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The standard form of a representation-finite algebra


<http://www.numdam.org/item?id=BSMF_1983__111__21_0>
THE STANDARD FORM 
OF A REPRESENTATION-FINITE ALGEBRA

BY

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RESUME. — On connait depuis [RI 1] et [BG] l'importance de la technique des revêtements pour l'étude des représentations des algèbres de dimension finie. Dans la définition du groupe fondamental $\Phi$ d'une algèbre $A$, qui est donnée dans [BG] lorsque $A$ a un nombre fini de représentations indécomposables à isomorphisme près, les modules indécomposables sont supposés connus. Dans ce qui suit, nous donnons une description directe de $\Phi$, du revêtement universel et de la forme standard de $A$.

ABSTRACT. — The importance of covering techniques for the investigation of the representations of finite-dimensional algebras is well-known since [RI 1] and [BG]. When the algebra $A$ admits only finitely many isomorphism classes of indecomposable representations, the definition of the fundamental group $\Phi$ of $A$ given in [BG] requires the knowledge of all indecomposables. In the sequel, we describe $\Phi$, the universal cover and the standard form of $A$ directly in terms of $A$.

A very convenient way to determine whether a given algebra $A$ is representation-finite (and to compute its representations) consists in finding a simply connected cover of a suitable degeneration of $A$ ([BG], [G 2], [G0]). In theorem 3.1, we describe the standard form $\overline{A}$ of $A$, which is the best possible degeneration, and we determine the universal Galois covering of $\overline{A}$ in terms of $A$ (Theorem 1.5 and Remark 3.3 a). Our results heavily rest on the existence of a multiplicative basis in $\overline{A}$, i.e. of a basis such that the product of two basis vectors is again a basis vector or else is zero. The existence of such bases in representation-finite algebras was first stated by KUPISCH in the symmetric case [K] (see also [KS]). It was then proved by BONGARTZ for algebras whose quiver contains no oriented cycle [B 1]. BONGARTZ' proof is easily extended to simply connected locally representation-finite "algebras" [BRL], a case to which our existence proof...
can be reduced. Our own researches were started by Roiter’s publication [RO], in which the existence of a multiplicative basis for general representation-finite algebras is stated. Clearly, Roiter’s work lies on a higher level of difficulty, and we have not been in a position to check all the points in his developments. Paraphrasing Picard’s comment on Poincaré’s Duality Theorem (Picard et Simart, Théorie des Fonctions Algébriques, tome I, chap. II, 1897), we might say: « Roiter a donné une démonstration générale de l’existence d’une base multiplicative. Sa démonstration repose sur des considérations entièrement différentes, mais peut-être plusieurs points auraient-ils besoin d’être complétés. Aussi, avons-nous suivi une autre voie, mais avons dû nous limiter au cas des algèbres standard. »

Well-read mathematicians will easily interpret our proofs as statements on singular homology of triangulated topological spaces associated with the considered algebras. In fact, our intuition is geometrical, and so were our original proofs. The translation from geometry to algebra has been worked out during the spring vacation 1982 in order to take care of pure algebraists. For them we replaced short references to algebraic topology (to [ML], IV, 11.5 or [GZ], Ap. II, 3.6 for instance) by longer elementary variations on the snake-lemma. We hope that these variations will not be dismissed as mere exercises for a first-year course in homological algebra, since they come from, rest on and elucidate a mass of examples in representation-theory ([NR], [BGP], [L], [SZ], [BR], [BG], [BRL], [B 2], [CG]).

The method used here was presented in June 1981 in Oberwolfach, where corollary 2.2, 2.6, 2.11 and 3.3 b were stated. The remaining results were presented in November 1981 in a lecture at the university of Trondheim. Since then, we received from Martinez and de la Peña the proof [MP1] of an older conjecture of ours which simplifies the demonstration of Theorem 1.5. Nevertheless, we maintained the first proof in paragraph 2 because of its own virtues (2.6-2.8). As they informed us at the beginning of March, Martinez and de la Peña also noticed that our Oberwolfach-proof for the existence of a multiplicative basis in Schurian algebras extends to standard algebras [MP 2].

Our investigations commenced with Bongartz and Roiter as a joint discussion which diverged too rapidly. We like to express our thanks to both of them.

The notations are those introduced in [G1], [BG] and [G2]. In particular, k always denotes an algebraically closed field.
1. The group of constraints of a representation-finite algebra

1.1. We first fix the notations: Let $\Lambda$ be a locally bounded $k$-category ([BG], 2.1). By $\Lambda(a, b)$ we denote the space of morphisms from $a$ to $b$ in $\Lambda$, by $\mathcal{R}^0 \Lambda(a, b) = \Lambda(a, b) \supseteq \mathcal{R}^1 \Lambda(a, b) \supseteq \mathcal{R}^2 \Lambda(a, b) \ldots$ the radical series of $\Lambda(a, b)$ considered as a bimodule over $\Lambda(b, b)$ and $\Lambda(a, a)$. We say that a morphism $\mu \in \Lambda(a, b)$ has level $n \in \mathbb{N}$ if $\mu \in \mathcal{R}^n \Lambda(a, b) \setminus \mathcal{R}^{n+1} \Lambda(a, b)$; the zero-morphism has level $\infty$ by definition.

In the sequel, we assume that $\Lambda$ is locally representation-finite ([BG], 2.2). We denote by $\mathcal{I} = \text{ind} \Lambda$ the category formed by chosen representatives of the isoclasses of indecomposable finite-dimensional $\Lambda$-modules, and we agree that $a^* = \Lambda(?, a)$ is chosen as a representative for each $a \in \Lambda$. Given two objects $a, b \in \mathcal{I}$, we set $g(a, b, \mathbf{n}) = \sup \{ p : \mathcal{R}^p I(a^*, b^*) = \mathcal{R}^n \Lambda(a, b)^* \}$ for each $n \in \mathbb{N} \cup \{ \infty \}$, where $I \supseteq \mathcal{R} \supseteq \mathcal{R}^2 \mathcal{I} \ldots$ is the radical series of $I$ ([BG], 2.1). We say that a morphism $\mu \in \Lambda(a, b)$ of level $n$ has grade $g(\mu) = g(a, b, n)$ in $\Lambda$. In case $g(\mu) = g < \infty$, this implies the existence of irreducible morphisms ([AR], § 1; [G1], 1.6)

$$a^* \xrightarrow{\mu_1} m_1 \xrightarrow{\mu_2} m_2 \ldots m_{g-1} \xrightarrow{\mu_g} b^*$$

of $I$ such that $\mu^* - \mu_1 \ldots \mu_2 \mu_1 \in \mathcal{R}^{n+1} \Lambda(a, b)^*$ (remember that $\mathcal{R}^{n+1} \Lambda(a, b)$ has codimension $\leq 1$ in $\mathcal{R}^n \Lambda(a, b)$; see [G2], 2.4).

In the following lemma, we denote by $m \to n$ the arrow of $\Gamma_\Lambda$ (= the Auslander-Reiten quiver of $\Lambda$) which is associated with an irreducible morphism $\nu : m \to n$.

**Lemma.** — With the above notations, the homotopy-class ([BG], 1.2) $S(a, b, n)$ of the path

$$a^* \xrightarrow{\tilde{\nu}_1} m_1 \xrightarrow{\tilde{\nu}_2} m_2 \ldots m_{g-1} \xrightarrow{\tilde{\nu}_g} b^*$$

of $\Gamma_\Lambda$ depends only on $a, b$ and $n$.

