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A THEOREM ON POLARISED PARTITION RELATIONS FOR SINGULAR CARDINALS

BY

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RÉSUMÉ. — Si κ est un nombre cardinal singulaire et une limite des nombres cardinales mesurable, pour chaque $\alpha < \kappa^+$ est valide :

$$\binom{\kappa^+}{\kappa} \rightarrow \binom{\alpha}{\kappa}.$$

ABSTRACT. — If κ is a measurable limit cardinal then

$$\binom{\kappa^+}{\kappa} \rightarrow \binom{\alpha}{\kappa} \quad \text{for any } \alpha < \kappa^+.$$

In [EHR], ERDŐS, HAJNAL and RADO discussed polarized partition relations for cardinal numbers. By an easy counterexample it is shown that for all cardinals κ we have $\binom{\kappa}{\kappa} \nrightarrow \binom{\kappa}{\kappa}$. So it is a natural question to ask for which cardinals κ the following relation is valid:

$$\binom{\kappa}{\kappa^+} \rightarrow \binom{\kappa}{\kappa}.$$

PRIKRY proved [PR] that the negation of the partition relation is consistent for all successor cardinals κ . The first author proved the partition relation

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$\binom{\kappa}{\kappa^+} \rightarrow \binom{\kappa}{\kappa}$ for all measurable cardinals in [CH], the second author proved the theorem for weakly compact cardinals [WO 2].

For singular cardinals, there is a positive result in [EHR] for cardinals with cofinality ω . Here we want to show the relation for measurable limit cardinals.

1. 1. Notation.

The set theoretical notations are standard, see [DR]. Small Greek letters denote ordinals, κ, λ are infinite cardinals.

An ultrafilter U is κ -complete iff for all n. e. sets $X \subset U, |X| < \kappa, \cap X \in U$.

The cardinal κ is measurable iff there exists a κ -complete non-principal ultrafilter on κ ; κ is a measurable limit cardinal iff there exists a strongly monotone increasing sequence of cardinals $(\kappa_\nu | \nu < \text{cf } \kappa)$, such that all κ_ν are measurable and $\lim_{\nu < \text{cf } \kappa} \kappa_\nu = \kappa$.

Let $\mathcal{P} = (P, \leq)$ be a partially ordered set. \mathcal{P} is a forcing set iff there exists a $O_p \in P$ such that for all $p \in P, O_p \leq p$. If $p, q \in P$ and $p \leq q$ then q is called an extension of p . A subset D of P is P -dense in \mathcal{P} iff every $p \in P$ has an extension in D .

Let $\mathcal{P} = (P, \leq)$ be a forcing set, and \mathcal{D} be a family of dense subsets of \mathcal{P} . A subset G of P is \mathcal{D} -generic iff:

- (i) for all $p \in G$ and $q \leq p, q \in G$;
- (ii) for all $p, q \in G, p$ and q have a common extension in G ;
- (iii) for all dense sets $D \in \mathcal{D}, G \cap D \neq \emptyset$.

A subset K of P is an α -chain iff (K, \leq) is a total ordering and has order type α .

For ordinals α, β, γ and δ , the polarised partition relation:

$$(1) \quad \binom{\alpha}{\beta} \rightarrow \binom{\gamma}{\delta},$$

has the following meaning:

(1') Let $\alpha \times \beta = I_0 \cup I_1$. Then there exists a subset $A \subset \alpha$, $\text{type}(A) = \gamma$ and a subset $B \subset \beta$, $\text{type}(B) = \delta$ and $A \times B \subset I_0$ or $A \times B \subset I_1$.

2. Two simple remarks

Using the axiom of choice we can trivially prove the following:

PROPOSITION 1. — *Let $\mathcal{P}=(P; \leq)$ be a forcing set and κ be an infinite cardinal. Let for all $\xi < \kappa$ and ξ -chain $K \subset P$ there exists a $p \in P \setminus K$ such that for all $q \in K, q \leq p$ (\mathcal{P} is closed under unions of chains of length $< \kappa$). If \mathcal{D} is a system of P -dense sets and $|\mathcal{D}| \leq \kappa$, then there exists a \mathcal{D} -generic set $G \subset P$.*

The following proposition is also clear:

PROPOSITION 2. — *We have:*

$$\binom{\alpha}{\beta} \rightarrow \binom{\gamma}{\delta},$$

iff for every family $(X_\nu : \nu < \beta)$ of subsets of α , there exists an $I \subset \beta$, $\text{type}(I) = \delta$ such that $\text{type}(\bigcap_{\nu \in I} X_\nu) \geq \gamma$ or $\text{type}(\bigcap_{\nu \in I} (\alpha - X_\nu)) \geq \gamma$.

