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ANALYTIC ISOMORPHISMS OF INFINITE DIMENSIONAL POLYDISCS AND AN APPLICATION

BY

REINHOLD MEISE and DIETMAR VOGT (*)

ABSTRACT. — Let $\Lambda(P)$ be a Köthe-Schwartz space. In its strong dual $\Lambda(P)'_b$ we consider open polydiscs $\mathbb{D}_a$ of finite radii and show that $\mathbb{D}_a$ and $\mathbb{D}_b$ are holomorphically equivalent iff there exists a linear automorphism φ of $\Lambda(P)'_b$ with $\varphi(\mathbb{D}_a)=\mathbb{D}_b$. For power series spaces resp. their duals a complete classification can be given and a parameter representation of the group $\mathrm{Aut}(\mathbb{D}_a)$ of all holomorphic automorphisms of $\mathbb{D}_a$ is obtained. As an application one gets a characterization of the isomorphism classes of the l. m. c. algebras $(H(\mathbb{D}_a), \tau_0)$ of all holomorphic functions on $\mathbb{D}_a$.

RÉSUMÉ. — Soit $\Lambda(P)$ un espace de Köthe-Schwartz. Dans son dual fort $\Lambda(P)'_b$ on considère des polydisques $\mathbb{D}_a$ ouverts avec des rayons finis et on montre que $\mathbb{D}_a$ et $\mathbb{D}_b$ sont holomorphiquement équivalents si et seulement s'il existe un automorphisme linéaire φ de $\Lambda(P)'_b$ satisfaisant $\varphi(\mathbb{D}_a)=\mathbb{D}_b$. Pour les espaces $\Lambda_R^\infty(\alpha)$ respectivement $\Lambda_R(\alpha)'_b$ on donne une classification complète et on obtient une paramétrisation du groupe $\mathrm{Aut}(\mathbb{D}_a)$ de tous les automorphismes holomorphes de $\mathbb{D}_a$. Comme application on donne une caractérisation des classes d'isomorphismes des algèbres l. m. c. $(H(\mathbb{D}_a), \tau_0)$ de toutes les fonctions holomorphes sur $\mathbb{D}_a$.

Preface

Let $\Lambda(P)$ be a reflexive Köthe-Schwartz space. Then its strong dual $\Lambda(P)'_b$ is a sequence space again and for $a>0$ varying in:

$$\Lambda^\infty(P) := \{ x \in \mathbb{C}^\mathbb{N} \mid \sup_{n\in\mathbb{N}} |x_n| p_n < \infty \text{ for all } p \in P \}$$

the set:

$$\mathbb{D}_a := \{ x \in \Lambda(P)'_b \mid \sup_{n\in\mathbb{N}} |x_n| a_n < 1 \}$$

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is an open subset of $\Lambda(P)'_b$, called an (infinite dimensional) open polydisc of finite radii. We investigate necessary and sufficient conditions for such open polydiscs $\mathbb{D}_a$ and $\mathbb{D}_b$ in $\Lambda(P)'_b$ to be holomorphically equivalent. The treatment of this question is motivated by a result of our article [7], showing that for every nuclear power series space $\Lambda_1(\alpha)$ there exist open polydiscs of finite radii $\mathbb{D}_a$ and $\mathbb{D}_b$ in $\Lambda_1(\alpha)'_b$ which are not holomorphically equivalent.

The main results of the present article are the following: First we show that two open polydiscs of finite radii $\mathbb{D}_a$ and $\mathbb{D}_b$ in $\Lambda(P)'_b$ are holomorphically equivalent, if and only if, there exists a linear topological automorphism φ of $\Lambda(P)'_b$ with $\varphi(\mathbb{D}_a)=\mathbb{D}_b$. Moreover, we determine the structure of all possible holomorphic equivalences. Then we derive a necessary condition for such automorphisms φ, which turns out to be also sufficient in the case of power series spaces, resp. their duals. From this we get a classification of the holomorphically equivalent open polydiscs of finite radii in $\Lambda_R(\alpha)'_b$ resp. $\Lambda_R^\infty(\alpha)$ for $R=1, \infty$ in the following way: $\mathbb{D}_a$ and $\mathbb{D}_b$ are holomorphically equivalent, if and only if, there exists a bijection π of $\mathbb{N}$ with:

$$0<\inf_{j\in\mathbb{N}} \frac{\alpha_{\pi(j)}}{\alpha_j} \leqslant \sup_{j\in\mathbb{N}} \frac{\alpha_{\pi(j)}}{\alpha_j} < \infty$$

such that $(a_j/b_{\pi(j)})_{j\in\mathbb{N}}$ and $(b_{\pi(j)}/a_j)_{j\in\mathbb{N}}$ belong to the diametral dimension $\Delta(\Lambda_R(\alpha))$ of $\Lambda_R(\alpha)$. This result has two further consequences. The first one is the existence of a continuum of open polydiscs in $\Lambda_R(\alpha)'_b$ resp. $\Lambda_R^\infty(\alpha)$ which are pairwise not holomorphically equivalent. The second one is that under the hypotheses given above, the locally convex algebras $(H(\mathbb{D}_a), \tau_0)$ and $(H(\mathbb{D}_b), \tau_0)$ of all holomorphic functions on $\mathbb{D}_a$ resp. $\mathbb{D}_b$ under the compact-open topology τ_0 are isomorphic, if and only if, the conditions in the classification result are satisfied. Hence there exists also a continuum of non-isomorphic nuclear Fréchet algebras $(H(\mathbb{D}_a), \tau_0)$.

Knowing the structure of holomorphic equivalences between open polydiscs of finite radii, one can use the arguments from the proof of the classification result in order to give a parameter representation of the group $\mathrm{Aut}(\mathbb{D}_a)$ of all holomorphic automorphisms of $\mathbb{D}_a$.

The proofs of our results are based on general arguments from functional analysis, on specific properties of sequence spaces and power series spaces and on ideas used by H. Cartan to determine the group of holomorphic automorphisms of bounded domains in $\mathbb{C}^n$.

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1. Preliminaries

The following conventions and definitions will be used throughout the whole article.

A l. c. space E means a locally convex Hausdorff complex vector space E; the topological dual of E is denoted by E'. For the usual l. c. topologies on E resp. E' we use the notation of HORVÁTH [5], resp. PIETSCH [11].

(*i*) *Sequence spaces*

Let P be a family of non-negative sequences $(p_n)_{n\in\mathbb{N}}$ with the following properties:

(1) For all $n\in\mathbb{N}$ there exists $p\in P$ with $p_n>0$.

(2) For all $p, q\in P$ there exist $c>0$ and $r\in P$ such that $p+q\leq cr$.

Then we define the *Köthe sequences spaces:*

$$\Lambda(P):=\{x\in\mathbb{C}^{\mathbb{N}}\mid \pi_p(x):=\textstyle\sum_{n=1}^{\infty}|x_n|p_n<\infty \text{ for all } p\in P\}$$

resp.:

$$\Lambda^{\infty}(P):=\{x\in\mathbb{C}^{\mathbb{N}}\mid \pi_p^{\infty}(x):=\sup_{n\in\mathbb{N}}|x_n|p_n<\infty \text{ for all } p\in P\},$$

which are given the natural l. c. topology induced by the semi-norm systems $(\pi_p)_{p\in P}$ resp. $(\pi_p^{\infty})_{p\in P}$. We use the symbol $\Lambda^*(P)$ for $\Lambda(P)$ and $\Lambda^{\infty}(P)$ if we don't want to distinguish between them. By $(e_j)_{j\in\mathbb{N}}$ we denote the canonical basis of $\Lambda(P)$, where $e_j=(\delta_{jn})_{n\in\mathbb{N}}$.

