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MEROMORPHIC EXTENSION OF HOLOMORPHIC FUNCTIONS WITH GROWTH CONDITIONS

BY

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ABSTRACT. — We prove that if f is a holomorphic function, defined on an open subset Ω of $\mathbb{C}^n$ except on some exceptional set P of sufficiently small Haudorff dimension, and if f satisfies certain integrability conditions in a neighbourhood of each point in P, then f extends meromorphically to Ω. We also get an upper bound for the order of the pole of the extension at each piont in P.

RÉSUMÉ. — Soit f une fonction holomorphe, définie dans un ouvert Ω dans $\mathbb{C}^n$ sauf sur un ensemble exceptionnel P ayant une dimension de Hausdorff suffisamment petite. Nous démontrons que si f satisfait à certaines conditions d'intégrabilité au voisinage de chaque point de P, alors f se prolonge en une fonction meromorphe dans Ω. Nous obtenons aussi une borne supérieure de l'ordre de pôle de l'extension à chaque point de P.

1. Introduction

Let Ω be an open subset of $\mathbb{C}^n$, $n \geqslant 1$, and denote by $O(\Omega)$ and $PSH(\Omega)$ the class of holomorphic functions on Ω and the class of plurisubharmonic functions on Ω, respectively. We do not include functions identically equal to $-\infty$ on some component of Ω in $PSH(\Omega)$.

DEFINITION 1.1. — *A meromorphic function on Ω is a collection $(V_i, g_i, h_i)_{i \in I}$ such that $(V_i)_{i \in I}$ is an open covering of Ω, $g_i, h_i \in O(V_i)$, g_i is not identically equal to 0 on any component of V_i and $g_i h_j = g_j h_i$ on $V_i \cap V_j$. The class of meromorphic functions on Ω is denoted by $M(\Omega)$.*

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Remark. — If $F = (V_i, g_i, h_i)_{i \in I} \in M(\Omega)$, then F is represented by $(f_i)_{i \in I}$ where $f_i = h_i/g_i$ at every point in V_i where g_i is different from 0. Since $g_i h_j = g_j h_i$ on $V_i \cap V_j$ we get that $f_i = f_j$ at every point in $V_i \cap V_j$ where g_i and g_j both are different from 0. Therefore F is represented by $f = h_i/g_i$ at every point in V_i where g_i is different from 0.

DEFINITION 1.2. — *Two meromorphic functions* $(V_i, g_i, h_i)_{i \in I}$ *and* $(W_j, G_j, H_j)_{j \in J}$ *are equal if* $G_j h_i = g_i H_j$ *on* $W_j \cap V_i$, $j \in J$ *and* $i \in I$.

Remark. — If $(W_j, G_j, H_j)_{j \in J}$ and $(V_i, g_i, h_i)_{i \in I}$ are meromorphic functions on Ω and if they are equal in the sense of Definition 1.2, then it follows from Definitions 1.1 and 1.2 and the remark above that $F = f$, where $F = H_j/G_j$ on W_j and $f = h_i/g_i$ on V_i are the above representations of $(W_j, G_j, H_j)_{j \in J}$ and $(V_i, g_i, h_i)_{i \in I}$, respectively, at every point where the denominators are different from 0.

Now, let $f \in M(\Omega)$ and let $f = h_i/g_i$ in V_i as above. If $z \in \Omega$, then there is an $i \in I$ such that $z \in V_i$, and since $h_i \in O(V_i)$ we have that $h_i \in L^r_{\text{loc}}(V_i)$ for all r such that $1 \leqslant r < +\infty$. Hence $f g_i \in L^r_{\text{loc}}(V_i)$ for all r, $1 \leqslant r < +\infty$. This can be stated as follows:

If $f \in M(\Omega)$, then given $z \in \Omega$ and r such that $1 \leqslant r < +\infty$, there exists a neighbourhood V of z in Ω and a function $\varphi \in PSH(V)$ such that $\int_V |f|^r e^{\varphi} \, d\lambda < +\infty$, where $d\lambda$ denotes Lebesgue measure on $\mathbb{C}^n$. (We have just to put $\varphi = r \log |g_i|$ in $V \subset V_i$.)