3. Our main result is the following theorem, which generalizes results in [CH] and [WO 1].

THEOREM. — *Let κ be a singular, measurable limit cardinal. Then for any $\alpha < (\text{cf } \kappa)^+ \cdot \kappa$:*

$$\binom{\kappa}{\kappa^+} \rightarrow \binom{\kappa}{\alpha}$$

Proof. — Let κ be a singular measurable limit cardinal, $\text{cf } \kappa < \kappa$ and let $(\kappa_\nu : \nu < \text{cf } \kappa)$ be a monotonic strictly increasing sequence of measurable cardinals such that:

$$\text{cf } \kappa < \kappa_0 < \dots < \kappa_\nu < \dots < \kappa: \quad \nu < \kappa;$$

$$\lim_{\mu < \nu} \kappa_\mu < \kappa_\nu \quad \text{for any } \nu < \text{cf } \kappa$$

and:

$$\lim_{\nu < \text{cf } \kappa} \kappa_\nu = \kappa.$$

Let $\kappa = \bigcup_{\nu < \text{cf } \kappa} M_\nu$, where $M_\nu = \kappa_\nu$ for all $\nu < \text{cf } \kappa$. Let U_ν be a κ_ν -complete non-principal ultrafilter on M_ν for any $\nu < \text{cf } \kappa$ and \mathcal{D}_0 be an uniform ultrafilter on $\text{cf } \kappa$.

We define a product ultrafilter \mathcal{D} on κ :

$$\mathcal{D} = \{ X \subset \kappa : \{ \nu < \text{cf } \kappa : X \cap M_\nu \in U_\nu \} \in \mathcal{D}_0 \}.$$

Let $(X_\rho : \rho < \kappa^+)$ be an arbitrary family of subsets of κ . According to Proposition 2 it is necessary to show that there exists such an $I \subset \kappa^+$, type $(I) = \alpha$ that:

$$|\bigcap_{\zeta \in I} X_\zeta| = \kappa \quad \text{or} \quad |\bigcap_{\zeta \in I} (\kappa \setminus X_\zeta)| = \kappa.$$

As \mathcal{D} is an ultrafilter on κ ; we can suppose without loss of generality that:

$$E_0 = \{ \zeta < \kappa^+ : X_\zeta \in \mathcal{D} \} \text{ has power } \kappa^+.$$

Thus for any $\zeta \in E_0$, $C_\zeta = \{ \gamma < \text{cf } \kappa : X_\zeta \cap M_\gamma \in U_\gamma \} \in \mathcal{D}_0$ and so C_ζ has power $\text{cf } \kappa$ for any $\zeta \in E_0$. Because $2^{\text{cf } \kappa} < \kappa$, there exists such $E_1 \subset E_0$, $|E_1| = \kappa^+$ that for all $\zeta_1, \zeta_2 \in E_1$:

$$C_{\zeta_1} = C_{\zeta_2} = C, \quad |C| = \text{cf } \kappa.$$

We denote for simplicity $A = \bigcup_{\mu \in C} M_\mu$ and $Y_\zeta = X_\zeta \cap A$ for $\zeta \in E_1$. Without loss of generality we can suppose that $C = \text{cf } \kappa$. Thus:

$$(2) \quad A = \bigcup_{\mu < \text{cf } \kappa} M_\mu$$

and for all $\rho \in E_1$

$$(3) \quad Y_\zeta \subseteq A \text{ and } Y_\zeta \cap M_\nu \in U_\nu \text{ for all } \nu < \text{cf } \kappa, \text{ where } |E_1| = \kappa^+.$$

Let $w \subseteq A$ and $w \subseteq \bigcup_{\nu < \mu} M_\nu$ for some $\mu < \text{cf } \kappa$. We denote:

$$T_w = \{ \zeta \in E_1 : w \subset Y_\zeta \}$$

and call w exceptional (symbolically, $w \in Ex$), if $|T_w| \leq \kappa$. Since κ is a strong limit cardinal, $\sum_{\alpha < \kappa} 2^\alpha = \kappa$, the number of all sets $w \subseteq \bigcup_{\nu < \mu} M_\nu$ for some $\mu < \text{cf } \kappa$, is at most κ .