Throughout the whole article we shall assume that $\Lambda^*(P)$ is a reflexive Schwartz space. We recall that the *Schwartz property* of $\Lambda^*(P)$ is characterized by the following property (S) of P

(S). For all $p\in P$ there exist $q\in P$ and a null-sequence c with $p\leqq cq$.

The dual space of $\Lambda(P)$ is again a sequence space, namely:

$$\Lambda(P)'=\{x\in\mathbb{C}^{\mathbb{N}}\mid \text{there exist } p\in P \text{ and } d>0 \text{ such that } |x_n|<dp_n \text{ for all } n\in\mathbb{N}\}.$$

For a reflexive Schwartz space $\Lambda(P)$, the strong dual $\Lambda(P)'_b$ is an ultrabornological Montel space. Hence a subset B of $\Lambda(P)'_b$ is relatively compact iff B is equicontinuous, and a fundamental system for the compact sets in $\Lambda(P)'_b$ is given by $\{N_b \mid b \in \Lambda(P)'\}$, where:

$$N_b := \{x \in \mathbb{C}^{\mathbb{N}} \mid |x_n| \leqslant |b_n| \text{ for all } n \in \mathbb{N}\}$$

denotes the *normal hull* of b. From this and a standard compactness argument it follows that the topology of $\Lambda(P)'_b$ coincides on the compact subsets of $\Lambda(P)'_b$ with the topology induced by $\mathbb{C}^{\mathbb{N}}$. Hence the dual basis $(e'_j)_{j \in \mathbb{N}}$ of the canonical basis is a basis of $\Lambda(P)'_b$.

(*ii*) *Open polydiscs in* $\Lambda(P)'_b$

For all $a \in \Lambda^\infty(P)$ and all $x \in \Lambda(P)'$ we have $\lim_{n \to \infty} a_n x_n = 0$. This follows easily from the description of $\Lambda(P)'$ given above and our general assumption that $\Lambda(P)$ is a reflexive Schwartz space. Hence $\pi_a : \Lambda(P)' \to \mathbb{R}_+$, $\pi_a(x) := \sup_{n \in \mathbb{N}} |x_n a_n|$, defines a semi-norm on $\Lambda(P)'$ which is bounded on the bounded subsets of $\Lambda(P)'_b$. Since $\Lambda(P)'_b$ is bornological, π_a is continuous on $\Lambda(P)'_b$. Assuming $a \geqslant 0$ for convencience, we call the set:

$$\mathbb{D}_a := \{x \in \Lambda(P)' \mid \pi_a(x) = \sup_{n \in \mathbb{N}} |x_n| a_n < 1\}$$

an *open polydisc* in $\Lambda(P)'_b$. From the considerations in part (i) it is obvious that $\{N_b \mid b \in \mathbb{D}_a\}$ is a fundamental system for the compact subsets of $\mathbb{D}_a$. The open polydisc $\mathbb{D}_a$ in $\Lambda(P)'_b$ is said to have finite radii if $a > 0$, i. e. $a_n > 0$ for all $n \in \mathbb{N}$. Since we are interested only in open polydiscs with finite radii we consider only such weight families P for which $\Lambda^\infty(P)$ contains elements $a > 0$ and we consider only open polydiscs $\mathbb{D}_a$ in $\Lambda(P)'_b$ of finite radii.

(*iii*) *Power series spaces*

Let α be an increasing unbounded sequence of positive real numbers (called exponent sequence) and let $0 < R \leqslant \infty$. Then the *power series spaces* $\Lambda^*(\alpha)$ are defined as:

$$\Lambda^*_R(\alpha) := \Lambda^*(P(R, \alpha)),$$

where:

$$P(R, \alpha) := \{(r^{\alpha_n})_{n \in \mathbb{N}} \mid 0 < r < R\}.$$

$\Lambda^*_R(\alpha)$ is called power series space of finite type, if $0 < R < \infty$ and of infinite type if $R = \infty$. For $0 < R < \infty$ all the spaces $\Lambda^*_R(\alpha)$ are isomorphic, hence it suffices to look at $\Lambda^*_1(\alpha)$, which is known to be not isomorphic to a power series space of infinite type.

Since the exponent sequences α are assumed to be unbounded, it follows that $\Lambda_R^*(\alpha)$ as well as $\Lambda_R^*(\alpha)'_b$ are reflexive Schwartz spaces. We remark that:

$$\Lambda_R^\infty(\alpha)=\Lambda(Q(R,\alpha))'_b,$$

where:

$$Q(R,\alpha)=\{q\in\Lambda_R^\infty(\alpha)\mid q\geqslant 0\}.$$

Let α be an exponent sequence and let π be a bijection of $\mathbb{N}$. It is easy to see that there exists a (linear topological) automorphism A_π of $\Lambda_R^*(\alpha)$ $(R=1,\infty)$ satisfying $A_\pi(e_j)=e_{\pi(j)}$ for all $j\in\mathbb{N}$, if and only if, π satisfies:

$$0<\inf_{j\in\mathbb{N}}\frac{\alpha_{\pi(j)}}{\alpha_j}\leqslant\sup_{j\in\mathbb{N}}\frac{\alpha_{\pi(j)}}{\alpha_j}<\infty.$$

By $\Pi(\alpha)$ we denote the set of all such permutations. Obviously, $\Pi(\alpha)$ is a group under composition. We remark that for every bijection π of $\mathbb{N}$ there exist exponent sequences β such that $\pi\in\Pi(\beta)$.

We recall that for $k\in\mathbb{N}$ the spaces $H(\mathbb{D}^k)$ resp. $H(\mathbb{C}^k)$ of all holomorphic functions on the open polydisc $\mathbb{D}^k$ resp. on $\mathbb{C}^k$ as well as the space s of rapidly decreasing sequences are classical examples of power series spaces, since $H(\mathbb{D}^k)\simeq\Lambda_1(\alpha^{(k)})$, $H(\mathbb{C}^k)\simeq\Lambda_\infty(\alpha^{(k)})$, where $\alpha^{(k)}=(\sqrt[k]{n})_{n\in\mathbb{N}}$ and $s\simeq\Lambda_\infty((\ln(n+1))_{n\in\mathbb{N}})$.

(*iv*) *Analytic mappings*

Let E and F be l. c. spaces and let Ω be an open subset of E. $f:\Omega\to F$ is called Gâteaux-analytic, if for any $y'\in F'$, any $a\in\Omega$ and any $b\in E$ the function $z\mapsto y'\circ f(a+zb)$ is a holomorphic functions in one variable on its natural domain of definition. f is called *holomorphic*, if it is Gâteaux-analytic and continuous and it is called *hypoanalytic*, if it is Gâteaux-analytic and continuous on the compact subsets of Ω. By $H(\Omega,F)$ resp. $H_{hy}(\Omega,F)$ we denote the vector space of all holomorphic resp. hypoanalytic functions on Ω with values in F, which will be endowed with the compact-open topology τ_0. We shall write $H(\Omega)$ resp. $H_{hy}(\Omega)$ instead of $H(\Omega,\mathbb{C})$ resp. $H_{hy}(\Omega,\mathbb{C})$. Obviously $(H_{hy}(\Omega),\tau_0)$ is a locally multiplicatively-convex (l. m. c.) topological algebra and $(H(\Omega),\tau_0)$ is a subalgebra of $(H_{hy}(\Omega),\tau_0)$, if the multiplication in $H_{hy}(\Omega)$ is defined by pointwise multiplication.