**Proof.** — Let $\pi : \tilde{\Gamma}_\Lambda \to \Gamma_\Lambda$ be a universal covering and $F : k(\tilde{\Gamma}_\Lambda) \to I$ a well-behaved functor ([RI2], 2.5; [BG], 3.1). Choose a sequence of morphisms

$$x \xrightarrow{\nu_1} z_1 \xrightarrow{\nu_2} z_2 \ldots z_{g-1} \xrightarrow{\nu_g} y$$

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of \( k(\tilde{\Gamma}_\Lambda) \) such that \( F v_i - \mu_i \in R^2 I \) for each \( i \). We then get \( F(v_g \ldots v_1) - \mu_g \ldots \mu_1 \in R^s + 1 I \). Since we have

\[
R^s I/R^s + 1 I(a^*, b^*): k] = 1
\]

and

\[
\bigoplus_{x, z = b} (R^s k(\tilde{\Gamma}_\Lambda)/R^s + 1 k(\tilde{\Gamma}_\Lambda))(x, z) \Rightarrow (R^s I/R^s + 1 I)(a^*, b^*),
\]

\( y \) is the unique point of \( \pi^{-1}(b^*) \) such that

\[
(R^s k(\tilde{\Gamma}_\Lambda)/R^s + 1 k(\tilde{\Gamma}_\Lambda))(x, y) \Rightarrow (R^s I/R^s + 1 I)(a^*, b^*).
\]

As a consequence, if we lift the path \( \mu_g \ldots \tilde{\mu}_1 \) of \( \Gamma_\Lambda \) to a path of \( \tilde{\Gamma}_\Lambda \) with prescribed origin \( x \), the terminus \( y \) of the lifted path depends only on \( a, b \) and \( g \).

**Remark.** – Consider an arrow \( a \rightarrow b \) of the quiver \( Q_\Lambda \) of \( \Lambda \) and assume that \( \mu \) is a representative of \( \tilde{\mu} \), i.e. that \( \mu \in R \Lambda(a, b) \setminus R^2 \Lambda(a, b) \). Then we simply write \( S \tilde{\mu} \) instead of \( S(a, b, 1) \), and we have:

\[
v = v_g \ldots v_1 \in R \tilde{\Lambda}(x, y) \setminus R^2 \tilde{\Lambda}(x, y),
\]

since \( F \) induces a covering \( \tilde{\Lambda} \rightarrow \Lambda \) (\( \tilde{\Lambda} \) denotes the universal cover of \( \Lambda \) which is defined in [G 2], 2.1). In other words, if we lift \( \tilde{\mu} \) to an arrow \( \tilde{v} \) of \( Q_\Lambda \) with tail \( x \), the head of \( \tilde{v} \) is the terminus of the lifted path \( \tilde{v}_g \ldots \tilde{v}_1 \).

1.2. Denote by \( GQ_\Lambda \) the free groupoid generated by \( Q_\Lambda \) ([GZ], II, § 6), by \( G \Gamma_\Lambda \) the fundamental groupoid of \( \Gamma_\Lambda \): the objects of \( GQ_\Lambda \) are the points of the quiver \( Q_\Lambda \) of \( \Lambda \); the (invertible) morphisms of \( GQ_\Lambda \) are formal compositions of arrows and of formal inverses of arrows of \( Q_\Lambda \). Similarly, the objects of \( G \Gamma_\Lambda \) are the vertices of \( \Gamma_\Lambda \), i.e. the objects of \( I \); the morphisms of \( G \Gamma_\Lambda \) are the homotopy classes of paths of \( \Gamma_\Lambda \).

The map \( \tilde{\mu} \mapsto S \tilde{\mu} \) described in 1.1 defines a functor \( S : GQ_\Lambda \rightarrow G \Gamma_\Lambda \) such that \( S a = a^* = \Lambda(?, a) \) if \( a \in \Lambda \). We denote by \( S_a = S(a, a) \) the induced group homomorphism from

\[
\Pi(Q_\Lambda, a) = (GQ_\Lambda)(a, a) \text{ to } \Pi(\Gamma_\Lambda, a^*) = (G \Gamma_\Lambda)(a^*, a^*).
\]

By definition, \( \Pi(Q_\Lambda, a) \) is the fundamental group of the quiver \( Q_\Lambda \) at \( a \), and \( \Pi(\Gamma_\Lambda, a^*) \) is the fundamental group of the translation-quiver \( \Gamma_\Lambda \) at \( a^* \) (or equivalently of \( \Lambda \) at \( a \)).

**Proposition.** – *Let \( \Lambda \) be locally representation-finite and connected, a a point of \( \Lambda \), \( \tilde{a} \) a point of the universal cover \( \tilde{\Lambda} \) which lies over \( a \). Then \( S_a \) gives rise to an exact group sequence:

\[
1 \rightarrow \Pi(Q_\Lambda, \tilde{a}) \rightarrow \Pi(Q_\Lambda, a) \rightarrow \Pi(\Gamma_\Lambda, a^*) \rightarrow 1.
\]

In particular, \( \Pi(Q_\Lambda, \tilde{a}) \) is identified with \( \text{Ker} S_a \).*
Proof. — Let \( \pi : Q \rightarrow Q_\Lambda \) be a covering of quivers and \( \Pi \) a group acting on \( Q \) from the right. Assume that \( Q \) is connected and that \( \Pi \) acts freely and transitively on \( \pi^{-1}(a) \). Then each \( b \in \pi^{-1}(a) \) determines an exact sequence:

\[
1 \rightarrow \Pi(Q, b) \xrightarrow{\pi} \Pi(Q_\Lambda, a) \xrightarrow{\tau} \Pi \rightarrow 1,
\]

where \( \pi \) is induced by \( \pi \) and \( \tau \) is defined by \( b \tau(w) = wb \) (= terminus of the lifting of \( w \) to \( Q \) with origin \( b \)).

This works in particular if \( Q = Q_\Lambda \), \( b = \tilde{a} \) and \( \Pi = \Pi(\Gamma_\Lambda, a^*) \). In this case, \( \pi \) is induced by the universal covering \( \tilde{\Gamma}_\Lambda \rightarrow \Gamma_\Lambda \), and we define the action of \( \Pi \) on \( \tilde{\Gamma}_\Lambda \) and \( Q \tilde{\Lambda} \) in such a way that \( \tilde{a}v = v\tilde{a} \) (= terminus of the lifting of \( v \) to \( \tilde{\Gamma}_\Lambda \) with origin \( \tilde{a} \)). It then remains to show that \( \tau(w) = S_a(w) \) for each \( w \in \Pi(Q_\Lambda, a) \), or equivalently that \( \tau(w) \tilde{a} = S_a(w) \tilde{a} \). Now \( \tau(w) \tilde{a} \) is the terminus of the lifting of \( w \) to \( Q_\Lambda \) with origin \( \tilde{a} \). Similarly, \( S_a(w) \tilde{a} \) is obtained by lifting \( S_a(w) \). Since \( w \) is a composition of arrows and of inverses of arrows of \( Q_\Lambda \), our statement follows from the remark at the end of 1.1.

1.3. In the sequel, the kernel of \( S_a(1.2) \) will be called the group of constraints of \( A \) at \( a \). It coincides with the group defined in [G 2], 2.2, and our purpose here is to produce a set of generators. For this sake, we first recall that a schurian category in the terminology of Roiter is a locally finite-dimensional \( k \)-category \( M \) such that \([M(a, b) : k]\leq 1 \) for all \( a, b \in M \). The terminology is justified by the fact that \( M \) is schurian if it is locally representation-finite and if \( M(a, a) \cong k \) for each \( a \in M \). This last condition is satisfied in particular if \( M \) is locally bounded and directed (i.e., the quiver \( Q_M \) contains no oriented cycle).

Assume \( M \) to be locally bounded and schurian. We say that a path

\[
\begin{align*}
x &= x_0 \overset{\tau_1}{\longrightarrow} x_1 \overset{\tau_2}{\longrightarrow} x_2 \ldots x_{p-1} \overset{\tau_p}{\longrightarrow} x_p &= y
\end{align*}
\]

of \( Q_M \) is a zero-path of \( M \) if the composition-map

\[
M(x_0, x_1) \times \ldots \times M(x_{p-1}, x_p) \rightarrow M(x_0, x_p)
\]

is zero. We call non-zero contour of \( M \) a pair \((v, w)\) of non-zero paths \( v \) and \( w \) with the same origin \( x \) and the same terminus \( y \) (compare with [RO]). We call the non-zero contour simple if \( x \) and \( y \) are the only common vertices of \( v \) and \( w \). Finally, we denote by \( C_a(v, w) \) the conjugacy class of the element \( u^{-1}w^{-1}uw \in \Pi(Q_M, a) \) associated with a walk \( u \) from \( a \in M \) to the origin of the contour \((v, w)\). This conjugacy class is independent of \( u \).
LEMMA. — Let $\Lambda$ be locally representation-finite and simply connected and $a \in \Lambda$. The fundamental group $\Pi(Q_\Lambda, a)$ is generated by the conjugacy classes $C_a(v, w)$ associated with the simple non-zero contours of $Q_\Lambda$.