Thus for:

$$E_2 = E_1 \setminus \bigcup_{w \in Ex} T_w,$$

we have:

$$|E_2| = \kappa^+.$$

We can assume without loss of generality that:

$$E_2 = \kappa^+.$$

In particular, if $w \subseteq A$ and w is bounded in κ and $w \subseteq Y_\zeta$ for some $\zeta < \kappa^+$, then w is not exceptional and $|\{ \eta < \kappa^+ : w \subseteq Y_\eta \}| = \kappa^+$.

Now we define a forcing set $\mathcal{P} = (P; \leq)$. Let P be the set of all pairs $\tau = (C; D)$ such that:

(i) there exists such $\xi < \text{cf } \kappa$, that $\xi \geq 1$:

$$(4) \quad C \subset \bigcup_{\mu < \xi} M_\mu$$

and:

- (5) $|C \cap M_\mu| = \kappa_\mu$ for all $\mu < \xi$;
- (ii) $D \subseteq \kappa^+$;
- (iii) $C \subseteq \bigcap_{\zeta \in D} Y_\zeta$;
- (iv) $|D| \leq |C|$;
- (v) $M_0 \cap Y_0 \subseteq C$ and $O \in D$.

Because $M_0 \cap Y_0 \in U_0$, $|M_0 \cap Y_0| = \kappa_0$, C is infinite. By (iii) and (v), $C \subset Y_0$, so if $(C; D) \in P$, then C is not exceptional.

For $\tau_i = (C_i; D_i) \in P$ we put $\tau_0 \leq \tau_1$ iff $C_0 \subseteq C_1$, $D_0 \subseteq D_1$. Then $\mathcal{P} = \{P; \leq\}$ is a forcing set with a minimal element $(M_0 \cap Y_0; \{0\}) \in P$.

LEMMA 3. — *The set \mathcal{P} is closed under union of chains of length $< \text{cf } \kappa$.*

Proof of lemma 3. — Let $\{\tau_v : v < \xi\}$ be a chain in \mathcal{P} of length ξ , $\xi < \text{cf } \kappa$, i. e. $\tau_{v_1} \leq \tau_{v_2}$ for $v_1 \leq v_2 < \xi$.

Case 1: $\xi = \eta + 1$. So $\tau_\eta = (C; D)$ is the greatest element in the chain. There exists such an $\rho < \text{cf } \kappa$, $1 \leq \rho$, that $C \subseteq \bigcup_{\mu < \rho} M_\mu$ and $|C \cap M_\mu| = \kappa_\mu$ for all $\mu < \rho$ and $C \subseteq \bigcap_{\zeta \in D} Y_\zeta$. As C is not exceptional by definition of \mathcal{P} and $|D| < \kappa$, there exists such a $\zeta_0 \in \kappa^+ \setminus D$, that $C \subseteq Y_{\zeta_0}$. Then $(C; D \cup \{\zeta_0\})$ is an element of P and $(C; D) \leq (C; D \cup \{\zeta_0\})$, where $(C; D) \neq (C; D \cup \{\zeta_0\})$.

Case 2: ξ is a limit ordinal. If $\tau_v = (C_v; D_v) : v < \xi$, then $(\bigcup_{v < \xi} C_v; \bigcup_{v < \xi} D_v)$ belongs to P as $|\bigcup_{v < \xi} D_v| < \kappa$ for $|D_v| < \kappa : v < \xi$ and $\xi < \text{cf } \kappa$. For all $\mu < \xi$, $\tau_\mu \leq (\bigcup_{v < \xi} C; \bigcup_{v < \xi} D_v)$ and Lemma 3 is proved. \square

We shall now define a family of dense subsets of \mathcal{P} in order to apply Proposition 1.