If U is open in E and V is open in F, U and V are said to be hypoanalytically equivalent, if there is a hypoanalytic map $f:U\to F$ which maps U bijectively onto V and for which $f^{-1}:V\to E$ is also hypoanalytic. The map f is called a hypoanalytic isomorphism (or equivalence) between U and V. For further details on analytic mappings and functions we refer to the books of DINEEN [4] and NOVERRAZ [10].

2. LEMMA. — *Let E, F and G be quasi-complete l. c. spaces, let $U \subset E$ and $V \subset F$ be open. Assume that $f \in H_{hy}(V, G)$, that $g \in H_{hy}(U, F)$ and that $g(U) \subset V$. Then $f \circ g \in H_{hy}(U, G)$ and the chain-rule holds, i. e.:*

$$(f \circ g)'(x) = f'(g(x))[g'(x)] \quad \text{for all} \quad x \in U;$$

where $f'(y) : F \to G$ and $g'(x) : E \to F$ are linear and continuous on compact sets.

The proof of this Lemma can be given by standard arguments if one remarks that the Taylor expansion of a Gâteaux-analytic function f (*see* e. g. NOVERRAZ [10], Thm. 1.2.4) converges locally uniformly on compact sets if f is hypoanalytic.

3. LEMMA. — *For any $w \in \mathbb{D}_a$ the mapping $\tau : \mathbb{D}_a \to \Lambda(P)'_b$ defined by:*

$$\tau(z) = \left(\frac{z_j - w_j}{1 - a_j^2 \bar{w}_j z_j} \right)_{j \in \mathbb{N}},$$

is a hypoanalytic automorphism of $\mathbb{D}_a$ and its inverse is the mapping $\sigma : \mathbb{D}_a \to \Lambda(P)'_b$, defined by:

$$\sigma(\zeta) = \left(\frac{\zeta_j + w_j}{1 + a_j^2 \bar{w}_j \zeta_j} \right)_{j \in \mathbb{N}}.$$

Proof. — In order to show that τ is continuous on the compact subsets of $\mathbb{D}_a$, we first prove that for any compact set K in $\mathbb{D}_a$ the set $\tau(K)$ is contained in a compact set of $\mathbb{D}_a$. This implies the desired continuity property of τ since the topology of $\Lambda(P)'_b$ coincides on its compact subsets with the topology of coordinatewise convergence.

If $K \subset \mathbb{D}_a$ is compact, then there exists $c \in \mathbb{D}_a$, $c \geqslant 0$, such that $K \subset N_c$. Since $\lim_{j \to \infty} |x_j| a_j = 0$ for all $x \in \Lambda(P)'$, there exists $J \in \mathbb{N}$ such that for all $j \geqslant J$ we have for all $z \in \mathbb{C}$ with $|z| \leqslant c_j$:

$$\left| \frac{z - w_j}{1 - a_j^2 \bar{w}_j z} \right| \leqslant \frac{1}{a_j} \frac{a_j c_j + a_j |w_j|}{1 - a_j c_j a_j |w_j|} \leqslant \frac{1}{2 a_j}.$$

Since the mapping $z \mapsto (z-w_j)/(1-a_j^2 \bar{w}_j z)$ maps the disc of radius $1/a_j$ around zero onto itself, it follows that there exists $d \in \mathbb{D}_a$ such that $N_d \supset \tau(N_c) \supset \tau(K)$. Hence $\tau(K)$ is contained in a compact subset of $\mathbb{D}_a$.

Since the Gâteaux-analyticity of τ is an easy consequence of the fact that all the component functions of τ are holomorphic, we have shown that τ is hypoanalytic and maps $\mathbb{D}_a$ into $\mathbb{D}_a$. Since σ is of the same form as τ, it has the same properties as τ. Hence the proof is completed by showing $\sigma = \tau^{-1}$, which is done by an easy calculation.

4. Lemma. – *Let $f: \mathbb{D}_a \to \Lambda(P)'_b$ be a hypoanalytic mapping of $\mathbb{D}_a$ into itself satisfying $f(0)=0$ and $f'(0)=id_{\Lambda(P)'}$. Then $f=id_{\Lambda(P)'}$.*

Proof. – For $n \in \mathbb{N}$ we define $i_n : \mathbb{C}^n \to \Lambda(P)'_b$ and $\pi_n : \Lambda(P)'_b \to \mathbb{C}^n$ by $i_n(z)=(z_1, \ldots, z_n, 0, \ldots)$ and $\pi_n(x)=(x_1, \ldots, x_n)$. Then $f_n = \pi_n \circ f \circ i_n$ is a holomorphic map of a bounded open polydisc $D_n \subset \mathbb{C}^n$ into itself, satisfying $f_n(0)=0$ and $f'_n(0)=\mathrm{id}_{\mathbb{C}^n}$. Hence $f_n=\mathrm{id}$ by a classical result of H. Cartan [1] (*see* also Narasimhan [9], p. 66). Since this holds for all $n \in \mathbb{N}$ and since $\bigcup_{n \in \mathbb{N}} (\mathbb{D}_a \cap \mathrm{Im}\, i_n)$ is dense in any compact subset of $\mathbb{D}_a$, we have $f=\mathrm{id}_{\Lambda(P)'}$.

Lemma 4 is a version of the classical Lemma of Schwarz, which we will use now together with an idea of proof of Cartan [1], (*see* Narasimhan [9], p. 67 f) to determine the structure of the bihypoanalytic isomorphisms between open polydiscs.

5. Theorem. – *Let $\Lambda(P)$ be a reflexive Schwartz space and let $a, b \in \Lambda^\infty(P)$ satisfy $a>0$ and $b>0$. A mapping $f: \mathbb{D}_a \to \Lambda(P)'_b$ is a hypoanalytic isomorphism between $\mathbb{D}_a$ and $\mathbb{D}_b$, if and only if, $f=\sigma \circ (\varphi \mid \mathbb{D}_a)$, where φ is a continuous linear automorphism of $\Lambda(P)'_b$ with $\varphi(\mathbb{D}_a)=\mathbb{D}_b$ and where $\sigma : \mathbb{D}_b \to \Lambda(P)'_b$ is defined by:*

$$\sigma(z)=\left(\frac{z_j+w_j}{1+b_j^2 \bar{w}_j z_j}\right)_{j \in \mathbb{N}} \qquad \text{for} \quad w=f(0) \in \mathbb{D}_b.$$

Proof. – From Lemma 2 and 3 it is clear, that f is a hypoanalytic isomorphism, if it is of the form $\sigma \circ (\varphi \mid \mathbb{D}_a)$.