Our original proof is given in §2.10 below. It uses Galois coverings whose groups are torsion-free and mainly rests on [G 2], 3.6. The following simpler proof rests on a conjecture recently proved by MARTINEZ and DE LA PEÑA.

Proof. — First consider the more general case where $\Lambda$ is schurian. Denote by $P$ the subgroup of $\Pi=\Pi(Q_\Lambda, a)$ generated by the conjugacy classes $C_a(v, w)$. Let $\tilde{Q}_\Lambda$ be the universal cover of $Q_\Lambda$, $Q$ the quotient $\tilde{Q}_\Lambda/P$ and $\pi : Q \to Q_\Lambda$ the canonical projection. Then, two paths of $Q$ with the same origin must have the same terminus, provided their projections on $Q_\Lambda$ make up a non-zero contour.

$Q$ is the quiver of a locally bounded $k$-category $\Lambda'$ whose morphism-spaces are defined as follows: $\Lambda'(x, y) = 0$ if the projections of all paths from $x$ to $y$ are zero-paths of $\Lambda$; otherwise, $\Lambda'(x, y)$ is identified with $\Lambda(\pi x, \pi y)$. The composition $\Lambda'(x, y) \times \Lambda'(y, z) \to \Lambda'(x, z)$ is identified with $\Lambda(\pi x, \pi y) \times \Lambda(\pi y, \pi z)$ whenever $\Lambda'(x, y) \neq 0$, $\Lambda'(y, z) \neq 0$ and $\Lambda'(x, z) \neq 0$; it is zero otherwise. By construction, the action of $\Pi/P$ on $Q$ extends to a free action of $\Pi/P$ on $\Lambda'$, and the quiver-morphism $\pi$ extends to a ($\Pi/P$)-invariant functor $F : \Lambda' \to \Lambda$. A path of $Q$ is non-zero in $\Lambda'$ iff its projection is non-zero in $\Lambda$. We infer that $F$ is a Galois covering with group $\Pi/P$. By [MP 1], $\Pi/P$ acts freely on ind $\Lambda'$. By [G 2], 3.6 $F$ induces a Galois covering $\text{ind} \, \Lambda' \to \text{ind} \, \Lambda$.

In the particular case where $\Lambda$ is simply connected (hence directed and schurian), the covering $\text{ind} \, \Lambda' \to \text{ind} \, \Lambda$ must be trivial. Hence $\Pi = P$.

1.4. Let us now turn to the general case of a connected locally representation-finite $\Lambda$.

LEMMA. — Let $a_0 \to a_1 \to a_2 \ldots a_{n-1} \to a_n$ be a path of $Q_\Lambda$ and $\alpha_i \in \mathcal{R} \Lambda(a_{i-1}, a_i)$ a representative of $\bar{\alpha}_i$ (1.1). The following statements are equivalent:

(i) $g(\alpha_n \ldots \alpha_2 \alpha_1) = \sum g(\alpha_i)$;
(ii) $\alpha_n \ldots \alpha_2 \alpha_1 \neq 0$ for all representatives $\alpha_i$ of the $\bar{\alpha}_i$;
(iii) $\bar{\alpha}_n \ldots \bar{\alpha}_2 \bar{\alpha}_1$ is the projection of a non-zero path of $\bar{\Lambda}$.
Proof. — Let \( x_0 \xymatrix{\ar[r] & x_1 \ar[r] & x_2 \ldots \ar[r] & x_{n-1} \ar[r] & x_n } \) be a lifting of \( \vec{\alpha}_n \ldots \vec{\alpha}_2 \vec{\alpha}_1 \) to \( Q_\Lambda \) and \( F : k(\Gamma_\Lambda) = \mathcal{I} \to I = \text{ind} \Lambda \) a well-behaved functor. The formulae:

\[
\bigcup_{F_x = F_y} (\mathcal{R}^p / \mathcal{R}^{p+1}) \mathcal{I}(x, z) \cong (\mathcal{R}^p I / \mathcal{R}^{p+1} I)(F x, F y)
\]

(see [BG], 3.2 and 3.3) imply that \( \alpha_i = F \xi_i + \sum_j F \rho_{ij} \) for some representative \( \xi_i \) of \( \xi_i \) and some morphisms \( \rho_{ij} \) with domain \( x_i \) and grade \( g(\alpha_i) = g(\xi_i) \). Hence \( \alpha_n \ldots \alpha_2 \alpha_1 = F (\xi_n \ldots \xi_2 \xi_1) + \rho \), where \( g(\rho) = \sum_{i=1}^n g(\xi_i) \) (\( F \) preserves the grade). Accordingly, statement (i) means that \( g(\alpha_n \ldots \alpha_2 \alpha_1) = \sum_{i=1}^n g(\xi_i) \), or equivalently that \( \xi_n \ldots \xi_2 \xi_1 \neq 0 \). (Remember that \( \mathcal{I} \cong \text{Gr} \mathcal{I} \) with the notations of [BG], 5.1). We infer that (i) \( \iff \) (iii).

Now statement (iii) is independent of the choice of the representatives \( \alpha_i \). Therefore, (iii) also implies \( g(\alpha'_n \ldots \alpha'_i) = \sum_{i=1}^n g(\alpha'_i) < \infty \) and \( \alpha'_n \ldots \alpha'_i \neq 0 \). Finally, choosing \( \alpha'_i = F \xi_i \), we see that (ii) implies \( \xi_n \ldots \xi_1 \neq 0 \). Hence (ii) \( \iff \) (iii).

1.5. We call a path \( a_0 \xymatrix{\ar[r] & a_1 \ar[r] & a_2 \ldots \ar[r] & a_{n-1} \ar[r] & a_n } \) of \( Q_\Lambda \) stable if it satisfies statement (ii) of lemma 1.4. If this is so, the level \( l \) of \( \alpha_n \ldots \alpha_2 \alpha_1 \) stays constant whatever \( \alpha_i \) we choose as representative of \( \vec{\alpha}_i \). This constant level \( l \) is by definition the level of the path \( \vec{\alpha}_n \ldots \vec{\alpha}_2 \vec{\alpha}_1 \) (compare with [G 2], 2.4). The associated number \( g(a_0, a_n, l) \) (1.1) is called the grade of \( \vec{\alpha}_n \ldots \vec{\alpha}_2 \vec{\alpha}_1 \).

A stable contour of \( \Lambda \) is a pair \( (v, w) \) of two stable paths of \( Q_\Lambda \) which have the same origin, the same terminus and the same level. With each stable contour \( (v, w) \) and each \( a \in \Lambda \) we associate a conjugacy class \( C_a(v, w) \) of \( \Pi(Q_\Lambda, a) \) as we did in 1.3.

**Theorem.** — The group of constraints \( K_\Lambda \) (1.3) of a locally representation-finite connected \( k \)-category \( \Lambda \) is generated by the conjugacy classes \( C_a(v, w) \) associated with the stable contours \( (v, w) \) of \( \Lambda \).

This immediately follows from 1.2, 1.3 and 1.4. Notice that the stable contours coincide with the non-zero contours if \( \Lambda \) is schurian.