We put for $\xi < \text{cf } \kappa$:

$$\Delta_\xi = \{(C; D) \in P : |C \cap M_\xi| = \kappa_\xi\}.$$

LEMMA 4. — *For any $\xi < \text{cf } \kappa$, Δ_ξ is dense in \mathcal{P} .*

Proof of lemma 4. — Let $\tau_0 = (C; D) \in P$ and $(C; D) \notin \Delta_\xi$. Then there exists an $\eta \leq \xi$ such that $C \subseteq \bigcup_{\mu < \eta} M_\mu$ and since $|D| \leq |C|$, $|D| \leq \sum_{\mu < \eta} \kappa_\mu < \kappa$. Because $Y_\zeta \cap M_\rho \in U_\rho$ for all $\zeta \in D$, $\eta \leq \rho \leq \xi$ and all U_ρ are κ_η -complete and uniform on $M_\rho : \eta \leq \rho \leq \xi$, we have:

$$\begin{aligned} \bigcap_{\zeta \in D} (Y_\zeta \cap M_\rho) \in U_\rho, \quad \eta \leq \rho \leq \xi, \\ \bigcap_{\zeta \in D} (Y_\zeta \cap M_\rho) = \kappa_\rho \quad \text{for } \eta \leq \rho \leq \xi. \end{aligned}$$

Thus $\tau_1 = (C \cup (\bigcap_{\zeta \in D} Y_\zeta \cap (\bigcup_{\eta \in \rho \leq \xi} M_\rho)), D)$ belongs to P and $\tau_1 \in \Delta_\xi$ by construction. Also $\tau_0 = (C; D) \leq \tau_1$ and Δ_ξ is dense. \square

Let's take a sequence $(\alpha_\lambda : \lambda < \kappa^+)$ such that $\alpha_\lambda \in \{\kappa_\nu : \nu < \text{cf } \kappa\}$ and each κ_ν has κ^+ many appearances in the sequence $(\alpha_\lambda : \lambda < \kappa^+)$.

LEMMA 5. — *There exists a sequence $(\xi_\lambda : \lambda < \kappa^+)$ of elements κ^+ , such that the sets:*

$$\nabla_\lambda = \{(C; D) \in P : |D \cap \{\zeta : \xi_\lambda \leq \zeta < \xi_{\lambda+1}\}| \geq \alpha_\lambda\}$$

are dense in \mathcal{P} for all $\lambda < \kappa^+$.

Proof of lemma 5. — Let $\xi_0 < \kappa^+$ be arbitrary. We shall construct the sequence $(\xi_\lambda : \lambda < \kappa^+)$ by induction.

Suppose that we have constructed $(\xi_\lambda : \lambda \leq \delta)$ for $\delta < \kappa^+$ such that all $\nabla_\zeta : \zeta < \delta$ are dense in \mathcal{P} . Let us suppose however that there are no $\xi_{\delta+1}$ such that ∇_δ is dense in \mathcal{P} . Then for any $\beta > \xi_\delta$, $\beta < \kappa^+$, there is $\tau_\beta = (C_\beta; D_\beta) \in P$ such that τ_β is not extended by a member of ∇_δ , where (in definition of ∇_δ) we put $\xi_{\delta+1} = \beta$.

For each $C_\beta : \xi_\delta < \beta < \kappa^+$ there exists an $\eta < \text{cf } \kappa$ with $C_\beta \subseteq \bigcup_{\mu < \eta} M_\mu$. Because κ is a strong limit, $\sum_{\alpha < \kappa} 2^\alpha \leq \kappa$, we can find $E \subset \kappa^+$, $|E| = \kappa^+$ with:

$$(6) \quad C_{\beta_1} = C_{\beta_2} = C \quad \text{for all } \beta_1, \beta_2 \in E.$$

Case 1: Let $|C| \geq \alpha_\delta$. Since C is not an exceptional set (by (6) and definition of P), $|\{\zeta > \xi_\delta : C \subseteq Y_\zeta\}| = \kappa^+$. Let us take $M \subseteq \kappa^+$ with:

$$M \subseteq \{\zeta > \xi_\delta : C \subseteq Y_\zeta\} \quad \text{and} \quad |M| = \alpha_\delta.$$