To prove the converse implication, let f be any given hypoanalytic isomorphism between $\mathbb{D}_a$ and $\mathbb{D}_b$. For $t \in \mathbb{R}$ we then denote the mapping $e^{it}\,\mathrm{id}_{\Lambda(P)'}$ by M_t and put $g_t := M_{-t} \circ \psi^{-1} \circ M_t \circ \psi$, where $\psi = \tau \circ f$ and where τ is the mapping defined in Lemma 3. By Lemma 2 and 3 g_t is a hypoanalytic automorphism of $\mathbb{D}_a$ satisfying $g_t(0)=0$, and the chain rule gives that $g_t'(0)=\mathrm{id}_{\Lambda(P)'}$. Hence Lemma 4 implies $g_t=\mathrm{id}_{\Lambda(P)'}$ and consequently $M_t \circ \psi = \psi \circ M_t$ for all $t \in \mathbb{R}$. Now fix $x \in \mathbb{D}_a$ and differentiate the identity $M_t(\psi(x))=\psi(M_t(x))$ with respect to t. Then the definition of M_t and the chain rule give:

$$ie^{it}\psi(x)=\psi'(e^{it}x)[ie^{it}x] \qquad \text{for all} \quad t \in \mathbb{R}.$$

Hence we have $\psi(x)=\psi'(e^{it}x)[x]$. Since the function $z \mapsto \psi'(zx)[x]$ is holomorphic in a neighbourhood of the closed unit disc, this implies $\psi(x)=\psi'(0)[x]$. By Lemma 2 $\varphi := \psi'(0)$: $\Lambda(P)_b' \to \Lambda(P)_b'$ is linear and continuous on compact sets. Consequently, φ is sequentially continuous and hence continuous since $\Lambda(P)_b'$ is bornological. Since the previous arguments apply also to ψ^{-1}, we get that φ is a linear automorphism of $\Lambda(P)_b'$ satisfying $\varphi(\mathbb{D}_a)=\mathbb{D}_b$. Since we have shown in Lemma 3 that $\tau^{-1}=\sigma$, we finally get:

$$f=\sigma\circ(\tau\circ f)=\sigma\circ\psi=\sigma\circ(\varphi \mid \mathbb{D}_a).$$

Theorem 5 describes the structure of the hypoanalytic isomorphisms between $\mathbb{D}_a$ and $\mathbb{D}_b$. In order to get more insight from it one has to know all linear automorphisms of $\Lambda(P)_b'$ satisfying $\varphi(\mathbb{D}_a)=\mathbb{D}_b$. The special form of these automorphisms is described by the following Lemma.

6. LEMMA. — *If φ is a linear topological isomorphism of $\Lambda(P)_b'$, satisfying $\varphi(\mathbb{D}_a)=\mathbb{D}_b$, then there exist a bijection $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence λ of complex numbers of modulus 1 such that for any $x=(x_j)_{j\in\mathbb{N}} \in \Lambda(P)_b'$:*

$$\varphi(x)=\sum_{j=1}^{\infty} \lambda_j \frac{a_j}{b_{\pi(j)}} x_j e_{\pi(j)}',$$

where e_k' denotes the k-th canonical basis vector of $\Lambda(P)_b'$.

Proof. — For $c \in \Lambda^\infty(P)$, $c>0$, it is easy to show that the polar $\mathbb{D}_c^0$ of $\mathbb{D}_c$ is the set:

$$\mathbb{D}_c^0=\left\{ x \in \Lambda(P) \,\Big|\, \sum_{j=1}^{\infty} |x_j| \frac{1}{c_j} \leqslant 1 \right\}$$

and that the set of extremal points of $\mathbb{D}_c^0$ is:

$$\operatorname{Ext} \mathbb{D}_c^0=\{\lambda c_j e_j \mid \lambda \in \mathbb{C}, |\lambda|=1, j \in \mathbb{N}\},$$

where e_j denotes the j-th canonical basis vector of $\Lambda(P)$.

Since φ is a linear topological isomorphism of $\Lambda(P)'_b$ with $\varphi(\mathbb{D}_a)=\mathbb{D}_b$, its adjoint ψ: $\Lambda(P)\to\Lambda(P)$ is an isomorphism satisfying $\psi(\mathbb{D}_b^0)=\mathbb{D}_a^0$. Hence we have $\psi^{-1}(\mathbb{D}_a^0)=\mathbb{D}_b^0$ and consequently $\psi^{-1}(\operatorname{Ext}\mathbb{D}_a^0)=\operatorname{Ext}\mathbb{D}_b^0$. Because of the preceding considerations this implies the existence of a sequence λ of complex numbers of modulus 1 and of a mapping π: $\mathbb{N}\to\mathbb{N}$ such that $\psi^{-1}(a_j e_j)=\lambda_j^{-1} b_{\pi(j)} e_{\pi(j)}$ for all $j\in\mathbb{N}$. Since ψ^{-1} is an isomorphism, π is a bijection. From this we get:

$$\psi(e_k)=\lambda_{\pi^{-1}(k)}\frac{a_{\pi^{-1}(k)}}{b_k}e_{\pi^{-1}(k)}.$$

Hence we have for any $j\in\mathbb{N}$ and any $k\in\mathbb{N}$:

$$\langle\varphi(e'_j), e_k\rangle=\langle e'_j, \psi(e_k)\rangle=\delta_{j\pi^{-1}(k)}\cdot\lambda_{\pi^{-1}(k)}\cdot\frac{a_{\pi^{-1}(k)}}{b_k}.$$

This implies:

$$\varphi(e'_j)=\lambda_j\frac{a_j}{b_{\pi(j)}}e_{\pi(j)}\qquad\text{for any}\quad j\in\mathbb{N}$$

and consequently:

$$\varphi(x)=\varphi\left(\textstyle\sum_{j=1}^{\infty}x_j e'_j\right)=\textstyle\sum_{j=1}^{\infty}x_j\,\varphi(e'_j)=\sum_{j=1}^{\infty}\lambda_j\dfrac{a_j}{b_{\pi(j)}}e'_{\pi(j)}.$$

We want to use Lemma 6 and Theorem 5 to give a complete description of the holomorphically equivalent polydiscs $\mathbb{D}_a$ in $\Lambda_R(\alpha)'_b$ resp. $\Lambda_R^\infty(\alpha)$ for $R=1$ and $R=\infty$. In order to be able to do this we need the following two Lemmas which extend [7], Lemma 3.1.

We recall that for a l. c. space E we denote by $\Delta(E)$ its diametral dimension and that $\Delta(\Lambda_1^*(\alpha))=\Lambda_1^\infty(\alpha)$ and $\Delta(\Lambda_\infty^*(\alpha))=\Lambda_\infty(\alpha)'$.

7. Lemma. — *For $R=1$ or $R=\infty$ a diagonal map D: $\Lambda_R^*(\alpha)\to\Lambda_R^*(\alpha)$, $D(x):=(d_j x_j)_{j\in\mathbb{N}}$, is an automorphism if and only if d and $1/d$ belong to $\Delta(\Lambda_R^*(\alpha))$.*

Proof. — It is easy to check that for a Köthe space $\Lambda^*(P)$ a diagonal map D: $\Lambda^*(P)\to\Lambda^*(P)$ is continuous, if and only if the corresponding sequence d satisfies: For all $p\in\mathbb{N}$ there exists $q\in\mathbb{N}$ and $C>0$ such that $|d_j|p_j\leqslant Cq_j$ for all $j\in\mathbb{N}$. Hence the Lemma is an immediate consequence of this and the definition of the diametral dimension.

8. LEMMA. — *Let $R=1$ or $R=\infty$, let π be a bijection of $\mathbb{N}$ and let $(d_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. Then there exists a linear automorphism A of $\Lambda_R^*(\alpha)$ satisfying $A(e_j)=d_j e_{\pi(j)}$ for all $j\in\mathbb{N}$, if and only if,* (1) *and* (2) *hold true:*

(1) $$0<\inf_{j\in\mathbb{N}}\frac{\alpha_{\pi(j)}}{\alpha_j}\leqslant\sup_{j\in\mathbb{N}}\frac{\alpha_{\pi(j)}}{\alpha_j}<\infty,\qquad \text{i.e.}\quad \pi\in\Pi(\alpha),$$

(2) $$d \textit{ and } \frac{1}{d} \textit{ belong to } \Delta(\Lambda_R^*(\alpha)).$$

Proof. — Assume that (1) and (2) hold true. From 1 (iii) we know that (1) implies that A_π: $x\mapsto\sum_{j=1}^{\infty}x_j e_{\pi(j)}$ is a linear automorphism of $\Lambda_R^*(\alpha)$. From this and (2) we get that $\tilde{d}:=(d_{\pi^{-1}(j)})_{j\in\mathbb{N}}$ and $1/\tilde{d}$ belong to $\Delta(\Lambda_R^*(\alpha))$. Hence we get from Lemma 7 that D: $\Lambda_R^*(\alpha)\to\Lambda_R^*(\alpha)$, $D(x):=(\tilde{d}_j x_j)_{j\in\mathbb{N}}$ is a linear automorphism of $\Lambda_R^*(\alpha)$ and hence $A=D\circ A_\pi$ has this property.