1.6. **Corollary.** — Under the assumptions of theorem 1.5, let \( v, w \) be two paths from \( x \) to \( y \) and \( u \) a path from \( a \) to \( x \) in \( Q_\Lambda \). Assume \( \nu \) to be stable with level \( n \). Then \( \omega \) is stable with level \( n \) iff \( \omega^{-1} w^{-1} v u \in K_\Lambda \).
Proof. — The condition is necessary by theorem 1.5. Conversely, suppose that \( u^{-1} w^{-1} vw \in K_\Lambda \). Let \( F : \tilde{A} \rightarrow \Lambda \) be a universal covering. If \( \tilde{x} \in \tilde{A} \) lies over \( x \), the liftings \( \tilde{v} \) and \( \tilde{w} \) of \( v \) and \( w \) to \( Q_\Lambda \) with origin \( \tilde{x} \) have a common terminus \( \tilde{y} \) by assumption. Since \( \tilde{v} \) is non-zero, so is \( \tilde{w} \) ([BRL], 5.1). Therefore, \( w \) is stable (1.4). Finally, \( \tilde{v} \) and \( \tilde{w} \) have the same grade. So do \( v \) and \( w \), since \( F \) preserves the grade.

1.7. For non-specialists we recall a celebrated example of Riedtmann: Let \( \Lambda \) be the \( k \)-category defined by the quiver \( b \xrightarrow{\sigma} a \xrightarrow{\rho} \rho \) and the relations \( \delta \sigma = \rho^2 \), \( \sigma \delta = \sigma \rho \delta \), \( \rho^4 = 0 \). The maximal stable paths are \( \sigma \rho \delta \), \( \rho^3 \), \( \delta \sigma \rho \) and \( \rho \delta \sigma \). So there is "essentially" one stable path, namely \( (\rho^2, \delta \sigma) \). The group \( \Pi(Q_\Lambda, a) \) is freely generated by \( \rho \) and \( \delta \sigma \). The group \( K_\Lambda \) is generated by the conjugates of \( \sigma^{-1} \delta^{-1} \rho^2 \).

For a generalization of this example see [W 1], [RI 3].

2. Directed and simply connected algebras

2.1. Let \( M \) be a schurian category (1.3) and \( S_n M \) the set of its non-degenerate \( n \)-simplices, i.e. of the sequences \( x_0, \ldots, x_n \) of distinct objects of \( M \) such that the composition

\[
M(x_0, x_1) \times \cdots \times M(x_{n-1}, x_n) \rightarrow M(x_0, x_n)
\]

is not zero. By definition, \( S_1 M \) is identified with the set of pairs \((x, y)\) such that \( M(x, y) \neq 0 \); for \( n = 0 \), we agree that \( S_0 M \) is the set of objects of \( M \). The family \( S_n M = (S_n M)_{n \in \mathbb{N}} \) is called the simplicial frame of \( M \).

We denote by \( C_n M \) the group of \( n \)-chains of \( M \), i.e. the free abelian group generated by \( S_n M \). The \( C_n M \) give rise to a differential complex

\[
\cdots \rightarrow C_2 M \xrightarrow{d} C_1 M \xrightarrow{d} C_0 M
\]

such that:

\[
d(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
\]

The homology groups of this complex \( C_* M \) are denoted by \( H_n M \) and are called the simplicial homology groups of \( M \). If \( Z \) is an abelian group, the cohomology groups of the complex \( \text{Hom}(C_* M, Z) \) are denoted by \( H^n(M, Z) \) and are called simplicial cohomology groups of \( M \) with coefficients in \( Z \).

2.2. Two schurian categories \( M, N \) are said to be equivalent if they have the same simplicial frame and if there is an isomorphism \( M \cong N \) which is the identity on the objects.
LEMMA. — Let $M$ be schurian. Then there is a natural bijection between $H^2(M, k^*)$ and the set of equivalence classes of schurian categories $N$ such that $S_\ast N = S_\ast M$.

Proof. — Let $N$ be schurian and such that $S_\ast N = S_\ast M$. Up to an equivalence we can assume that $N(x, y) = M(x, y)$ for all $x, y \in S_0 M$. For each $(x, y) \in S_1 M$ we choose a basis $(y|x)$ of $M(x, y)$. For each $(x, y, z) \in S_2 M$, the composition of $N$ defines a non-zero scalar $c_N(x, y, z)$ such that:

$$(z|y|x) = c_N(x, y, z)(z|x).$$

The cochain $c_N : S_2 M \to k^*, (x, y, z) \mapsto c_N(x, y, z)$ obtained in this way is a 2-cocycle, because the composition is associative. The residue class $\bar{c}_N \in H^2(M, k^*)$ is independent of the choice of the basis vectors $(y|x)$. The map $N \mapsto \bar{c}_N$ yields the desired bijection.

2.3. On account of lemma 2.2, we have to determine second cohomology groups. This presupposes some information about $H_0 M$ and $H_1 M$.

Clearly, if we map the homology class $\bar{x} \in H_0 M$ of an object $x \in S_0 M$ onto the connected component of $M$ in which $x$ lies, we obtain an isomorphism between $H_0 M$ and the free abelian group which has the connected components of $M$ as basis. From a dual point of view, $H^0(M, Z)$ is identified with the space of functions $S_0 M \to Z$ which are constant on the connected components.

For the interpretation of $H_1 M$, we focus on the relevant case in which $M$ is connected and locally bounded. We denote by $\Pi = \Pi(Q_M, a)$ the fundamental group of the quiver $Q_M$ of $M$ at some vertex $a$, by $P$ the normal subgroup of $\Pi$ generated by the conjugacy classes $C(v, w)$ assigned to the non-zero contours $(v, w)$ of $M$ (1.3). We construct a homomorphism $\xi : \Pi/P \to H_1 M$ according to the following prescription: If $w$ is a closed walk of $Q_M$ from $a$ to $a$, consider the 1-cycle $\xi w = \sum_{t \varphi} (w^+ \varphi - w^- \varphi)$ $(t \varphi, h \varphi) \in C_1 M$, where $t \varphi \to h \varphi$ is an arbitrary arrow of $Q_M$ and $w^\pm \varphi$ (resp. $w^- \varphi$) the number of steps from $t \varphi$ to $h \varphi$ (resp. from $h \varphi$ to $t \varphi$) in $w$. The homomorphism $\xi$ is obtained from $w \mapsto \xi w$ by passing to the quotients.

LEMMA. — Assume $M$ to be connected, locally bounded and schurian. For each abelian group $Z$, the homomorphism $\xi : \Pi/P \to H_1 M$ induces isomorphisms

$$H^1(M, Z) \cong \text{Hom}(H_1 M, Z) \cong \text{Hom}(\Pi/P, Z).$$
Proof. — The differential complex $C_* M$ yields exact sequences

$$0 \to Z_1 M \to C_1 M \xrightarrow{d} C_0 M \to H_0 M \to 0$$

and

$$C_2 M \to Z_1 M \to H_1 M \to 0,$$

where $Z_1 M = \text{Ker} d$. Since $C_1 M$, $C_0 M$ and $H_0 M$ are free abelian groups, the first sequence splits and therefore remains exact when acted upon by an additive functor. Applying the functor $X \mapsto \text{Hom}(X, Z) = X^*$, we thus get exact sequences

$$0 \to Z_1 M^* \to C_1 M^* \to C_0 M^* \to H_0 M^* \to 0$$

and

$$C_2 M^* \to Z_1 M^* \to H_1 M^* \to 0,$$

which induce the first canonical isomorphism $H^1(M, Z) \simeq \text{Hom}(H_1 M, Z)$.

The composition $H^1(M, Z) \to \text{Hom}(\Pi/P, Z)$ of this isomorphism with $\text{Hom}(\mathcal{G}, Z)$ is induced by the map which assigns the value $\sum_{\varphi}(w^+ \varphi - w^- \varphi) f(t \varphi, h \varphi) \in Z$ to each 1-cocycle $f$ of $\text{Hom}(C_* M, Z)$ and each closed walk $w$ from $a$ to $a$ in $Q_M$. In order to prove that this composition is bijective, we just produce the reciprocal map: Choose a walk $w$ from $a$ to each vertex $x \in Q_M$; for each $(x, y) \in S_1 M$ choose a non-zero path $\mathcal{P}_{xy}$ from $x$ to $y$; denote by $\overline{\mathcal{P}}_{xy}$ the image of the closed walk $w^{-1} \mathcal{P}_{xy} w$ in $\Pi/P$. The reciprocal map sends $g \in \text{Hom}(\Pi/P, Z)$ onto the cohomology class of the 1-cocycle $(x, y) \mapsto g(\overline{\mathcal{P}}_{xy})$.