In this paper we study the converse situation. More precisely: let Ω be an open subset of $\mathbb{C}^n$ and P a subset of Ω. Given $f \in O(\Omega \setminus P)$ such that for each $z \in P$ there exists a neighbourhood V of z in Ω, an r, $1 \leqslant r < +\infty$, and a function $\varphi \in PSH(V)$ such that $\int_{V \setminus P} |f|^r e^{\varphi} \, d\lambda < +\infty$, find sufficient conditions on P and r such that this estimate leads to a meromorphic extension of f to Ω. Furthermore, if such an extension exists, what can be said about the order of the pole of this extension at a point $z \in P$? (Definition 2.1 explains what we mean by the order of the pole of a meromorphic function at a point.)

I wish to express my gratitude to Christer Kiselman and Urban Cegrell for many helpful discussions and valuable remarks on the subject treated in this paper.

2. Meromorphic extension

In this section Ω will always denote an open subset of $\mathbb{C}^n$, $n \geqslant 1$. Let H_d denote the d-dimensional Hausdorff measure on $\mathbb{C}^n$, or in fact on $\mathbb{R}^{2n}$, since this does not involve the complex structure on $\mathbb{C}^n$. (*Cf.* L. CARLESON [1] for an introduction to Hausdorff measures.) If $\varphi \in PSH(\Omega)$ we denote by $\nu_\varphi(z)$ the Lelong number (or density) of φ at $z \in \Omega$ (P. LELONG [7]).

DEFINITION 2.1. — *If $f \in M(\Omega)$ we define the order of the pole of f at $z \in \Omega$ as inf $\nu_{\log|g|}(z)$, where g varies over $\{ g \in O(V)$: $gf \in O(V)$, V a neighbourhood of z in Ω and g is not identically equal to 0 in any component of $V \}$.*

THEOREM 2.2. — *Let P be a closed subset of an open set Ω in $\mathbb{C}^n$ such that $H_{2n-q}(P)$ is locally finite for some $q > 1$. Let $f \in O(\Omega \setminus P)$ and assume that for each $z_0 \in P$ there exists a neighbourhood V of z_0 in Ω, an $r > q/(q-1)$ and a function $\varphi \in PSH(V)$ such that $\int_{V \setminus P} |f(z)|^r e^{\varphi(z)} \, d\lambda(z) < +\infty$. Then f can be extended meromorphically to Ω and the order of the pole of this extension at z_0 is at most $\nu_\varphi(z_0)/(r-2)$.*

Before the proof we need a few lemmas:

LEMMA 2.3. — (R. HARVEY *and* J. POLKING [4, *p.* 703, *Theorem* 1.1 (*d*) *and* (*e*), *respectively*]. *Part* (*b*) *was first proved by* B. SCHIFFMAN *in* [8].)

Suppose P is a closed subset of Ω and that $f \in O(\Omega \setminus P)$.

(*a*) *If $f \in L^p_{\text{loc}}(\Omega)$ for some p, $2 \leqslant p < +\infty$, and if $H_{2n-q}(P)$ is locally finite, $1/q + 1/p = 1$, then f extends holomorphically to Ω.*

(*b*) *If $H_{2n-2}(P) = 0$, then f extends holomorphically to Ω.*

Before the next lemma we need a definition.

DEFINITION 2.4.

(a) *A subset P of $\mathbb{R}^{2n}$ is called* polar *if there exists a subharmonic function φ on $\mathbb{R}^{2n}$ such that the restriction of φ to P is equal to $-\infty$.*

(b) *A subset P of $\mathbb{C}^n$ is called* pluripolar *if there exists a function $\varphi \in PSH(\mathbb{C}^n)$ such that the restriction of φ to P is equal to $-\infty$.*

Remark. — A pluripolar subset of $\mathbb{C}^n$ is polar as a subset of $\mathbb{R}^{2n}$, since plurisubharmonic functions on $\mathbb{C}^n$ are subharmonic on $\mathbb{R}^{2n}$.