On the other hand, if A is a linear automorphism of $\Lambda_R^*(\alpha)$ which satisfies (1), then A_π defined as above is an automorphism. Then $D:=A\circ(A_\pi)^{-1}$ is a diagonal map and an automorphism of $\Lambda_R^*(\alpha)$, hence (2) is satisfied because of Lemma 7. Hence it suffices to show that A satisfies (1).

In order to prove this we remark that the continuity of A and A^{-1} imply: For any $0<r<R$ there exist $C=C(r)>0$ and $0<\rho=\rho(r)<R$ such that:

(3) $$|d_j|\,r^{\alpha_{\pi(j)}}\leqslant C\rho^{\alpha_j},$$

(4) $$r^{\alpha_j}\leqslant C\,|d_j|\,\rho^{\alpha_{\pi(j)}},$$

are true for all $j\in\mathbb{N}$. Now fix r with $0<r<R$ and choose s with $\rho(r)<s<R$. Dividing (3) with s by (4) with r then gives:

(5) $$\left(\frac{s}{\rho(r)}\right)^{\alpha_{\pi(j)}}\leqslant C(r)\,C(s)\left(\frac{\rho(s)}{r}\right)^{\alpha_j}\quad\text{for all } j\in\mathbb{N}.$$

Since $s/\rho(r)>1$ we get from (5) by taking logarithms that $\sup_{j\in\mathbb{N}}\alpha_{\pi(j)}/\alpha_j<\infty$. The same argument applied to A^{-1} gives $\sup_{j\in\mathbb{N}}\alpha_{\pi^{-1}(j)}/\alpha_j<\infty$ and hence $\inf_{j\in\mathbb{N}}\alpha_{\pi(j)}/\alpha_j>0$, which in total implies (1).

From Theorem 5, Lemma 8 and the proof of Lemma 6 it is now clear how to prove the following Proposition.

9. Proposition. — *Let $R=1$ or $R=\infty$ and let $\mathbb{D}_a$ and $\mathbb{D}_b$ be open polydiscs in $\Lambda_R(\alpha)'_b$ resp. in $\Lambda_R^\infty(\alpha)$. Then $\mathbb{D}_a$ and $\mathbb{D}_b$ are holomorphically equivalent, if and only if, there exists $\pi\in\Pi(\alpha)$ such that:*

$$(\star)\qquad \left(\frac{a_j}{b_{\pi(j)}}\right)_{j\in\mathbb{N}} \text{ and } \left(\frac{b_{\pi(j)}}{a_j}\right)_{j\in\mathbb{N}} \text{ belong to } \Delta(\Lambda_R^*(\alpha)).$$

Remark. — In condition $(\star)$ of Proposition 9 one cannot get rid of the permutation. This is a consequence of the following example:

Let α be an exponent sequence satisfying $\sup_{n\in\mathbb{N}}\alpha_{n+1}/\alpha_n<\infty$. Then for a fixed r with $0<r<1$ the sequences $a:=(1, r^{\alpha_1}, 1, r^{\alpha_3}, 1, r^{\alpha_5}, \ldots)$ and $b:=(r^{\alpha_1}, 1, r^{\alpha_3}, 1, r^{\alpha_5}, 1, \ldots)$ belong to $\Lambda_1^\infty(\alpha)$ and the bijection π of $\mathbb{N}$, defined by $\pi(2n):=2n-1$ and $\pi(2n-1):=2n$ is in $\Pi(\alpha)$ and satisfies 9 $(\star)$. Hence $\mathbb{D}_a$ and $\mathbb{D}_b$ are holomorphically equivalent. However, a/b and b/a don't belong to $\Lambda_1^\infty(\alpha)=\Delta(\Lambda_1^\infty(\alpha))$.

10. Corollary. — *For $R=1$ and $R=\infty$ there exists a continuum of open polydiscs with finite radii in $\Lambda_R(\alpha)'_b$ resp. in $\Lambda_R^\infty(\alpha)$ which are pairwise not holomorphically equivalent.*

Proof. — We distinguish two cases.

Case 1: $\Lambda_1(\alpha)'_b$ and $\Lambda_1^\infty(\alpha)$.

At first we remark that without any restriction we may assume α to be strictly increasing. Since $\Lambda_1^\infty(\alpha)=\Lambda(Q(1,\alpha))'_b$ for

$$Q(1,\alpha)=\{q\in\Lambda_1^\infty(\alpha)\,|\,q\geqslant 0\},$$

it is easy to see that for all $R>1$ the sequence $a_R:=(R^{-\alpha_j})_{j\in\mathbb{N}}$ is in $\Lambda^\infty(Q(1,\alpha))$ and also in $\Lambda_1^\infty(\alpha)$. Hence $D_R:=\mathbb{D}_{a_R}$ is an open polydisc in $\Lambda_1^\infty(\alpha)$ resp. in $\Lambda_1(\alpha)'_b$. If we assume that for some $R>S>1$ the polydiscs D_R and D_S are holomorphically equivalent, we get from Proposition 9 the existence of a bijection π of $\mathbb{N}$ such that $(R^{\alpha_{\pi(j)}}/S^{\alpha_j})_{j\in\mathbb{N}}$ is in $\Lambda_1^\infty(\alpha)$. This implies $\limsup_{j\to\infty} R^{\alpha_{\pi(j)}/\alpha_j}/S\leqq 1$ and consequently:

$$\limsup_{j\to\infty}\frac{\alpha_{\pi(j)}}{\alpha_j}\ln R\leqslant \ln S<\ln R.$$

Hence we have:

$$\limsup_{j\to\infty}\frac{\alpha_{\pi(j)}}{\alpha_j}=p<1.$$

Now we choose q with $p<q<1$.

Then there exists $J\in\mathbb{N}$ such that $\alpha_{\pi(j)}<q\alpha_j$ for all $j\leqq J$. Since α is strictly increasing we get for $j_1:=1+\max\{J, \max_{j\leqslant J}\pi(j)\}$ that $\pi\{1, \ldots, j_1\}\subset\{1, \ldots, j_1-1\}$, which implies that π cannot be a bijection of $\mathbb{N}$. From this contradiction we get that D_R and D_S are not holomorphically equivalent.

Case 2: $\Lambda_\infty(\alpha)'_b$ and $\Lambda_\infty^\infty(\alpha)$.