2.4. For practical purposes, the following interpretation of $H^1(M, Z)$ turns out to be useful: Denote by $Q^1_M$ the set of arrows of $Q_M$. A function $f : Q^1_M \to Z$ is called closed if $\sum_i f(v_i) = \sum_j f(w_j)$ for each non-zero contour $(v, w)$ of $M$ (the $v_i$ and $w_j$ denote the arrows occurring in $v$ and $w$ respectively). The function $f$ is called exact if there is a function $e : S_0 M \to Z$ such that $f(\varphi) = e(h \varphi) - e(t \varphi)$ for all $\varphi \in Q^1_M$.

Lemma. — Let $M$ be schurian and locally bounded. The canonical injection $Q^1_M \to S_1 M$, $\varphi \mapsto (t \varphi, h \varphi)$ induces an isomorphism from $H^1(M, Z)$ onto the quotient of the space of $Z$-valued closed functions by the subspace of exact functions.

Clear.

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2.5. Lemma. — Assume \( \Lambda \) to be connected, locally representation-finite. Then \( \Lambda \) is simply connected iff \( \Lambda \) is schurian and \( H_1 \Lambda \) is a torsion-group.

Proof. — First assume \( \Lambda = M \) to be simply connected. Then \( \Lambda \) is directed, hence schurian. Denote by \( tH_1 \Lambda \) the torsion subgroup of \( H_1 \Lambda \). By 2.3 \( H_1 \Lambda \) is identified with the largest abelian factor-group of \( \Pi/P \). We infer that \( H_1 \Lambda/tH_1 \Lambda \cong \Pi/K \) for some invariant subgroup \( K \) of \( \Pi \) which contains \( P \). By [G2], 3.1, the connected Galois covering \( \Lambda' \to \Lambda \) with group \( \Pi/P \) constructed in the proof of lemma 1.3 yields a connected Galois covering \( \Lambda'/\left(K/P\right) \to \Lambda \) with group \( \Pi/K = H_1 \Lambda/tH_1 \Lambda \). Since \( \Pi/K \) is torsion-free, the Galois covering \( \Lambda'/\left(K/P\right) \to \Lambda \) yields a Galois covering of \( \text{ind} \Lambda \) ([G2], 3.6) with group \( \Pi/K \). Since \( \Lambda \) is simply connected, \( \Pi/K \) is trivial and \( H_1 \Lambda = tH_1 \Lambda \).

Conversely, assume \( \Lambda = M \) schurian. By 2.3, \( H_1 \Lambda/tH_1 \Lambda \) is the largest factor-group of \( \Pi/P \) which is both abelian and torsion-free. So, if \( H_1 \Lambda \) is torsion, \( \{1\} \) is the only torsion-free abelian factor-group of \( \Pi/P \). On the other hand, the group of constraints \( K_\Lambda \) contains \( P \) by 1.2 and 1.4 (or equivalently by [G2], 2.4). Therefore, the fundamental group \( \Pi(\Gamma_\Lambda, a^*) \) of \( \Lambda \) is a factor-group of \( \Pi/P \); it is free non-commutative by [BG], 4.2. If it was not trivial, \( \Pi(\Gamma_\Lambda, a^*) \) and \( \Pi/P \) would admit a non-trivial torsion-free abelian factor-group.

2.6. Our main theorem in this section rests on some definitions, which we introduce now: Let \( Q \) be a directed quiver (i.e. a quiver without oriented cycles). Given two vertices \( x, y \) of \( Q \), the inequality \( x \leq y \) means that \( Q \) contains a path from \( x \) to \( y \). We denote by \([x, y]\) the interval \( \{z \in Q : x \leq z \leq y\} \) and call \( Q \) interval-finite if all intervals \([x, y]\) are finite. We call a set \( C \) of vertices of \( Q \) convex if it contains \([x, y]\) whenever \( x, y \in C \).

Theorem. — Assume \( \Lambda \) locally representation-finite and \( Q_\Lambda \) directed and interval-finite. Then \( H_1 \Lambda \) is a free abelian group, and \( H_i \Lambda = 0 \) for \( i \geq 2 \). If \( \Lambda' \) is a full subcategory of \( \Lambda \) whose set of objects is convex in \( Q_\Lambda \), the inclusion \( \Lambda' \to \Lambda \) induces an isomorphism of \( H_1 \Lambda' \) onto a pure subgroup of \( H_1 \Lambda \).

Proof. — First we consider a partially ordered set \( \Sigma \) and the associated \( k \)-category \( \Sigma_k \) : Set \( \Sigma_k(x, y) = k(y|x) = 1 \)-dimensional space with basis \( (y|x) \) or \( \Sigma_k(x, y) = 0 \) according as \( x \leq y \) or \( x \not\leq y \); set \( (z|y) \circ (y|x) = (z|x) \) if \( x \leq y \leq z \). Assume that \( s \) is the largest element of \( \Sigma \) and denote by \( h_n : C_n \Sigma_k \to C_{n+1} \Sigma_k \) the map sending \((x_0, \ldots, x_n)\) to \((x_0, \ldots, x_n, s)\) if \( x_n \neq s \).
or else to 0. Then we have \( dh_n c = h_{n-1} dc - (-1)^n c \) for each \( n \geq 1 \) and each \( c \in C_n \Sigma_k \). We infer that \( H_n \Sigma_k = 0 \) if \( n \geq 1 \).

Now consider the case where \( \Sigma_k = M \) is representation-finite. Then we have \( H_1 \Sigma_k = 0 \): Indeed, assume \( M \) to be connected. In [G 2], 2.5 we have shown that the fundamental group \( \Pi = \Pi(Q_M, a) \) is generated by the conjugacy classes attached to the contours of \( Q_M \). With the notations of 2.3, this means that \( \Pi/P = \{1\} \). Accordingly, the equality \( H_1 \Sigma_k = 0 \) follows from lemma 2.3.

In a third step of our proof, we suppose that \( \Lambda \) contains a maximal element \( s \) and denote by \( \Lambda' \) the full subcategory formed by the objects of \( \Lambda \) other than \( s \). We set \( \Sigma_s = \{ t \in \Lambda : \Lambda(t, s) \neq 0 \} \) and \( \Sigma'_s = \Sigma_s \setminus \{s\} \), and we partially order these sets by setting \( t \leq t' \) iff \( \Lambda(t', s) \circ \Lambda(t, t') \neq 0 \). The associated \( k \)-categories \( \Sigma_{sk} \) and \( \Sigma'_{sk} \) are clearly identified with subcategories of \( \Lambda \), and we have a short exact sequence of differential complexes:

\[
\begin{array}{c}
0 \rightarrow C_n \Sigma'_{sk} \rightarrow C_n \Sigma_{sk} \oplus C_n \Lambda' \rightarrow C_n \Lambda \rightarrow 0,
\end{array}
\]

where \( u, v, i \) and \( j \) are induced by the inclusion-functors. Passing to homology, we obtain the celebrated Mayer-Vietoris sequence:

\[
\ldots H_n \Sigma'_{sk} \rightarrow H_n \Sigma_{sk} \oplus H_n \Lambda' \rightarrow H_n \Lambda \rightarrow H_{n-1} \Sigma'_{sk} \ldots
\]

In a fourth step, we assume that \( \Lambda \) is associated with a finite partially ordered set. Then, using induction on the cardinality, we suppose that \( H_n \Sigma'_{sk} = H_n \Lambda' = 0 \) for \( n \geq 1 \). Since \( \Sigma_s \) has only one maximal element, we have \( H_n \Sigma_{sk} = 0 \) for \( n \geq 1 \) by our first step. As a consequence, the Mayer-Vietoris sequence tells us that \( H_n \Lambda = 0 \) for \( n \geq 2 \). Together with step two, this proves that \( H_n \Lambda = 0 \) for \( n \geq 1 \).