LEMMA 2.5. — (*Cf.* L. CARLESON [1, *p.* 91, *Theorem* 2]). *If P is a polar subset of $\mathbb{R}^{2n}$, then $H_{2n-q}(P)$ is locally finite for all q, $1 \leqslant q < 2$, in fact $H_{2n-q}(P) = 0$ for all such q.*

LEMMA 2.6. — (Y.-T. SIU [10, *p.* 1201, *Lemma* 1.1]). *Let* $\varphi \in PSH(\Omega)$ *and suppose that* P *is a submanifold of* Ω *of codimension* 1 *such that the ideal sheaf of* P *is generated by a single function* $g \in O(\Omega)$. *If* $\nu_\varphi \geqslant c$ *on* P, *then* $\varphi - c \log |g| \in PSH(\Omega)$.

LEMMA 2.7. — (B. SCHIFFMAN [9, *p.* 338, *Theorem* 3 (*ii*)]). *Suppose that* P *is a closed subset of* Ω *and that* $\varphi \in PSH(\Omega \setminus P)$. *If* $H_{2n-2}(P) = 0$ *then* φ *extends to a plurisubharmonic function on* Ω.

Proof of Theorem 2.2. — Because of Lemma 2.3 (*b*) we can assume that $1 < q \leqslant 2$ and, since $H_{2n-q}(P)$ is locally finite, P has Lebesgue measure 0. Thus $\int_V |f|^r e^\varphi \, d\lambda = \int_{V \setminus P} |f|^r e^\varphi \, d\lambda$ (if f is given any values on P).

We shall now prove the existence of a meromorphic extension of f to Ω by finding for each $z_0 \in P$ a neighbourhood W of z_0 in Ω and a function $g \in O(W)$ such that g is not identically equal to 0 in any component of W and $gf \in L^p_{\text{loc}}(W)$ where $1/p + 1/q = 1$. Applying Lemma 2.3 (*a*) we then get that gf extends holomorphically to W and, since $z_0 \in P$ was arbitrarily chosen, this shows that f extends meromorphically to Ω.

Now, since $r > q/(q-1)$ we get $r > p \geqslant 2$. Put $q' = r/p > 1$ and choose p' such that $1/q' + 1/p' = 1$, i.e. $p' = r/(r-p)$. From HÖRMANDER [5, p. 98, Corollary 4.4.6 and p. 96, Theorem 4.4.4] it follows that there exists a function $G \in O(V)$ such that G is not identically equal to 0 on any component of V and $G \in L^2_{\text{loc}}(V, e^{-p'\varphi/q'}) = \{ h \colon \Omega \to \mathbb{C} \colon |h|^2 e^{-p'\varphi/q'} \in L^1_{\text{loc}}(V) \}$. Now, let W be a neighbourhood of z_0 contained in V such that $\int_W |G|^2 e^{-p'\varphi/q'} \, d\lambda < +\infty$ and $\sup_{z \in W} |G(z)| < +\infty$.

Defining $g(z) = G(z)/\sup_W |G|$ we get a function $g \in O(W)$ such that $\int_W |g|^2 e^{-p'\varphi/q'} \, d\lambda < +\infty$ and $|g(z)| \leqslant 1$ if $z \in W$.