Then we take $\beta \in E$ such that $\beta > M$, i. e. for any $\zeta \in M$, $\beta > \zeta$. We obtain by $|C| \geq \alpha_\delta$:

$$|D_\beta \cup M| \leq |C_\beta| + \alpha_\delta = |C| + \alpha_\delta = |C|;$$

and by our choice of M :

$$C \subseteq \bigcap_{\zeta \in (D_\beta \cup M)} Y_\zeta.$$

In other words, $(C; D_\beta \cup M) \in P$. Further, for $C_\beta = C : \beta \in E$, we have $\tau_\beta \leq (C; D_\beta \cup M)$. But $(C; D_\beta \cup M) \in \nabla_\delta$, where we put $\xi_{\delta+1} = \beta$, and so we come to a contradiction.

Case 2: $|C| < \alpha_{\delta}$, where $\alpha_{\delta} = \kappa_v$. Let ρ be the least ordinal with $|C| < \kappa_{\rho}$; $\rho \leq v$.

Let $\beta_0 \in E$ be arbitrary; since $|D_{\beta_0}| \leq |C|$ and $C \subseteq \bigcup_{\mu < \rho} M_{\mu}$, $|D_{\beta_0}| \leq \sum_{\mu < \rho} \kappa_{\mu} < \kappa_{\rho}$. Consequently, if:

$$C_1 := \bigcup_{\rho \leq \mu \leq v} \bigcap_{\xi \in D_{\beta_0}} (M_{\mu} \cap Y_{\xi}),$$

then $C \cup C_1$ is not exceptional as $C \cup C_1 = C_{\beta_0} \cup C_1 \in Y_{\xi}$ for any $\xi \in D_{\beta_0}$. From this it follows that:

$$|\{\zeta > \xi_{\delta} : C \cup C_1 \subseteq Y_{\zeta}\}| = \kappa^+;$$

we choose $M \subseteq \{\zeta > \xi_{\delta} : C \cup C_1 \subseteq Y_{\zeta}\}$; $|M| = \kappa_v = \alpha_{\delta}$ and $\beta \in E$ with $\beta > M$.

From $|D_{\beta}| \leq |C|$ it follows $|D_{\beta}| < \kappa_{\rho}$ and $\bigcap_{\zeta \in D_{\beta}} (M_{\mu} \cap Y_{\zeta}) \in U_{\mu}$ for all μ , $\rho \leq \mu \leq v$. Then by definition of C_1 , $(M_{\mu} \cap C_1) \in U_{\mu}$, $\rho \leq \mu \leq v$. We put:

$$D_2 = D_{\beta} \cup M; \quad C_2 = C \cup \bigcup_{\rho \leq \mu \leq v} \bigcap_{\xi \in D_{\beta}} (M_{\mu} \cap Y_{\xi} \cap C_1).$$

In these notations, $|D_2| \leq \kappa_v \leq |C_2|$ and $C_2 \subseteq \bigcap_{\zeta \in D_2} Y_{\zeta}$ and $(C_{\beta}; D_{\beta}) \leq (C_2; D_2)$. But $|D_2 \cap \{\zeta : \zeta_{\delta} \leq \zeta < \beta\}| = \kappa_v$ and this contradicts the construction of $\tau_{\beta} = (C_{\beta}; D_{\beta})$. Lemma 5 is proved. \square

Now we can prove our theorem. Let $\alpha < (\text{cf } \kappa)^+ \cdot \kappa$; then there is a sequence $\sum_{i \in I} \xi_{\lambda_i} \geq \alpha$.

We consider the following family:

$$\mathcal{D} = \{ \Delta_{\xi} : \xi < \text{cf } \kappa \} \cup \{ \nabla_{\lambda_i} : i \in I \},$$

of $\text{cf } \kappa$ dense subsets of \mathcal{P} .

By Proposition 1 there exists a \mathcal{D} -generic set $G \subset P$. We put:

$$I = \bigcup \text{rg } G = \bigcup \{ D : (C; D) \in G \};$$

$$J = \bigcup \text{dom } G = \bigcup \{ C : (C; D) \in G \}.$$

Then $I < \kappa^+$, $\text{typ } (I) \geq \alpha$; $J < \kappa$, $|J| = \kappa$ and by Proposition 2, the theorem is proved.

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