In step five, we turn to the case where \( \Lambda \) has finitely many objects. The categories \( \Sigma'_{sk} \) and \( \Sigma_{sk} \) are representation-finite because the extension by 0 yields full embeddings from their module-categories into that of \( \Lambda \). Therefore we have \( H_n \Sigma'_{sk} = H_n \Sigma_{sk} = 0 \) for \( n \geq 1 \) by step four, and the Mayer-Vietoris sequence reduces to \( H_n \Lambda' \Rightarrow H_n \Lambda \) for \( n \geq 2 \) and to the exact sequence:

\[
0 \rightarrow H_1 \Lambda' \rightarrow H_1 \Lambda \rightarrow H_0 \Sigma'_{sk} \rightarrow H_0 \Sigma_{sk} \oplus H_0 \Lambda' \rightarrow H_0 \Lambda \rightarrow 0.
\]

By induction, we infer that \( H_n \Lambda = 0 \) for \( n \geq 2 \). Moreover, \( H_0 \Sigma'_{sk} \) is free abelian by 2.3 and so is \( \text{Ker } i \). By induction we get that \( H_1 \Lambda \cong H_1 \Lambda' \oplus \mathbb{Z}' \) is free abelian.
Step six: Let $A$ have infinitely many objects. We may suppose that $A$ is connected. By lemma 2.7 below, $A$ is the “union” of an increasing sequence of full finite subcategories $A_n$ such that $A_{n+1}$ is obtained by adding to $A_n$ an extremal element of $Q_{A_n}$. Therefore, $H_1 A \cong \lim_{\to n} H_1 A_n$ can be constructed step by step by successively adding new basis vectors (step 5). Hence $H_1 A$ is free. Moreover, $H_1 A \cong \lim_{\to n} H_1 A_n$ is zero for $i \geq 2$.

Step seven: Let us now turn to the last statement. Suppose that $A'$ has finitely many objects. By lemma 2.7 below, $A$ is the “union” of an increasing sequence of full finite subcategories $A_n$ such that $A_0 = A'$ and that $A_{n+1}$ is obtained by adding to $A_n$ an extremal element of $Q_{A_n}$. Therefore, $H_1 A'$ is obtained by successively adding new basis vectors to $H_1 A = H_1 A_0$. Accordingly, $H_1 A'$ is identified with a direct summand of $H_1 A$.

Last step: Let $A'$ have infinitely many objects. By lemma 2.7 below, $A'$ is the “union” of an increasing sequence of full finite subcategories $A'_n$ such that... We infer that $H_1 A'$ is identified with the union of an increasing sequence of direct summands $H_1 A'_n$ of $H_1 A$. Therefore, $H_1 A'$ is pure in $H_1 A$.

2.7. Lemma. Let $C$ be a finite convex set of vertices of a directed, interval-finite, connected and locally finite (BG, 2.1) quiver $Q$. Then, the set of vertices of $Q$ is the union of an increasing sequence $C = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ of finite convex sets such that, for each $n \in \mathbb{N}$, the difference $C_{n+1} \setminus C_n$ is empty or consists of one extremal (= maximal or minimal) element of $C_{n+1}$.

Proof. — Let $q_1, \ldots, q_m, \ldots$ be the vertices of $Q$ (a countable set). Let $D_m$ be the union of the intervals $[x, y]$, where $x, y \in C \cup \{ q_i : i \leq m \}$. Then $D_m$ is convex and finite, it contains $C$, and the set of vertices of $Q$ equals $\bigcup_{m \in \mathbb{N}} D_m$. It is therefore sufficient to construct for each $m$ a finite sequence $D_m = D^0_m \subseteq D^1_m \subseteq \ldots \subseteq D^r_m = D_{m+1}$ of convex sets such that each $D^{s+1}_m$ is obtained from $D^s_m$ by addition of a maximal or of a minimal point. In fact, we construct $D^{s+1}_m$ from $D^s_m$ as follows:

If one of the sets

$$\{ x \in D_{m+1} \setminus D^s_m : x \geq y \text{ for some } y \in D^s_m \}$$

or

$$\{ x \in D_{m+1} \setminus D^s_m : x \leq y \text{ for some } y \in D^s_m \},$$
is not empty, we adjoin to $D_m$ a minimal point of the first set or a maximal point of the second. If both are empty, we can add any point of $D_{m+1} \setminus D_m$ to $D_m$, or else we have $D_m = D_{m+1}$.

2.8. COROLLARY. — Let $\Lambda$ be locally representation-finite and simply connected. A full subcategory $\Lambda'$ of $\Lambda$ is simply connected provided its set of objects is convex in $Q_\Lambda$.

Proof. — $Q_\Lambda$ is interval-finite by [BG], 1.6. By 2.5 and 2.6 we have $H_1 \Lambda = 0$. This implies $H_1 \Lambda' = 0$ by 2.6 and our statement by 2.5.

2.9. Assume $\Lambda$ to be locally bounded and directed. For each $s \in \Lambda$, denote by $Q_s$ the full subquiver of $Q_\Lambda$ formed by the vertices $t \geq s$, by $\Sigma_s$ the partially ordered set $\{ t \leq s : \Lambda(t, s) \neq 0 \}$ (where $t \leq t'$ iff $\Lambda(t', s) \circ \Lambda(t, t') \neq 0$; see the proof of theorem 2.6). We call $s$ separating if each connected component of $Q_s$ contains at most one connected component of $\Sigma_s$.

COROLLARY (Separation-criterion of Bautista-Larrion). — Let $\Lambda$ be representation-finite and directed. Then $\Lambda$ is simply connected iff each of its objects is separating.

Proof. — First we assume $\Lambda$ simply connected. By 2.5 and 2.6 we have $H_1 \Lambda = 0$. We denote by $\Lambda_s$ and $\Lambda_s'$ the full subcategories of $\Lambda$ supported by $Q_s$ and $Q_s \cup \{ s \}$ respectively, by $\Sigma_s$ the partially ordered set $\{ t \leq s : \Lambda(t, s) \neq 0 \}$. Since the set of objects of $\Lambda_s$ is convex in $Q_s$, we have $H_1 \Lambda_s \subseteq H_1 \Lambda = 0$ by 2.6. The Mayer-Vietoris sequence (see 2.6)

$$H_1 \Lambda_s \xrightarrow{i} H_0 \Sigma_s \xrightarrow{j} H_0 \Sigma_s \oplus H_0 \Lambda_s'$$

tells us that $i$ is injective. By 2.3, we have $H_0 \Sigma_s \cong \bigoplus Z C$, $H_0 \Lambda_s' \cong \bigoplus Z D$ and $H_0 \Sigma_s \cong Z \Sigma_s$, where $C$ and $D$ range over the connected components of $\Sigma_s'$ and $Q_s$ respectively. Furthermore, $i$ maps $C$ onto $\Sigma_s + C \in Z \Sigma_s \oplus (\bigoplus Z D)$, where $\Sigma_s$ denotes the connected component of $Q_s$ containing $C$. Therefore, $i$ is injective iff the map $C \mapsto \Sigma_s$ is injective. We infer that $s$ is separating.

Conversely, suppose that the objects $x_1, \ldots, x_n$ of $\Lambda$ are separating. We may assume that $i < j$ implies $x_i \neq x_j$. Denoting then by $\Lambda_p$ the full subcategory of $\Lambda$ formed by $x_1, \ldots, x_p$, we shall prove by induction on $p$ that $H_1 \Lambda_p = 0$ and then apply 2.5. Actually, assume $\Lambda_{p-1}$ simply connected and set $s = x_p$. On account of the Mayer-Vietoris sequence:

$$0 = H_1 \Lambda_{p-1} \rightarrow H_1 \Lambda_p \xrightarrow{i} H_0 \Sigma_{p-1} \xrightarrow{j} H_0 \Sigma_{p-1} \oplus H_0 \Lambda_{p-1},$$
we have to show that $i$ is injective, or equivalently that the map $C \mapsto C'$ is injective, where $C'$ is the connected component of $\Lambda_{p-1}$ containing $C$. But this is true, because $s$ is separating and $\Lambda_{p-1} \subseteq \Lambda'$. 

