BULLETIN DE LA S. M. F.

L. MAKAR-LIMANOV

On automorphisms of Weyl algebra

Bulletin de la S. M. F., tome 112 (1984), p. 359-363

<http://www.numdam.org/item?id=BSMF_1984__112__359_0>

Bull. Soc. math. France,
112, 1984, p. 359-363

ON AUTOMORPHISMS OF WEYL ALGEBRA

BY

L. MAKAR-LIMANOV (*)

RÉSUMÉ. — Le but de cette note est de généraliser un théorème de Jacques Dixmier qui décrit le groupe des automorphismes de l'algèbre de Weyl dans le cas de caractéristique positive et de donner une nouvelle démonstration dans le cas de caractéristique zéro.

ABSTRACT. — In this note I am going to generalize a theorem of J. Dixmier [1] describing the group of automorphisms of the Weyl algebra to the case of non-zero characteristic and give a new proof of it in the case of zero characteristic.

Let A be the Weyl algebra with generators p and q ($[p, q]=pq-qp=1$) over a field K. Let the characteristic of K be χ.

LEMMA 1. — *If $\chi \neq 0$ then p^{χ} and q^{χ} generate (over K) the center C of A.*

Proof. — It is clear that p^{χ} and $q^{\chi} \in C$ ($[p^{\chi}, q]=\chi p^{\chi-1}=0$). Let $r=\sum_{i,j} k_{i,j} p^i q^j$ belong to C, where $k_{i,j} \in K$ (as is well known every element of A can be represented in such a form). Then

$$[p, r]=\sum j k_{i,j} p^i q^{j-1}=0$$

which means that $jk_{i,j}=0$, i.e. $j \equiv 0 \pmod{\chi}$ if $k_{i,j} \neq 0$. Analogously from $[r, q]=0$ it follows that $i \equiv 0 \pmod{\chi}$ if $k_{i,j} \neq 0$. So $r \in K[p^{\chi}, q^{\chi}]$.

The group H of automorphisms of the polynomial ring $K[x, y]$ is generated by linear automorphisms

$$x \to ax+by, \qquad y \to cx+dy \left(\begin{vmatrix} a & c \\ b & d \end{vmatrix} \neq 0 \right)$$

(*) The author is supported by NSF grant MCS-8201115.

Texte reçu le 8 juillet 1983.

L. MAKAR-LIMANOV, Wayne State University, Department of Mathematics, Detroit, Mich. 48202, U.S.A.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE — 0037-9434/1984/03 359 5/$ 2.50

and triangular automorphisms $x \to x$, $y \to y+f(x)$ where $f(x) \in k[x]$ (this is proven by H. W. E. Jung in case char $K=0$ [2] and by W. van der Kulk for arbitrary char K [3]). So the images $X=\tau(x)$ and $Y=\tau(y)$ of x and $y(\tau \in H)$ have the following property.

Let $X=\sum_{i=0}^{m} X_i$ and $Y=\sum_{j=0}^{n} Y_j$ where X_i and Y_j are homogeneous polynomials of x and y of degree i (j correspondingly) with respect to the degree given by deg $x=1$, deg $y=1$ $(X_m \neq 0,\ Y_n \neq 0)$. Let us denote X_m by $\bar{X}$ and Y_n by $\bar{Y}$.

LEMMA 2. — *Either* $\bar{X}=c(\bar{Y})^r$, *or* $\bar{Y}=d(\bar{X})^r$ *where* $c, d \in K$ *and* r *is a natural number*, *or* deg $\bar{X}=$deg $\bar{Y}=1$.

