_1^2(\alpha)=\Lambda_1(\alpha)$ has $(\tilde{\Omega}_{N_a^2})$. Since N_a^2 is a Hilbert ball in $\Lambda_1^2(\alpha)$ the proof of Theorem 10 shows that N_a^2 is non-polar in $\Lambda_1^2(\alpha)$. Since $N_a^2 \subset N_a$, N_a is non-polar in $\Lambda_1(\alpha)$.

If N_a is non-polar then it follows from Theorem 7 that $\Lambda_1^0(\alpha)=\Lambda_1(\alpha)$ has to have $(\tilde{\Omega}_{N_a})$. Hence it follows from Proposition 4 (*b*) that for $p\in\mathbb{N}$ there exists $q>p$, $C>0$ and $d>0$ such that:

$$r_p^{\alpha_j d} \leqslant C a_j r_q^{\alpha_j(1+d)} \quad \text{for all } j\in\mathbb{N}.$$

This implies $a>0$ and also:

$$\left(\frac{r_p}{r_q}\right)^d \frac{1}{r_q} \leqslant C^{1/\alpha_j} a_j^{1/\alpha_j} \quad \text{for all } j \in \mathbb{N}.$$

Hence:

$$\liminf_{j \to \infty} a_j^{1/\alpha_j} > 0.$$

Concluding we give a class of Fréchet spaces which in general are not Schwartz spaces but have property $(\tilde{\Omega}_b)$.

Let X be a locally compact, σ-compact Hausdorff space. Let $A=(a_k)_{k \in \mathbb{N}}$ be an increasing system of non-negative continuous functions on X which satisfy the following conditions:

(i) For each $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\operatorname{Supp} a_k \subset \overset{\circ}{\operatorname{Supp}}\, a_n$;

(ii) $\bigcup_{k \in \mathbb{N}} \overset{\circ}{\operatorname{Supp}}\, a_k = X$;

(iii) For each $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for each $\varepsilon>0$ there exists a compact subset K in X such that $a_k(x) \leqslant \varepsilon\, a_n(x)$ for all $x \in X \setminus K$.

Let F be an arbitrary Banach space. It is easy to see that the space:

$$C(X, A; F) := \{ f \mid f : X \to F \text{ is continuous and } \| f \|_k := \sup_{x \in X} \| f(x) a_k(x) \| < \infty \text{ for all } k \in \mathbb{N} \}$$

endowed with the topology induced by $(\| . \|_k)_{k \in \mathbb{N}}$ is a Fréchet space which in general is neither nuclear nor Schwartz.

13. Proposition. — *For the space $C(X, A; F)$ introduced above, the following conditions are equivalent:*

(1) *$C(X, A; F)$ contains a bounded subset which is not uniformly polar;*

(2) *$C(X, A; F)$ has $(\tilde{\Omega}_b)$;*

(3) *$C(X, A; F)$ has $(\tilde{\Omega})$.*

Proof. — By Corollary 8 (a), (1) implies (2), while (2) implies (3) trivially. Hence it remains to show that (3) implies (1).

In order to show this we remark that only $F \neq \{0\}$ is relevant and that in this case $C(X, A; \mathbb{C})$ is a complemented subspace of $C(X, A; F)$. Hence (3) implies that $C(X, A; \mathbb{C})$ has $(\tilde{\Omega})$.

If we apply this to the point-evaluations δ_x, we get, for each $p \in \mathbb{N}$, there exists $q > p$ and $d > 0$ such that for all $k \in \mathbb{N}$ there exists $C_k > 0$ such that for all $x \in X$:

$$\| \delta_x \|_q^{*\,1+d} \leqslant C_k \| \delta_x \|_k^* \| \delta_x \|_p^{*\,d}.$$

This implies that for all $x \in Y_p := \{ y \in X \mid a_p(y) > 0 \}$ we have:

$$a_q(x)^{-(1+d)} \leqslant C_k a_k(x)^{-1} a_p(x)^{-d}$$

i. e.:

$$\frac{a_p(x)^d}{a_q(x)^{1+d}} a_k(x) \leqslant C_k \quad \text{for all } x \in Y_p \text{ and all } k \in \mathbb{N}.$$

By condition (i) this implies that:

$$g_p: \quad x \mapsto \frac{a_p(x)^d}{a_q(x)^{1+d}}$$

belongs to $C(X, A; \mathbb{C})$ if q is chosen appropriately. It is easy to see that there exists $a \in C(X, A; \mathbb{C})$, $a > 0$, such that for each $p \in \mathbb{N}$ there exists $M_p > 0$ with $g_p \leqslant M_p a$. Hence we get:

(1) There exists $a \in C(X, A; \mathbb{C})$, $a > 0$, such that for every $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$, $C > 0$ and $d > 0$ such that $a_p^d \leqslant C a a_q^{1+d}$.

In order to show that:

$$B = \{ f \in C(X, A; F) \mid \| f(x) \| \leqslant a(x) \text{ for all } x \in X \}$$

is not uniformly polar, we let:

$C_0(Y_p, a_p; F) := \{ f \mid f: Y_p \to F,\ f$ is continuous, for every $\varepsilon > 0$ there exists $K \subset\subset Y_p$ such that $\sup \{ \| f(x) a_p(x) \| \mid x \in Y_p \setminus K \} < \varepsilon \}$.