An application of Hölder's inequality gives

$$\int_W |fg|^p \, d\lambda \leqslant \left(\int_W |f|^{p \cdot q'} e^\varphi \, d\lambda \right)^{1/q'} \left(\int_W |g|^{p \cdot p'} e^{-p'\varphi/q'} \, d\lambda \right)^{1/p'}$$
$$\leqslant \left(\int_V |f|^r e^\varphi \, d\lambda \right)^{1/q'} \left(\int_W |g|^2 e^{-p'\varphi/q'} \, d\lambda \right)^{1/p'},$$

where the last inequality follows from the fact that $pq' = r$, $pp' > 2$ and $|g| \leqslant 1$ in W. Since, by assumption on f and the choice of g, the right-hand side of this expression is finite, it follows that $fg \in L^p(W)$. So, f has a meromorphic extension to Ω, which we also denote by f.

The next thing to prove is that the order of the pole of f at z_0 is at most $\nu_\varphi(z_0)/(r-2)$. To do this we first recall a result of SKODA [12, pp. 389-394] which says that if ψ is plurisubharmonic in an open subset D of $\mathbb{C}^n$, then $e^{-\psi}$ is summable in a neighbourhood of each point $z \in D$ where $\nu_\psi(z) < 2$. Thus $e^{-p'\varphi/q'}$ is summable in a neighbourhood of z_0 if $\nu_\varphi(z_0) < 2q'/p'$. Hence, we can, shrinking W if necessary, choose g identically equal to one and we get that f is holomorphic in a neighbourhood of z_0 if $\nu_\varphi(z_0) < 2q'/p'$.

Now, let $z_0 \in P$ and assume that $\nu_\varphi(z_0) \geqslant 2q'/p'$. Choose g and h holomorphic in a connected neighbourhood W of z_0 contained in V, such g is not identically equal to 0, $gf = h$ in W and the germs of g and h at z_0 have no common irreductible factor. The last condition on g and h is possible to fulfil since the ring of germs of holomorphic functions at z_0 is a unique factorization domain (L. HÖRMANDER [5, p. 152, Theorem 6.3.2]) and Noetherian (L. HÖRMANDER [5, p. 154, Theorem 6.3.3]). (Note that this g is not always an element of $L^2_{loc}(W, e^{-p'\varphi/q'})$.) The fact that f is holomorphic outside $\{z \in W: \nu_\varphi(z) \geqslant 2q'/p'\}$ shows that

$$g^{-1}(0) \subset h^{-1}(0) \cup \{z \in W: \nu_\varphi(z) \geqslant 2q'/p'\}.$$

But $\{z \in W: \nu_\varphi(z) \geqslant 2q'/p'\}$ is an analytic subset of W (Y.-T. SIU [11, pp. 98-104, Main theorem for (1,1)-currents]; see also C. O. KISELMAN [6, p. 296, Théorème de Siu] for a much shorter proof). Thus the choice of g and h shows that there is a connected neighbourhood U of z_0 contained in W such that $\{z \in U: g(z) = 0\} \subset \{z \in U: \nu_\varphi(z) \geqslant 2q'/p'\}$. (This is easily seen if we first choose U such that g and h are relatively prime in $O(U)$ and then write $\{z \in U: h(z) = 0\}$, $\{z \in U: g(z) = 0\}$ and $\{z \in U: \nu_\varphi(z) \geqslant 2q'/p'\}$ as unions of their irreducible components. Note also that, because of this inclusion, f is holomorphic in U if the codimension of $\{z \in U: \nu_\varphi(z) \geqslant 2q'/p'\}$ is $\geqslant 2$.)

Since $\{z \in U: \nu_\varphi(z) \geqslant c\}$, $c > 0$, is an analytic subset of U and thereby polar, Lemma 2.5 gives that $H_{2n-q_1}(\{z \in U: \nu_\varphi(z) \geqslant c\}) = 0$ for all q_1, $1 \leqslant q_1 < 2$. Now, let $\varepsilon > 0$ be given such that $2 + \varepsilon < r$ and put $p_1 = 2 + \varepsilon$. Choose q_1 such that $1/p_1 + 1/q_1 = 1$ and choose p' and q' as above with p and q replaced by p_1 and q_1, respectively, and put $c = 2q'/p'$. We have that $H_{2n-q_1}(\{z \in U: \nu_\varphi(z) \geqslant 2q'/p'\}) = 0$ for all $\varepsilon > 0$. The function g chosen above does not depend on ε and in the same way as before we get that $\{ z \in U\colon g(z) = 0 \} \subset \{ z \in U\colon \nu_\varphi(z) \geqslant 2q'/p' \}$ for all $\varepsilon > 0$ and hence $\{ z \in U\colon g(z) = 0 \} \subset \{ z \in U\colon \nu_\varphi(z) \geqslant r - 2 \}$.