Proof. — This is a rather well known statement and more or less every description of H is based on something like this (*see:* [2], [3], [4]). I'll outline how it can be deduced from the above description of H. Let us represent $\tau \in H$ as a shortest possible product of the linear and triangular automorphisms: $\tau=\Delta_t \ldots L_1 \Delta_1 L_0$. It is easy to see that if L is an upper triangular matrix then $L\Delta=\tilde{\Delta}L$. So we can assume that each L_i with $i \neq 0$ has one of the forms $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Also we can assume that deg $\Delta_i(y)>1$ if $i \neq t$ because $L\Delta L'=\hat{\Delta}\hat{L}$ if deg $\Delta(y)=1$. Now if $t=1$, then deg $\bar{X}=$deg $\bar{Y}=1$ if deg $\Delta_1(y)=1$, and $\bar{Y}=d(\bar{X})^r$ if deg $\Delta_1(y)=r>1$. If $t>1$ then deg $\Delta_i(y)>1$ for $i<t$ and it is easy to show by induction (using of course the special form of the linear transformations L_i) that $\bar{Y}'=d\bar{X}'^r$ where $X'=\Delta_{t-1} \ldots L_0(x)$, $Y'=\Delta_{t-1} \ldots L_0(y)$, and $r=\prod_{i=1}^{t-1}$ deg $\Delta_i(y)$. If deg $\Delta_t(y)>1$ then it is sufficient to make one more induction step. If, however, deg $\Delta_t(y)=1$ then $\Delta_t L_{t-1}=\Delta' L'$ where $\Delta'(y)=y+e\,(e \in K)$, and depending on L' we can have either $\bar{Y}=d'(\bar{X})^r$ or $\bar{X}=d'(\bar{Y})^r$ or $\bar{Y}=d'\bar{X}$.

Let us call the automorphism τ of $K[x, y]$ normalized if the determinant of the Jacoby matrix of $\tau(x)$, $\tau(y)$ is equal to 1. Let us denote the group of all such automorphisms by $\mathrm{Aut}_n(K[x, y])$. One more denotation. Any element $R=\sum r_{ij} p^i q^j \in A$ also can be rewritten as $\sum_{i=0}^{h} R_i$ where the total degree of R_j is i. Let us denote R_h by $\bar{R}$. It is easy to see from the definition of multiplication in A that $\bar{R}$ is well defined and that $R \to \bar{R}$ gives a multiplicative homomorphism of A into the ring of commuting polynomials.

THEOREM 1. — *If* $\chi \neq 0$ *then* $\mathrm{Aut}(A) \cong \mathrm{Aut}_n(K[x, y]))$.

Proof. — Let σ be an automorphism of A and $P=\sigma(p)$, $Q=\sigma(q)$. Let us denote p^{α} by x^{α}, q^{α} by y^{α}, P^{α} by X^{α}, and Q^{α} by Y^{α} where x, y, X, and Y are commuting. Then (Lemma 1) $C=K[x^{\alpha}, y^{\alpha}]$. On the other hand X^{α} and Y^{α} also generate the center C of A. Thus the mapping $x^{\alpha}\to X^{\alpha}$, $y^{\alpha}\to Y^{\alpha}$ gives an automorphism of $K[x^{\alpha}, y^{\alpha}]$. So we can apply Lemma 2 to $(\overline{X^{\alpha}})=(\bar{P})^{\alpha}$ and $(\overline{Y^{\alpha}})=(\bar{Q})^{\alpha}$ and obtain that if $\deg P<\deg Q$ then $\overline{Y^{\alpha}}=c(\overline{X^{\alpha}})^{r}$. Therefore $\bar{Y}=c'(\bar{X})^{r}$ (where $c'\in K$ because c' is the element of $K(x, y)$ such that $(c')^{\alpha}=c$). Thus the degree of $Q_1=Q-c'P^{r}$ is smaller than $\deg Q$. But $p\to P_1=P$, $q\to Q_1$ also generates an automorphism. We can repeat such a transformation of P and Q only a finite number of times because $\deg P+\deg Q$ drops with every step. If after some step, e. g. in the very beginning, $\deg Q_i<\deg P_i$, then we'll consider the automorphism $p\to Q_i$, $q\to -P_i$. So after a finite number of triangular automorphisms and linear automorphisms $p\to q$, $q\to -p$ we reduce P and Q to elements P' and Q' of degree 1. Then $P'=a_1 p+a_2 q+a_3$, $Q'=b_1 p+b_2 q+b_3$ where $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=1$. So both groups $\mathrm{Aut}(A)$ and $\mathrm{Aut}_n(K[x, y])$ are spanned by triangular automorphisms and linear automorphisms with determinant equal to 1. Let us consider now the free algebra $K\langle a, b\rangle$. Triangular automorphisms and "normalized" linear automorphisms span a subgroup $\mathrm{Aut}_n(K\langle a, b\rangle)$ of the group of automorphisms of $K\langle a, b\rangle$. It is clear that

$$\mathrm{Aut}_n(K\langle a, b\rangle)\cong \mathrm{Aut}_n(K[x, y]),$$

and that $\mathrm{Aut}_n(K\langle a, b\rangle)\cong \mathrm{Aut}(A)$ because there exist natural surjective homomorphisms from the left sides to the right sides in both cases. For checking that the kernels are trivial, it is sufficient to investigate the homogeneous forms of the images of biggest degrees, which can be done along the same lines as in lemma 2. So $\mathrm{Aut}_n(k[x, y])\cong \mathrm{Aut}(A)$.