2.10. An alternative proof for lemma 1.3. Let $\Lambda$ be locally representation-finite and simply connected, and let $a_0 - a_1 - \ldots - a_n - a_{n+1} = a_0$, $n \geq 2$, be a simple closed walk in $Q_\Lambda$. We have to show that it can be “decomposed” into non-zero simple contours of $\Lambda$. Now, the union of the intervals $[a_i, a_j]$ is a finite convex set of objects of $Q_\Lambda$. Replacing $\Lambda$ if necessary by the full subcategory formed by this union and applying 2.8, we may assume that $\Lambda$ has a finite number of objects and proceed by induction on this number.

Let $s$ be a maximal point of $Q_\Lambda$. If $a_i \neq s$ for all $i$, we apply our induction hypothesis to the full subcategory $\Lambda' = \Lambda'_s$ of $\Lambda$ formed by the points other than $s$. Otherwise, we may assume that $a_0 = s$ and set $\Sigma' = \Sigma'_s = \{ t \in \Lambda' : \Lambda(t, s) \neq 0 \}$. Since $a_1 - \ldots - a_n$ is a walk of $Q_\Lambda$, $a_1$ and $a_n$ lie in the same connected component of $\Lambda'$. Since $s$ is separating, $a_1$ and $a_n$ are in the same connected component of $\Sigma'$, and of course they are maximal in $\Sigma'$. The following cases only can occur:

1. $\Sigma'$ contains a point $b$ such that $a_1 \geq b \leq a_n$. If $v$ and $w$ are paths from $b$ to $a_1$ and $a_n$, the given closed walk decomposes into the simple “contour” $b - a_1 \leftarrow a_0 \rightarrow a_n \leftarrow w$ and the closed walk $b - a_1 - \ldots - a_n \leftarrow b$ in $Q_\Lambda$. Induction applies.

2. $\Sigma'$ is connected and has a third maximal point $m$ besides $a_1$ and $a_n$. (Remember that $Q_\Lambda$ has at most 3 arrows heading for $a_0 = s$.) Furthermore, $\Sigma'$ contains two points $c$ and $d$ such that $a_1 \geq c \leq m \leq d \leq a_n$. In this case, we choose two paths $v$ and $v'$ from $c$ to $a_1$ and $m$ on one hand, and two paths $w$ and $w'$ from $d$ to $m$ and $a_n$ on the other. The given closed walk then decomposes into two simple “contours”

$\begin{align*}
c &\rightarrow v - a_1 \leftarrow a_0 \leftarrow m - c, \\
d &\rightarrow w - m \leftarrow a_0 \leftarrow a_n \leftarrow d 
\end{align*}$

and the closed walk $a_1 - \ldots - a_n \leftarrow d - m - c - a$. Again induction applies.

2.11. Corollary. — Assume $\Lambda$ to be schurian, connected and locally representation-finite. Let $\Phi = \Pi(\Gamma_\Lambda, a^*)$ be the fundamental group of $\Gamma_\Lambda$ at some projective vertex $a^*$ and $Z$ an abelian group. Then $H^1(\Lambda, Z)$ is canonically isomorphic to $\text{Hom}(\Phi, Z)$.

Proof. — By Theorem 1.5, $\Phi$ is identified with the group $\Pi/P$ of 2.3. Our statement therefore follows from lemma 2.3.
2.12. Corollary. — Assume $\Lambda$ to be schurian, connected and locally representation-finite. Let $Z$ be a non-zero abelian group. Then $\Lambda$ is simply connected iff $H_1 \Lambda = 0$, or equivalently iff $H^1(\Lambda, Z) = 0$.

Proof. — Being free non-commutative ([BG], 4.2), $\Phi$ is trivial iff $\text{Hom}(\Phi, Z) \cong H^1(\Lambda, Z)$ is zero (2.11), or equivalently iff the largest abelian factor group $H_1 \Lambda$ of $\Phi = \Pi/P$ is trivial (2.3).

3. The standard form of a locally representation-finite algebra

Let $\Lambda$ be locally representation-finite.

3.1. The standard form $\overline{\Lambda}$ of $\Lambda$ is by definition the full subcategory formed by the projective points of the mesh-category $k(\Gamma_\Lambda)$ ([BG], 2.2 and 5.1). We know by [BG] that $\overline{\Lambda}$ is locally representation-finite, has the same Auslander-Reiten quiver as $\Lambda$ and is a degeneration of $\Lambda$ in the sense of algebraic geometry. The description of $\overline{\Lambda}$ is important from a practical point of view, because some concrete algebras are proved to be representation-finite by first exhibiting a representation-finite degeneration and then applying [GO], 4.2.

Let $x, y$ be two objects of $\Lambda$. A morphism from $x$ to $y$ in the quiver-category $kQ_\Lambda$ is by definition a formal linear combination of paths from $x$ to $y$ in the quiver $Q_\Lambda$. We consider the subspace $I_\Lambda(x, y)$ of $(kQ_\Lambda)(x, y)$ which is generated by the non-stable paths and by the differences $v - w$, where $(v, w)$ ranges over the stable contours with origin $x$ and terminus $y$ (1.5). In view of lemma 1.4, it is clear that the family $(I_\Lambda(x, y))_{x, y \in \Lambda}$ is an ideal of $kQ_\Lambda$; we call it the standard ideal.

Theorem. — The standard form $\overline{\Lambda}$ of $\Lambda$ is isomorphic to $kQ_\Lambda/I_\Lambda$.

Proof. — We first consider the case where $\Lambda$ is simply connected and therefore isomorphic to $\overline{\Lambda}$. In this case, we have $H_n \Lambda = 0$ for $n \geq 1$ (2.5 and 2.6); equivalently, the sequence of free abelian groups

$$\ldots \rightarrow C_2 \Lambda \rightarrow C_1 \Lambda \rightarrow C_0 \Lambda \rightarrow H_0 \Lambda \rightarrow 0$$

is exact and splits. If we let the functor $\text{Hom}(?, Z)$ act on it, the induced sequence remains exact and yields $H^*(\Lambda, Z) = 0$ for $n \geq 1$. In particular, we have $H^2(\Lambda, k^*) = 1$. By 2.2 this means that $\Lambda$ admits a multiplicative basis (in the terminology of [RO]), i.e. that we can choose a basic-vector $(y|x)$ in each non-zero morphism space $\Lambda(x, y)$ in such a way that $(z|y) \circ (y|x) = (z|x)$ if $\Lambda(y, z) \circ \Lambda(x, y) \neq 0$. This clearly proves our theorem in the considered case (for historical comments, see the introduction).
In the general case, we may assume $\Lambda$ connected. We then denote by $\Phi$ the fundamental group of the Auslander-Reiten quiver $\Gamma_\Lambda$ at some projective point $a^\ast$. The universal covering $\pi : \tilde{\Gamma}_\Lambda \to \Gamma_\Lambda$ induces Galois coverings

$$\begin{align*}
\bar{\Lambda} & \subset k(\tilde{\Gamma}_\Lambda) \\
\downarrow & \\
\Lambda & \subset k(\Gamma_\Lambda)
\end{align*}$$

with group $\Phi$. We infer that $\bar{\Lambda}$ is identified with $\bar{\Lambda}/\Phi([G, 2], 3.1)$. On the other hand, $\pi$ induces a Galois covering $k Q_\Lambda/I_\Lambda \to k Q_\Lambda/I_\Lambda$ (apply lemma 1.4). Accordingly, $k Q_\Lambda/I_\Lambda$ is identified with $(k Q_\Lambda/I_\Lambda)/\Phi$. Now we already know that $k Q_\Lambda/I_\Lambda \simeq \bar{\Lambda}$. The trouble is that we lack a $\Phi$-equivariant isomorphism, which would induce an isomorphism between the quotients of $k Q_\Lambda/I_\Lambda$ and $\bar{\Lambda}$ by $\Phi$.