Since $\Lambda_\infty^\infty(\alpha)=\Lambda(Q(\infty, \alpha))'_b$ for $Q(\infty, \alpha)=\{q\in\Lambda_\infty^\times(\alpha)\mid q\geqslant 0\}$, it is easy to see that for every increasing unbounded sequence $R=(R_j)_{j\in\mathbb{N}}$ of positive numbers the sequence $a(R):=(\exp(-R_j\alpha_j))_{j\in\mathbb{N}}$ is in $\Lambda^\infty(Q(\infty, \alpha))$ and also in $\Lambda_\infty^\infty(\alpha)$. Hence $\mathbb{D}_{a(R)}$ is an open polydisc in $\Lambda_\infty^\infty(\alpha)$ resp. in $\Lambda_\infty^\infty(\alpha)'_b$. If $\mathbb{D}_{a(R)}$ and $\mathbb{D}_{a(S)}$ are holomorphically equivalent, we get from Proposition 9 the existence of $\pi\in\Pi(\alpha)$ such that $(a(S)_{\pi(j)}/a(R)_j)_{j\in\mathbb{N}}$ is in $\Lambda_\infty(\alpha)'$. This implies that for some $T>0$, some $C>0$ and all $j\in\mathbb{N}$:

$$\exp(-S_{\pi(j)}\alpha_{\pi(j)})\leqslant C\exp((T-R_j)\alpha_j).$$

Since $\pi\in\Pi(\alpha)$ we get from this by taking logarithms:

$$M\geqq\frac{\alpha_{\pi(j)}}{\alpha_j}\geqq\frac{1}{S_{\pi(j)}}\left(R_j-\frac{\ln C}{\alpha_j}-T\right).$$

Since $\lim_{j\to\infty}\alpha_j=\infty$, this implies the existence of some $L>0$ such that:

(1) $$R_j\leqslant LS_{\pi(j)}\quad\text{for all } j\in\mathbb{N}.$$

We claim that (1) implies that the set $E:=\{j\in\mathbb{N}\mid R_j\leqslant LS_j\}$ is infinite. In order to prove this let us assume that E is finite. Then there exists $J\in\mathbb{N}$ such that:

(2) $$R_j>LS_j\quad\text{for all } j\geqslant J.$$

Now take $j\geqslant J$ and assume $\pi(j)\leqslant j$. Since the sequence S is increasing, this implies $S_{\pi(j)}\leqslant S_j$. Hence we get from (1) and (2) the contradiction:

$$R_j\leqslant LS_{\pi(j)}\leqslant LS_j<R_j.$$

Consequently we have $\pi(\{j\in\mathbb{N}\mid j\geqslant J\})\subset\{j\in\mathbb{N}\mid j>J\}$, which contradicts the fact that π is a bijection. Hence we have shown that E is infinite.

Now we define for $p \geqslant 1$ the sequences $R^{(p)} := (j^p)_{j \in \mathbb{N}}$. Then the preceding considerations show that for $p > q$ the open polydiscs $\mathbb{D}_{a(R^{(p)})}$ and $\mathbb{D}_{a(R^{(q)})}$ are not holomorphically equivalent, since:

$$\lim_{j \to \infty} j^{p-q} = \infty.$$

Remark. – The assertion of Corollary 10, which does not hold in the finite dimensional situation, might be considered as an outcome of the fact that there are too many open polydiscs with finite radii in $\Lambda_R(\alpha)'_b$ resp. in $\Lambda_R^{\mathscr{L}}(\alpha)$. However, this argument is not striking, as the following example shows: Let φ denote the space $(\mathbb{C}^{\mathbb{N}})'_b$ of all finite sequences. Every sequence $a > 0$ defines an open polydisc $\mathbb{D}_a$ in φ. However, all these open polydiscs are holomorphically equivalent, since for $a > 0$ and $b > 0$ the mapping $A: \varphi \to \varphi, Ax := (x_j a_j / b_j)_{j \in \mathbb{N}}$, is a linear automorphism of φ with $A(\mathbb{D}_a) = \mathbb{D}_b$.

We also want to remark that φ has the following property: For every reflexive Schwartz space $\Lambda(P)$ and every open polydisc $\mathbb{D}_a(\Lambda(P)'_b)$ in $\Lambda(P)'_b$ the set $\mathbb{D}_a(\Lambda(P)'_b) \cap \varphi$ is the open polydisc $\mathbb{D}_a(\varphi)$. Furthermore we have that for every hypoanalytic equivalence ψ between $\mathbb{D}_a(\Lambda(P)'_b)$ and $\mathbb{D}_b(\Lambda(P)'_b)$ the restriction of ψ to $\mathbb{D}_a(\varphi)$ is a holomorphic equivalence between $\mathbb{D}_a(\varphi)$ and $\mathbb{D}_b(\varphi)$.

In order to derive further consequences of Theorem 5 and Lemma 8, we introduce the following notation.

11. Notation. – (*a*) Let E be a l. c. space and let $G \neq \emptyset$ be an open subset of E. By $\mathrm{Aut}(G)$ we denote the group of all hypoanalytic automorphisms of G, and we endow $\mathrm{Aut}(G)$ with the compact open topology τ_0.

(*b*) Let α be an arbitrary exponent sequence, let $R = 1$ or $R = \infty$ and let a denote a sequence of positive numbers. We put:

$$\Pi(R, \alpha, a) := \left\{ \pi \in \Pi(\alpha) \;\middle|\; \left(\frac{a_{\pi(j)}}{a_j}\right)_{j \in \mathbb{N}} \text{ and } \left(\frac{a_j}{a_{\pi(j)}}\right)_{j \in \mathbb{N}} \text{ belong to } \Delta(\Lambda_R^*(\alpha)) \right\}.$$

Remark. – We remark that:

$$\Pi(1, \alpha, a) = \left\{ \pi \in \Pi(\alpha) \mid \lim_{j \to \infty} \sqrt[\alpha_j]{\frac{a_{\pi(j)}}{a_j}} = 1 \right\}$$

and that:

$$\Pi(\infty, \alpha, a) = \left\{ \pi \in \Pi(\alpha) \mid 0 < \inf_{j \in \mathbb{N}} \sqrt[\alpha_j]{\frac{a_{\pi(j)}}{a_j}} \leqslant \sup_{j \in \mathbb{N}} \sqrt[\alpha_j]{\frac{a_{\pi(j)}}{a_j}} < \infty \right\}.$$

From this it follows easily that $\Pi(R, \alpha, a)$ is a subgroup of $\Pi(\alpha)$.

In the same way as Proposition 9 one obtains:

12. PROPOSITION. — *For $R=1$ or $R=\infty$ let $\mathbb{D}_a$ be an open polydisc in $\Lambda_R(\alpha)'_b$ resp. in $\Lambda_R^\infty(\alpha)$.*

(*a*) *A mapping f belongs to $Aut(\mathbb{D}_a)$, if and only if, there exist $\mu\in(S^1)^{\mathbb{N}}$, $w\in a\,\mathbb{D}_a$ and $\pi\in\Pi(R, \alpha, a)$ such that for all $x\in\mathbb{D}_a$:*

$$f(x)=f_{(\mu,\, w,\, \pi)}(x):=\left(\frac{\mu_j}{a_j}\,\frac{a_{\pi(j)}x_{\pi(j)}+w_j}{1+\bar{w}_j a_{\pi(j)}x_{\pi(j)}}\right)_{j\in\mathbb{N}}.$$

(*b*) *The mapping F: $(S^1)^{\mathbb{N}}\times a\,\mathbb{D}_a\times\Pi(R, \alpha, a)\to Aut(\mathbb{D}_a)$ defined by $F(\mu, w, \pi):=f_{(\mu,\, w,\, \pi)}$ is a group isomorphism if the multiplication in $(S^1)^{\mathbb{N}}\times a\,\mathbb{D}_a\times\Pi(R, \alpha, a)$ is defined by:*

$$(\nu, y, \sigma)\circ(\mu, w, \pi):=(\nu\mu_\sigma,\ M_{w_\sigma}(y\bar{\mu}_\sigma),\ \pi\circ\sigma),$$

where $b_\sigma=(b_{\sigma(j)})_{j\in\mathbb{N}}$ for a sequence b and where:

$$M_c(d)=\left(\frac{d_j+c_j}{1+\bar{c}_j d_j}\right)_{j\in\mathbb{N}}$$

for sequences c, d in $D^{\mathbb{N}}$.