Now the Dixmier theorem.

THEOREM 2. — *If* char $K=0$ *then* $\mathrm{Aut}(A)\cong \mathrm{Aut}_n(K[x, y])$.

Proof. — Let $\mathbf{R}$ be the field of rational numbers and Z be the ring of integers; the field K contains $\mathbf{R}$. Let us fix $\tau\in \mathrm{Aut}(A)$. Let us consider the subfield k of K generated by all coefficients of $P=\tau(p)$ and $Q=\tau(q)$ and $\tau^{-1}(p)$, $\tau^{-1}(q)$. (Here one can have in mind a "polynomiallike" representation of P and Q as in lemma 1 though it is not essential because the commutator relation involves only rational numbers.) Let $u_1, \ldots, u_r$ be a transcendence basis of k over $\mathbf{R}$. Then k is algebraic over the field

$E=\mathbf{R}(u_1, \ldots, u_r)$ and there exists an element $\theta \in k$ which is integral over the ring $Z[u_1, \ldots, u_r]$ such that $k=E[\theta]$. Now for any prime $\chi \in \mathbf{R}$ we can define a partial homomorphism π_χ from k onto some field k' with char $k'=\chi$: π_χ is defined on the subring of $\mathbf{R}$ consisting of all elements $\not\equiv \infty \pmod{\chi}$ and maps this subring naturally onto Z_χ (field of residues modulo χ). Then π_χ can be extended to the corresponding subring of k by the formulas $\pi_\chi(u_i)=u_i'$ where u_i' are transcendental over Z_χ and $\pi_\chi(\theta)=\theta'$, where θ' is a root of the polynomial obtained from $\operatorname{irr}_E(\theta)$ by applying π_χ to its coefficients.

Let us consider the Weyl algebra A' over k'. The mapping π_χ can be extended to some subring of A (which depends on χ) by $\pi_\chi(p)=p'$, $\pi_\chi(q)=q'$; and $\pi_\chi(A)=A'$. Now if π_χ is defined on $\tau(p)$ and $\tau(q)$ and $\tau^{-1}(p)$, $\tau^{-1}(q)$ then $P'=\pi_\chi\tau(p)$ and $Q'=\pi_\chi\tau(q)$ are the images of p' and q' under some automorphism of A'. So from Theorem 1 $\pi_\chi(\bar{Q}\bar{P}^{-r}) \in \pi_\chi(k)$ when defined. The image of a nonconstant rational function under π_χ may be a constant only for finite number of χ. So $R=\bar{Q}\bar{P}^{-r}$ itself must belong to k because $\pi_\chi(R)$ is defined and constant for all but finite number of χ. Therefore the analog of Lemma 2 is proved and the proof of Theorem 1 can be repeated word by word in this case.

Remarks. — The "place" technique which was used here may be used for proving that the skew field D of fractions of A has a nonabelian free subgroup in every noncentral normal subgroup of D^*. This may be done by reducing the problem to the finite dimensional situation and then applying the finite dimensional result (*see* [5]; the statement on free subgroups of D^* is proved among other things in [6]). It seems rather natural that reduction of the infinite dimensional situation to the finite dimensional can be very helpful. So any answer to the following question would be of considerable importance. Is it true that for any infinite dimensional (over its center) skew field B there exists a sufficient number of partial homomorphisms to finite dimensional (over their centers which need not be the images of the center of B) skew fields?

REFERENCES

[1] J. DIXMIER, Sur les algèbres de Weyl, Bull. Soc. Math. France, 96 (1968), 209-242.

[2] H. W. E. JUNG, Uber ganze birationale Transformationen der Eben, J. Reine angew. Math., 184 (1942), 161-174.

[3] W. VAN DER KULK, On polynomial rings in two variables, Nieuw Arch. Wisk., 1 (1953), 33-41.

[4] S. ABHYANKAR, T. MOH, Embedding of the line in the plane, J. Reine Angew Math., 276 (1975), 148-166.

[5] L. MAKAR-LIMANOV, On free subgroups of skew fields, preprint.

[6] A. LICHTMAN, On matrix rings and linear groups over a field of fractions of enveloping algebras and group rings, preprint.