This becomes a Banach space under the norm:

$$\| f \|_p^{\sim} := \sup_{x \in Y_p} \| f(x) a_p(x) \|.$$

Using condition (iii) it is easy to see that for every $f \in C(X, A; F)$ the restriction $f \mid Y_p$ is in $C_0(Y_p, a_p; F)$ and that for:

$$\pi_p: \quad C(X, A; F) \to C_0(Y_p, a_p; F), \qquad \pi_p(f) := f \mid Y_p$$

we have $\| \pi_p(f) \|_p^{\sim} = \| f \|_p$. Since the continuous functions with compact support are dense in $C_0(Y_p, a_p; F)$, this shows that $C_0(Y_p, a_p; F)$ can be identified with the canonical space $C(X, A; F)_p$.

Now let f be a psh. function on $C(X, A; F)_p$ for which $f\circ\pi_p$ is psh. on $C(X, A; F)$. Then there exists $g\in C(X, A; F)$ with $f(\pi_p(g))>-\infty$.

Let:

$$U:=\{x\in Y_p \mid \|g(x)\|<a(x)\}$$

and

$$V:=\left\{x\in Y_p \mid \|g(x)\|>\frac{a(x)}{2}\right\};$$

and choose a continuous partition of unity $1=\varphi_U+\varphi_V$ where $\operatorname{Supp}\varphi_U\subset U$, $\operatorname{Supp}\varphi_V\subset V$.

Let $G:=\{z\in\mathbb{C}\mid \operatorname{Re} z>0\}$ and define $h: Y_p\times G\to F$ by:

$$h(x, z):=\varphi_U(x)g(x)+\varphi_V(x)\frac{1}{a_p(x)}\frac{g(x)}{\|g(x)\|}\|g(x)a_p(x)\|^z.$$

It is easy to check that $h(., z)\in C_0(Y_p, a_p; F)$ and that:

$$H:\quad G\to C_0(Y_p, a_p; F),\qquad H(z):=h(., z),$$

is holomorphic.

We remark that from (1) we obtain $q>p$, $C>0$ and $d>0$ such that:

$$a_p(x)^{1+d}\leqslant C\,a(x)\,a_p(x)\,a_q(x)^{1+d}.$$

Hence we have, with $\varepsilon=1/(1+d)$:

$$(2)\qquad \|g(x)a_p(x)\|\leqslant C^\varepsilon(a(x)a_p(x))^\varepsilon\|g(x)a_q(x)\|.$$

By condition (iii) there exists a compact set K in X such that:

$$\sup\{a(x)a_p(x)\mid x\in X\setminus K\}<1$$

and

$$\sup\{\|g(x)a_q(x)\|C^\varepsilon\mid x\in X\setminus K\}\leqslant 1.$$

Then for all $x\in X\setminus K$ and all $t>1/\varepsilon$:

$$(3)\qquad \|g(x)a_p(x)\|\leqslant(a(x)a_p(x))^\varepsilon\leqslant(a(x)a_p(x))^{1/t}.$$

Now we let $G_0:=\{z\in G\mid 1/\varepsilon<\operatorname{Re} z<\tau\}$ for some $\tau>1/\varepsilon$ and we show that there exists $\lambda>1$ such that $H(G_0)\subset\pi_p(\lambda B)$. If $x\in U$ we get from the definition of U that:

$$\|\varphi_U(x)g(x)\|\leqslant\|g(x)\|\leqslant a(x).$$

If $x \in V$ and $x \notin Y_p$, then:

$$\varphi_V(x) g(x) \| g(x) a_p(x) \|^{z-1} = 0.$$

If $x \in (V \setminus K) \cap Y_p$ then we have for all $z \in G_0$:

$$\left\| \varphi_V(x) \frac{1}{a_p(x)} \frac{g(x)}{\| g(x) \|} \| g(x) a_p(x) \|^z \right\| \leqslant a(x).$$

Since K is compact, $a > 0$ and all the functions involved are continuous we can find $M > 1$ such that for all $z \in G_0$ and all $x \in K$ we have:

$$\| \varphi_V(x) g(x) \| g(x) a_p(x) \|^{z-1} \| \leqslant M a(x).$$

If we let $\lambda = 1 + M$ it follows that $H(z) \in \pi_p(\lambda B)$ for all $z \in G_0$, since it is easy to check that for all $z \in G_0$ the function $h(., z)$ actually is the restriction of a function in λB.

Since $H : G \to C_0(Y_p, a_p; F)$ is holomorphic, $f \circ H$ is subharmonic in G. Since:

$$f \circ H(1) = f(h(., 1)) = f(\pi_p(g)) > -\infty,$$

$f \circ H$ is not identically $-\infty$. Hence there exists $z_0 \in G_0$ with $f(H(z_0)) > -\infty$. Since we have already remarked that $H(z_0) = \pi_p(g_0)$ for some $g_0 \in \lambda B$, this shows that $C(X, A; F)_B$ and consequently B is not uniformly polar.

14. Corollary. — *For a Fréchet-Schwartz space $\lambda^0(A)$ the following are equivalent:*

(1) *In $\lambda^0(A)$ there exists a bounded set which is not uniformly polar;*

(2) $\lambda^0(A)$ *has* $(\tilde{\Omega})$.

This follows from Proposition 13 by taking X as the discrete topological space $\mathbb{N}$ and by noting that (i)-(iii) are satisfied since $\lambda^0(A)$ is Schwartz.

For other examples and further information on polar sets in l. c. spaces we refer to our article [7].

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