Let $\varepsilon : k Q_\Lambda/I_\Lambda \simeq \bar{\Lambda}$ be an isomorphism. If $x \neq y$ and $\bar{\Lambda}(x, y) \neq 0$, the paths from $x$ to $y$ have a common image $\varepsilon_x$ in $\bar{\Lambda}(x, y)$. The basis vectors $\varepsilon_x$ produced by $\varepsilon$ satisfy $\varepsilon_y \circ \varepsilon_x = \varepsilon_x$ whenever $(x, y, z) \in S_2 \bar{\Lambda}$, i.e. they form a multiplicative basis $(\varepsilon_x)$ of $\bar{\Lambda}$. As a matter of fact, the map $\varepsilon \mapsto (\varepsilon_x)$ yields a bijection between the isomorphisms and the multiplicative bases. In this bijection the $\Phi$-equivariant isomorphisms are associated with the $\Phi$-invariant multiplicative bases, i.e. with the multiplicative bases $(\varepsilon_x)$ such that $\varepsilon_x \circ \phi = \varepsilon_x \circ \phi$ for all $x, y$ and each $\phi \in \Phi$. The existence of a $\Phi$-invariant multiplicative basis is proved in 3.2 below.

### 3.2. Construction of a $\Phi$-invariant multiplicative basis of $\bar{\Lambda}$.

First we choose a vector $\eta_x \neq 0$ in each $\bar{\Lambda}(x, y) \neq 0$ such that $x \neq y$. We assume that $\varepsilon_x \circ \eta_\phi = \varepsilon_y \circ \eta_x \phi$ for all $\phi \in \Phi$. This can be done because the action of $\Phi$ on $\bar{\Lambda}$ is free. In case $(x, y, z) \in S_2 \bar{\Lambda}$, we then have $\varepsilon_y \circ \eta_z = c(x, y, z) \eta_x$, where $c \in \text{Hom}(C_2 \bar{\Lambda}, \mathbb{K}^*)$ is a $\Phi$-invariant 2-cocycle, i.e. a 2-cocycle such that $c(x \phi, y \phi, z \phi) = c(x, y, z)$ for all $\phi \in \Phi$ and all $(x, y, z) \in S_2 \bar{\Lambda}$. From the lemma farther on, we infer the existence of scalars $b(x, y) \in k^*$ defined whenever $(x, y) \in S_1 \bar{\Lambda}$ and satisfying $b(x \phi, y \phi) = b(x, y)$ and $c(x, y, z) = b(y, z) b(x, z)^{-1} b(x, y)$. The required $\Phi$-invariant multiplicative basis is determined by $\varepsilon_x = b(x, y)^{-1} \eta_x$.

**Lemma.** — *Let $\mathbb{Z}$ be an abelian group. For each $n \geq 2$ and each $\Phi$-invariant $n$-cocycle $c \in \text{Hom}(C_n \bar{\Lambda}, \mathbb{Z})$, there exists a $\Phi$-invariant $b \in \text{Hom}(C_{n-1} \bar{\Lambda}, \mathbb{Z})$ whose coboundary is $c$.*
Proof. — Since \( \Phi \) acts freely on \( S^* \tilde{\Lambda} \) and \( H^* \tilde{\Lambda} \) is 0 for \( n \geq 1 \) (2.5 and 2.6), the differential complex \( C^* \tilde{\Lambda} \) yields a free resolution of the trivial \( \Phi \)-module \( Z \cong H^0 \tilde{\Lambda} \). Accordingly, the \( n \)-th cohomology group of the differential complex \( \text{Hom}_\Phi(C^* \tilde{\Lambda}, Z) \) is identified with the \( n \)-th cohomology group \( H^*(\Phi, Z) \) of \( \Phi \) with coefficients in \( Z \). This group vanishes for \( n \geq 2 \) because \( \Phi \) is free ([ML], IV, 7.3). Our lemma follows from the fact that \( \text{Hom}_\Phi(C^* \tilde{\Lambda}, Z) \) is identified with the complex of \( \Phi \)-invariant cochains.

3.3. Remarks. (a) Among other things, theorem 3.1 yields a description of the universal cover \( \tilde{\Lambda} \) ([G2], 2.1) of \( \Lambda \): Assume \( \Lambda \) to be connected. Let \( \tilde{Q}_\Lambda \) be the universal cover of \( Q_\Lambda^* \), \( K_\Lambda \) the group of constraints of \( \Lambda \), which is determined by the stable contours of \( \Lambda \) (1.5). Then \( Q_\Lambda \) equals \( Q_\Lambda^*/K_\Lambda \). The non-zero paths of \( \tilde{\Lambda} \) are the paths of \( Q_\Lambda \) whose projections on \( Q_\Lambda \) are stable (1.4). By 3.1 we obtain \( \tilde{\Lambda} \) from the quiver-category \( kQ_\Lambda^* \) by annihilating zero-paths and equalizing non-zero paths with the same ends.

(b) With the notations of 3.1, assume \( \Lambda \) to be schurian. Then \( S^* \tilde{\Lambda} \) is clearly identified with \( S^*_\Lambda/\Phi \) and \( \text{Hom}(C^*_n \Lambda, Z) \) with \( \text{Hom}_\Phi(C^*_n \tilde{\Lambda}, Z) \). As a result, lemma 3.2 tells us that \( H^2(\Lambda, k^*)=1 \), i.e. \( \Lambda \) has a multiplicative basis and is identified with its standard form \( \overline{\Lambda} \).

We leave it to the reader to interpret \( S^*_n \tilde{\Lambda}/\Phi \) and \( \text{Hom}_\Phi(C^*_n \tilde{\Lambda}, Z) \) in terms of \( \Lambda \) in the general case.

(c) With the notations of 3.1, assume that there exists a Galois covering \( F: M \rightarrow \Lambda \) with group \( G \) such that \( M \) is standard. Then \( \Lambda \) is standard: Indeed, by restriction to the connected components of \( \Lambda \) and \( M \), we can reduce the problem to the connected case. If \( \tilde{M} \) is the universal cover of \( M \), the stable paths of \( \Lambda \) are the projections of the non-zero paths of \( \tilde{M} \) by lemma 1.4 (indeed, \( F \) induces a Galois covering \( \text{Ind} M \rightarrow \text{Ind} \Lambda \) by \([\text{MP1}]\) and \([\text{G2}]\), 3.6; therefore, \( \tilde{r}_\Lambda \) is identified with \( \tilde{r}_M \), and \( \tilde{\Lambda} \) with \( \tilde{M} \)). We infer that the stable paths of \( \Lambda \) are the projections of the stable paths of \( M \). This implies isomorphism (2) in the series:

\[
\overline{\Lambda} \cong kQ_\Lambda/I_\Lambda \cong (kQ_M/I_M)/G \cong \overline{M}/G \cong M/G \cong \Lambda,
\]

where (1) and (3) follow from 3.1, and (4) from \([\text{G2}]\), 3.1.

(d) Assume that \( \Lambda \) is representation-finite and has a directed quiver. In 1979 Bongartz and Riedtmann constructed a polyhedron which is in fact the geometric realization of the 2-dimensional skeleton of \( S^*_\Lambda \) (2.1). They
conjectured that $\Lambda$ is simply-connected iff their polyhedron is so (private communication). Theorem 1.5 shows that $\Gamma_{\Lambda}$ and the polyhedron have the same fundamental group.

On this occasion, we like to repair a lack of precision in a reference to BONGARTZ' contribution to [G 2]. Knowing about the theory developed in [G 2], he had the idea of truncating the universal cover in order to get simple projectives and to start the efficient construction of the AUSLANDER-REITEN quiver which goes back to BAUTISTA and was exploited and transmitted to us by RINGEL ([G2], 4.2).

(e) Suppose that $\Lambda$ is finite and self-injective. The universal cover $\tilde{\Lambda}$ is described in [RI 2] when $\Lambda$ is of tree-class $A_\alpha$. In case $D_\alpha$, Riedtmann presented a description of $\tilde{\Lambda}$ using her classification of the translation quivers $\tilde{\Gamma}_{\Lambda}$ (Ottawa, August 1979). In Puebla (August 1980), Waschbüsch proposed a direct description of $\tilde{\Lambda}$ in the "regular" selfinjective case (see [W 2]). Some of his proofs seem defective, but it remains that Waschbüsch was the first to propose the use of contours — considered independently by Roiter [RO] — for a description of $\tilde{\Lambda}$.

REFERENCES