Under the hypotheses of Proposition 12 it is easy to check that the multiplication and the inversion in $\mathrm{Aut}(\mathbb{D}_a)$ is sequentially continuous with respect to the topology introduced in 11 *a*).

For bounded open polydiscs D in $\mathbb{C}^n$ it was shown by H. CARTAN [2] (*see* e. g. NARASIMHAN [9], p. 77) that the connected component of the identity in $\mathrm{Aut}(D)$ is open and contains no permutations. We show that the arcwise connected component of the identity in $\mathrm{Aut}(\mathbb{D}_a)$ contains no permutations and give an example showing that every neighbourhood of the identity may contain permutations.

13. PROPOSITION. — *Under the hypotheses of Proposition 12 the arcwise connected component of the identity in $Aut(\mathbb{D}_a)$ is the normal subgroup:*

$$\mathcal{M}(\mathbb{D}_a):=\{F(\mu, w, \mathrm{id}_{\mathbb{N}})\mid \mu\in(S^1)^{\mathbb{N}},\ w\in a\,\mathbb{D}_a\}.$$

Proof. — It is straightforward to show that $\mathcal{M}(\mathbb{D}_a)$ is arcwise connected and by the inversion formula $(F(\lambda, b, \pi))^{-1}=F(\bar{\lambda}_{\pi^{-1}}, -b_{\pi^{-1}}\lambda_{\pi^{-1}}, \pi^{-1})$ is easy to check that $\mathcal{M}(\mathbb{D}_a)$ is a normal subgroup of $\mathrm{Aut}(\mathbb{D}_a)$.

The following remark implies that $\mathcal{M}(\mathbb{D}_a)$ is really the arcwise connected component of $\operatorname{Aut}(\mathbb{D}_a)$: For every $n\in\mathbb{N}$ the mapping B_n: $\operatorname{Aut}(\mathbb{D}_a)\to\mathbb{N}$, $B_n(f_{(\mu,\omega,\pi)}):=\pi(n)$ is continuous.

In order to prove this, we remark that for every $n\in\mathbb{N}$ and every $x\in\mathbb{D}_a$ the function $f\mapsto(f(x))_n$ is continuous on $\operatorname{Aut}(\mathbb{D}_a)$ and that for $f=f_{(\mu,\omega,\pi)}$ we have:

$$a_{\pi(n)}\,|x_{\pi(n)}|=\frac{|(f(x))_n-(f(0))_n|\,a_n}{|1-a_n^2(f(x))_n\overline{(f(0))_n}|}.$$

Hence the function $f_{(\mu,\omega,\pi)}\mapsto|x_{\pi(n)}|$ is continuous on $\operatorname{Aut}(\mathbb{D}_a)$. From this it follows by an appropriate choice of $x\in\mathbb{D}_a$ that B_n is locally constant.

14. Examples. — (1) If α is an unstable exponent sequence i.e.:

$$\lim_{j\to\infty}\frac{\alpha_{j+1}}{\alpha_j}=\infty,$$

then we have:

$$\Pi(R,\alpha,a)=\Pi(\alpha)=\{\pi\,|\,\pi \text{ is a bijection of } \mathbb{N} \text{ with } \pi(j)\neq j \text{ only for finitely many } j\in\mathbb{N}\},$$

for $R=1$, ∞ and all positive sequences a.

(2) Let α be a given exponent sequence and put $a_R:=(R^{-\alpha_n})_{n\in\mathbb{N}}$ for $R>1$. In Corollary 10 we have already remarked that a_R belongs to $\Lambda_1^\infty(\alpha)$ and also to $\Lambda^\infty(Q(1,\alpha))$. We claim that for all $R>1$:

$$\Pi(1,\alpha,a_R)=\left\{\pi\,|\,\pi \text{ is a bijection of } \mathbb{N} \text{ with } \lim_{j\to\infty}\frac{\alpha_{\pi(j)}}{\alpha_j}=1\right\}=:\Pi_1(\alpha).$$

It is easy to check that $\Pi_1(\alpha)\subset\Pi(1,\alpha,a_R)$. On the other hand, $\pi\in\Pi(1,\alpha,a_R)$ implies that $(R^{\alpha_{\pi(j)}-\alpha_j})_{j\in\mathbb{N}}$ and $(R^{\alpha_j-\alpha_{\pi(j)}})_{j\in\mathbb{N}}$ are in $\Lambda_1^\infty(\alpha)$. Hence we have:

$$\limsup_{j\to\infty}R^{\alpha_{\pi(j)}/\alpha_j}\leqslant R \qquad\text{and}\qquad \limsup_{j\to\infty}R^{-\alpha_{\pi(j)}/\alpha_j}\leqslant R^{-1}.$$

From this we get:

$$\limsup_{j\to\infty}\frac{\alpha_{\pi(j)}}{\alpha_j}\leqslant 1 \qquad\text{and}\qquad \liminf_{j\to\infty}\frac{\alpha_{\pi(j)}}{\alpha_j}\geqslant 1,$$

and hence $\lim_{j\to\infty}\alpha_{\pi(j)}/\alpha_j=1$.

In connection with Corollary 10 this example shows that for the open polydiscs $\mathbb{D}_{a_R}$ which are not holomorphically equivalent the automorphism groups $\operatorname{Aut}(\mathbb{D}_{a_R})$ have the same "permutation part":

$$\operatorname{Aut}(\mathbb{D}_{a_R})/\mathscr{M}(\mathbb{D}_{a_R}) \simeq \Pi_1(\alpha).$$

(3) Let α be shift-stable, i. e. $\sup_{j\in\mathbb{N}} \alpha_{j+1}/\alpha_j = M < \infty$, and look at the open polydisc $\mathbb{D}_1$ in $\Lambda_\infty^\infty(\alpha)$. We claim that for every neighbourhood U of $\mathrm{id}_{\mathbb{D}_1}$ the set $U \cap \{F(1, 0, \pi) \mid \pi \in \Pi(\alpha) \setminus \mathrm{id}_{\mathbb{N}}\}$ is not empty.

In order to see this, we remark that for $a \in \mathbb{D}_1$, $S > 1$ and $\varepsilon > 0$ the sets:

$$U(a, S, \varepsilon) := \{f \in \operatorname{Aut}(\mathbb{D}_1) \mid \sup_{x\in N_a} \sup_{j\in\mathbb{N}} |x_j - f_j(x)| S^{\alpha_j} \leqslant \varepsilon\}$$

form a neighbourhood basis of $\mathrm{id}_{\mathbb{D}_1}$.