Before we proceed we need a lemma.

LEMMA 2.8. — *Let D be an open subset of $\mathbb{C}^n$ and let $\varphi \in PSH(D)$. Suppose that $g \in O(D)$ is such that $|g|^{-r} e^\varphi \in L^1_{loc}(D)$ for some $r > 2$. Then $\varphi - (r-2) \log |g| \in PSH(D)$.*

Before the proof of Lemma 2.8 we continue with the proof of Theorem 2.2.

We have that $|h/g|^r e^\varphi \in L^1(U)$ and from this it follows that $|g|^{-r} e^\varphi \in L^1_{loc}(U \setminus [h^{-1}(0) \cap g^{-1}(0)])$. An application of Lemma 2.8 gives that $\varphi - (r-2) \log |g| \in PSH(U \setminus [g^{-1}(0) \cap h^{-1}(0)])$. The choice of g, h and U shows that $H_{2n-2}(h^{-1}(0) \cap g^{-1}(0) \cap U) = 0$ and Lemma 2.7 gives that $\varphi - (r-2) \log |g| \in PSH(U)$. Since a plurisubharmonic function has non-negative Lelong number, we get that $\nu_{\log|g|}(z_0) \leqslant \nu_\varphi(z_0)/(r-2)$ and, by Definition 2.1, the order of the pole of f at z_0 is at most $\nu_\varphi(z_0)/(r-2)$. This ends the proof of Theorem 2.2

Proof of Lemma 2.8. — It follows from the part of the proof of Theorem 2.2 preceding Lemma 2.8 that $g^{-1}(0) \subset \{ z \in D : \nu_\varphi(z) \geqslant r - 2 \}$. Since plurisubharmonicity is a local property, it is enough to prove that $\varphi - (r-2) \log |g|$ is plurisubharmonic in a neighbourhood of each point in D. Let $z_0 \in D$ be given and let $g_1, \ldots, g_m$ be the irreducible factors of the germ of g at z_0, i.e. $g = g_1^{k_1} \ldots g_m^{k_m}$, for some $k_i \in \mathbb{N} \setminus \{0\}$, in a neighbourhood U of z_0. We choose U such that $g_1, \ldots, g_m$ are irreducible in $O(U)$ and g_i and g_j are relatively prime in $O(U)$, $i \neq j$, $1 \leqslant i, j \leqslant m$, and such that $\int_U |g|^{-r} e^\varphi \, d\lambda < +\infty$.

We shall now look at three different cases for the values of m and the k_i:s.

(1) $g = g_1$, i.e. the germ of g at z_0 is irreducible;

(2) $g = g_1^{k_1}$ for some $k_1 \geqslant 2$;

(3) $g = g_1^{k_1} \ldots g_m^{k_m}$, $m \geqslant 2$ and $k_i \in \mathbb{N} \setminus \{0\}$, $1 \leqslant i \leqslant m$.

(1) Put $X = \{ z \in U : g(z) = 0 \}$ and denote by X_{reg} and X_{sing} the set of all regular points in X and the set of all singular points in X, respectively. The assumption on g and the choice of U shows that

$$\{ h \in O(U) : h(z) = 0 \text{ if } z \in X \}$$

is generated by g. Since $H_{2n-2}(X_{sing}) = 0$, Lemma 2.3 (*b*) shows that $\{ h \in O(U \setminus X_{sing}) : h(z) = 0 \text{ if } z \in X \setminus X_{sing} = X_{reg} \}$ is also generated by

$g \in O(U \setminus X_{sing})$. But X_{reg} is a submanifold of $U \setminus X_{sing}$, so Lemma 2.6 gives that $\varphi - (r-2) \log |g| \in PSH(U \setminus X_{sing})$ and, since $H_{2n-2}(X_{sing}) = 0$, Lemma 2.7 shows that $\varphi - (r-2) \log |g| \in PSH(U)$.

(2) We get that $\int_U |g_1|^{-rk_1} e^{\varphi} \, d\lambda < +\infty$, so the part of the proof of Theorem 2.2 preceding Lemma 2.8 shows that

$$\{ z \in U : g_1(z) = 0 \} \subset \{ z \in U : \nu_\varphi(z) \geqslant rk_1 - 2 \}.$$

Part (1) above now shows that $\varphi - (rk_1 - 2) \log |g_1| \in PSH(U)$. Since $k_1 \geqslant 2$ we have that $(2k_1 - 2) \log |g_1| \in PSH(U)$, so

$$\varphi - (r-2) \log |g| = \varphi - (rk_1 - 2) \log |g_1| + (2k_1 - 2) \log |g_1| \in PSH(U).$$

(3) Put $X = \{ z \in U : g(z) = 0 \}$. The assumption on g and the choice of U shows that $\{ h \in O(U) : h(z) = 0 \text{ if } z \in X \}$ is generated by $g_1 \ldots g_m$.

We have that

$$H_{2n-2}\Big(\bigcup_{\substack{i,j=1 \\ i \neq j}}^{m} (g_i^{-1}(0) \cap g_j^{-1}(0)) \Big) = 0$$

and therefore Lemma 2.3 (*b*) shows that $\{ h \in O(U \setminus A) : h(z) = 0 \text{ if } z \in X \setminus A \}$ is also generated by $g_1 \ldots g_m$, where

$$A = \bigcup_{\substack{i,j=1 \\ i \neq j}}^{m} (g_i^{-1}(0) \cap g_j^{-1}(0)).$$

With the same arguments as in (1) and (2) we get that

$$\varphi - \sum_{i=1}^{m} (rk_i - 2) \log |g_i| \in PSH(U),$$

and since $k_i \geqslant 1$, $1 \leqslant i \leqslant m$,

$$\varphi - (r-2) \log |g| = \varphi - \sum_{i=1}^{m} (rk_i - 2) \log |g_i| + \sum_{i=1}^{m} (2k_i - 2) \log |g_i| \in PSH(U).$$

Remark. — From the proof of Lemma 2.8 it easily follows that if the denominator g of f is equal to $g_1^{k_1} \ldots g_m^{k_m}$ in a neighbourhood of z_0 and if $g_i(z_0) = 0$, $k_i > 1$ for some i, then the order of the pole of f at z_0 is strictly less than $\nu_\varphi(z_0)/(r-2)$.

COROLLARY 2.9. — *Let Ω be an open subset of $\mathbb{C}^n$ and $(q, r) \in \mathbb{R}^2$. Let P be a closed subset of Ω and $f \in O(\Omega \setminus P)$. Suppose that P and f satisfies the following two conditions*

(C_q): $H_{2n-q}(P)$ is locally finite (possibly $H_{2n-q}(P)=0$),

(C'_r): there exists a function $\varphi \in PSH(\Omega)$ such that for every $z_0 \in P$ there is a neighbourhood V of z_0 in Ω such that

$$\int_{V \setminus P} |f(z)|^r e^{\varphi(z)} \, d\lambda(z) < +\infty.$$

Then f can be extended

(1) holomorphically to Ω if

(a) $H_{2n-2}(P)=0$ (in particular if $q>2$) or

(b) $q>1$, $r \geqslant q/(q-1)$ and φ an be chosen identically equal to 0, i.e. if $f \in L^p_{loc}(\Omega)$ where $1/q+1/p=1$;

(2) meromorphically to Ω if $q>1$ and $r>q/(q-1)$. The order of the pole of the extension at z_0 is at most $\nu_\varphi(z_0)/(r-2)$.