Now we fix a, S and ε and choose $R > S$ such that $S^M \leqslant R$. Furthermore we choose $m \in \mathbb{N}$ such that $a_j R^{\alpha_j} \leqslant \varepsilon/2$ for all $j \geqslant m$. Then we define $\pi \in \Pi(\alpha)$ by $\pi(j) := j$ for $j \neq m, m+1$, $\pi(m) := m+1$ and $\pi(m+1) := m$. From our choices we get for all $x \in N_a$ and all $j \in \mathbb{N}$.

$$\begin{aligned} |x_j - (A_\pi(x))_j| S^{\alpha_j} &\leqslant \max(|x_m - x_{m+1}| S^{\alpha_m}, |x_{m+1} - x_m| S^{\alpha_{m+1}}) \\ &\leqslant \max(|a_m| R^{\alpha_m} + |a_{m+1}| R^{\alpha_m}, |a_{m+1}| R^{\alpha_{m+1}} + |a_m| S^{\alpha_{m+1}}) \leqslant \varepsilon, \end{aligned}$$

since:

$$|a_m| S^{\alpha_{m+1}} = |a_m| R^{\alpha_m} \left(\frac{S^{\alpha_{m+1}/\alpha_m}}{R}\right)^{\alpha_m} \leqslant |a_m| R^{\alpha_m}.$$

Concluding, we show how Theorem 5 and Proposition 9 can be used to characterize the algebra isomorphism classes of the l. m. c. algebras $(H_{hy}(\mathbb{D}_a), \tau_0)$. In order to be able to do this, we need some preparations.

If A is an l. m. c. algebra, then $M(A) \subset A'$ denotes the set of all nonzero continuous multiplicative linear functionals on A.

15. Lemma. — *Let E be a complete reflexive Schwartz space and let $U \subset E'_b$ be open. Assume that any element of $M(H_{hy}(U), \tau_0)$ is a point-evaluation δ_z for some $z \in U$. Then the mapping $\Delta_U \colon U \to (H_{hy}(U), \tau_0)'_\sigma$, $\Delta_U(z) := \delta_z$, is hypoanalytic and $\Delta_U^{-1} \colon M(H_{hy}(U), \tau_0) \to U$ equals ${}^t i \mid M(H_{hy}(U), \tau_0))$, where $i \colon E \to H_{hy}(U)$ is the canonical inclusion defined by $i(x) \colon y \mapsto y(x)$.*

Proof. — Since the topology $\sigma = \sigma((H_{hy}(U), \tau_0); (H_{hy}(U), \tau_0))$ has the property $((H_{hy}(U), \tau_0)'_\sigma)' = H_{hy}(U)$, it follows immediately that $f \circ \Delta_U$ is

Gâteaux-analytic on U and continuous on the compact subsets of U for every $f \in ((H_{hy}(U), \tau_0)'_\sigma)'$. Hence Δ_U belongs to $H_{hy}(U, (H_{hy}(U), \tau_0)'_\sigma)$.

For any $z \in U$ and any $x \in E$ we have:

$$\langle {}^t i(\delta_z), x \rangle = \langle \delta_z, i(x) \rangle = i(x)[z] = \langle z, x \rangle,$$

which shows ${}^t i \circ \Delta_U(z) = z$ for all $z \in U$. This completes the proof since by hypothesis Δ_U is a bijection between U and $M(H_{hy}(U), \tau_0)$.

16. Proposition. — *Let E and F be complete reflexive Schwartz spaces, let $U \subset E'_b$ and $V \subset F'_b$ be open subsets and assume that $M(H_{hy}(U), \tau_0) = \{\delta_z \mid z \in U\}$ and that $M(H_{hy}(V), \tau_0) = \{\delta_z \mid z \in V\}$. Then $\Phi: (H_{hy}(U), \tau_0) \to (H_{hy}(V), \tau_0)$ is a topological algebra isomorphism iff there exists a hypoanalytic equivalence φ between V and U such that $\Phi(f) = f \circ \varphi$ for all $f \in H_{hy}(U)$.*

Proof. — It is easy to check that Φ is a topological algebra isomorphism, if Φ is of the form given above. Hence let us assume that Φ is a topological algebra isomorphism. Then we denote by i_E resp. i_F the canonical inclusion of E resp. F in $H_{hy}(U)$ resp. $H_{hy}(V)$ and we define $\varphi: V \to E'_b$ resp. $\psi: U \to F'_b$ by $\varphi := {}^t i_E \circ {}^t\Phi \circ \Delta_V$ resp. $\psi := {}^t i_F \circ {}^t(\Phi^{-1}) \circ \Delta_U$. Then the hypothesis on Φ together with Lemma 15 implies $\varphi \in H_{hy}(V, E'_\sigma)$. Since E and E'_b are Montel spaces, this implies $\varphi \in H_{hy}(V, E'_b)$. Because of the same arguments we have $\psi \in H_{hy}(U, F'_b)$.

Since ${}^t\Phi$ and ${}^t(\Phi^{-1})$ map multiplicative linear functionals into multiplicative linear functionals, we get from Lemma 15 and the hypotheses that $\varphi \circ \psi = \mathrm{id}_U$ and $\psi \circ \varphi = \mathrm{id}_V$. This shows that φ is a hypoanalytic equivalence between V and U. Furthermore we have for any $v \in V$ and any $f \in H_{hy}(U)$:

$$\begin{aligned}\Phi(f)[v] = \langle \Phi(f), \delta_v \rangle = \langle f, {}^t\Phi \circ \Delta_V(v) \rangle &= \langle f, \Delta_U \circ {}^t i_E \circ {}^t\Phi \circ \Delta_V(v) \rangle \\ &= \langle f, \Delta_U \circ \varphi(v) \rangle = f(\varphi(v)),\end{aligned}$$

which completes the proof.

17. Proposition. — *Let $\Lambda(P)$ be a reflexive Schwartz space and let $a, b \in \Lambda^\infty(P)$ satisfy $a > 0$ and $b > 0$. Then the l. m. c. algebras $(H_{hy}(\mathbb{D}_a), \tau_0)$ and $(H_{hy}(\mathbb{D}_b), \tau_0)$ are topologically isomorphic iff there exists a topological isomorphism $\Phi: (H_{hy}(\mathbb{D}_a), \tau_0) \to (H_{hy}(\mathbb{D}_b), \tau_0)$ which is of the form $\Phi(f) = f \circ \psi$, where ψ is a hypoanalytic isomorphism between $\mathbb{D}_a$ and $\mathbb{D}_b$.*

Proof. — The proof follows from Proposition 16 as soon as one knows that $M(H_{hy}(\mathbb{D}_c), \tau_0) = \{\delta_z \mid z \in \mathbb{D}_c\}$ for all $c \in \Lambda^\infty(P)$, $c > 0$. This is a consequence of ISIDRO [6], Prop. 4, but can also be derived easily from the fact that the polynomials in the coordinate functions are dense in $(H_{hy}(\mathbb{D}_c), \tau_0)$.

18. COROLLARY. — *Let $R = 1$ or $R = \infty$ and let $\mathbb{D}_a$ and $\mathbb{D}_b$ be open polydiscs in $\Lambda_R(\alpha)'_b$ resp. in $\Lambda_R^\infty(\alpha)$. Then the l.m.c. algebras $(H(\mathbb{D}_a), \tau_0)$ and $(H(\mathbb{D}_b), \tau_0)$ are topologically isomorphic, if and only if, there exists $\pi \in \Pi(\alpha)$ satisfying condition* 9 $(\star)$.

Remark. — From Corollary 18 it follows that for an open polydisc $\mathbb{D}_a$ in $\Lambda_1(\alpha)'_b$ the l. m. c. algebras $(H(\mathbb{D}_a), \tau_0)$ and $(H(\mathbb{D}_1), \tau_0)$ are isomorphic, if and only if, $1/a \in \Lambda_1^\infty(\alpha)$. For nuclear spaces $\Lambda_1(\alpha)$ even more is known. In [7], Thm. 3.3, a linear topological invariant has been used to show that the l. c. spaces $(H(\mathbb{D}_a), \tau_0)$ and $(H(\mathbb{D}_1), \tau_0)$ are isomorphic, if and only if, $1/a \in \Lambda_1^\infty(\alpha) = \Lambda_1(\alpha)$.

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