Corollary 2.9 is just a restatement of Lemmas 2.3 (*a*) and (*b*) and Theorem 2.2 when $\varphi \in PSH(\Omega)$.

In order that f shall have an extension that is not holomorphic it is obviously necessary that $q \leqslant 2$, since P must contain the set of poles of the extension and, for the same reason, it is also necessary that the set $\{z \in \Omega : \nu_\varphi(z) \geqslant r-2\}$ has codimension 1.

Remark. — If we in the statement of Theorem 2.2 let $f \in M(\Omega \setminus P)$ instead of $f \in O(\Omega \setminus P)$, then Theorem 2.2 is still valid, i.e. f extends meromorphically to Ω and the order of the pole at $z_0 \in P$ is at most $\nu_\varphi(z_0)/(r-2)$. This follows from the fact that the set $A=\{z \in \Omega$: f is not holomorphic in a neighbourhood of $z\}$ has locally finite $(2n-2)$-dimensional Hausdorff measure, so if we just replace P by $P'=P \cup A$ then the proof of Theorem 2.2 given above applies. (Note that, since the envelope of holomorphy is the same as the envelope of meromorphy, we can still assume that $q \leqslant 2$.)

Remark. — The upper bound of the order of the pole of f given in Theorem 2.2 is the best possible in the following sense:

Given $r>2$ and $z_0 \in \Omega$, there exists a function $f \in M(\Omega)$, a neighbourhood V of z_0 and a function $\varphi \in PSH(V)$ such that $\int_V |f|^r e^\varphi \, d\lambda < +\infty$ and the order of the pole of f at z_0 is strictly greater than $\nu_\varphi(z_0)/(r-2)-1$, where the Lelong number is defined such that $\nu_{\log|g|}$ is always a natural number if g is holomorphic (P. Thie [13, p. 309, theorem 5.1]).

To prove this we choose $g \in O(\Omega)$ such that $g(z_0)=0$, e.g. let us choose $g(z)=(z_1-a_1)^m$ where $m \in \mathbb{N} \setminus \{0\}$, $z=(z_1, \ldots, z_n)$ and

$z_0 = (a_1, \ldots, a_n)$. Then $\nu_{\log|g|}(z_0) = m$ with the above definition of the Lelong number. Put $\varphi = s \log |g|$ for some s, $0 < s < r$. We get that φ, $r \log |g| - \varphi \in PSH(\Omega)$ and

$$r\nu_{\log|g|}(z_0) - \nu_\varphi(z_0) = m(r - s)$$

so, if $s > r - 2/m$, then $|g|^{-r} e^\varphi$ is summable in a neighbourhood V of z_0. Furthermore, $\nu_\varphi(z_0)/(r-2) = sm/(r-2)$. We now want to choose s such that $sm/(r-2) < 1 + m$. This is possible if $r - 2/m < (r-2)(1 + 1/m)$ i.e. if $r > 2m$. Hence, if we choose $m \in \mathbb{N} \setminus \{0\}$ such that $r > 2m$ and g as above, then we can find s such that $f = 1/g$ and $\varphi = s \log |g|$ have the required properties.

COROLLARY 2.10. — *Let P be a closed subset of Ω such that $H_{2n-q}(P)$ is locally finite for some $q > 1$. Suppose that $f \in O(\Omega \setminus P)$ is such that for each $z_0 \in P$ there exists a neighbourhood V of z_0 in Ω and a function $\varphi \in PSH(V)$ such that $\log |f| \leqslant -\varphi$ in $V \setminus P$. Then f extends meromorphically to Ω and the order of the pole of this extension at z_0 is at most $\nu_\varphi(z_0)$.*

Proof. — As in the proof of Theorem 2.2 we can assume that $q \leqslant 2$. Put $\psi = r\varphi$ where $r > q/(q-1)$, then we get that $\int_V |f|^r e^\psi \, d\lambda < +\infty$ and thus Theorem 2.2 gives that f extends meromorphically to Ω. We denote this extension also by f. Theorem 2.2 also gives that the order of the pole of f at z_0 is at most $\nu_\psi(z_0)/(r-2) = r\nu_\varphi(z_0)/(r-2)$. Since $r > q/(q-1)$ was arbitrarily chosen we get, letting $r \to +\infty$, that the order of the pole of f at z_0 is at most $\nu_\varphi(z_0)$.

Remark. — The assumption that $H_{2n-q}(P)$ is locally finite in Corollary 2.10 is essential. This is seen by the following example :

Let $\Omega = \mathbb{C} \setminus B(0, 1)$ and $P = \{z \in \mathbb{C} : z < 0\}$, then $H_{2n-1}(P) = H_1(P)$ is locally finite. Let f be the principal branch of $\log z$, then $f \in O(\Omega \setminus P)$ and $\log |f(z)| \leqslant \log |z| + \pi$ in $\Omega \setminus P$. $-\log |z| - \pi \in PSH(\Omega)$, but f has no meromorphic extension to Ω.

This example also shows that the assumption in Theorem 2.2 that $H_{2n-q}(P)$ is locally finite for some $q > 1$ is essential.

Remark. — It is possible to weaken the assumptions made on the plurisubharmonic function φ in Corollary 2.10: if we assume that $H_{2n-q}(P)$ is locally finite for some $q > 1$ and that P is a removable singularity set for the bounded plurisubharmonic functions (U. CEGRELL [2]), then we only need to assume that $\varphi \in PSH(V \setminus P)$ and $e^{2p\varphi} \in L^1_{loc}(V)$, where $1/p + 1/q = 1$ (U. CEGRELL [3]). The method of the proof of the existence of an extension under these assumptions is similar, but does not use Theorem 2.2, and we do not, in general, get an estimate of the order of the pole of f as in corollary 2.10.

COROLLARY 2.11. — *Let P be a closed subset of* Ω *such that* $H_{2n-q}(P)$ *is locally finite for some* $q > 1$. *Suppose that f is a function on* Ω *such that* $f \in O(\Omega \setminus P)$ *and* $\log|f| \in \delta PSH(\Omega)$ (*i.e.* $\log|f| = \varphi - \psi$ *a.e. where* $\varphi, \psi \in PSH(\Omega)$). *Then f extends meromorphically to* Ω.

Proof. — As in the proof of Theorem 2.2 we can assume that $q \leqslant 2$. Let $z_0 \in P$ be given. If $\log|f| = \varphi - \psi$ a.e., then there exists a neighbourhood V of z_0 in Ω and a real constant c such that $\log|f| \leqslant c - \psi$ in V. We rewrite this inequality as $\log|f| \leqslant -\psi'$ where $\psi' = -c + \psi \in PSH(V)$. From Corollary 2.10 it now follows that f extends meromorphically to Ω.

Remark. — The assumption that $H_{2n-q}(P)$ is locally finite for some $q > 1$ is essential in Corollary 2.11. This is seen by the following example : Let $n = 1$ and $\Omega = \mathbb{C} \setminus \{0\}$. Put

$$f(z) = \begin{cases} z & \text{if } |z| > 1 \\ 1/z & \text{if } |z| < 1 \quad \text{and} \quad z \neq 0. \end{cases}$$

Then $\log|f(z)| = \sup(\log|z|, -\log|z|)$ in Ω so $\log|f| \in PSH(\Omega) \subset \delta PSH(\Omega)$, but f has no meromorphic continuation to Ω. (Here $P = \{z \in \mathbb{C} : |z| = 1\}$ and thus $H_{2n-1}(P) = H_1(P)$ is locally finite.)

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