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ELEMENTARY PROOF OF A THEOREM OF BRUHAT-TITS-ROUSSEAU AND OF A THEOREM OF TITS

BY

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RÉSUMÉ. — Nous donnerons une démonstration élémentaire d'un théorème de Bruhat, Tits et Rousseau, et aussi d'un théorème de Tits.

ABSTRACT. — We give an elementary proof of a theorem of Bruhat, Tits and Rousseau, and also of a theorem of Tits.

Let k be a field with a non-trivial non-archimedean valuation v. We shall assume that the valuation v has a (up to equivalence) unique extension to any finite field extension of k, or, equivalently, k is *henselian* for v (i. e. the Hensel's lemma holds in k with respect to v). We fix an algebraic closure $\mathscr{K}$ of k and shall denote the unique valuation on it, which extends the given valuation on k, again by v. Let K be the separable closure of k in $\mathscr{K}$; the extended valuation on K is obviously invariant under the Galois group $\mathrm{Gal}(K/k)$.

Let V be a finite dimensional k-vector space. Let G be a connected reductive k-subgroup of $\mathbf{SL}(V)$. For any extension L of k contained in $\mathscr{K}$, let $G(L)$ be the group of L-rational points of G endowed with the Hausdorff topology and the bornology induced by the valuation on L. Let $G(k)^+$ be the normal subgroup of $G(k)$ generated by the k-rational points of the unipotent radicals of parabolic k-subgroups of G.

G is said to be *isotropic* over k if G contains a non-trivial k-split torus, and *k-anisotropic* (or anisotropic over k) otherwise.

The object of this note is to give a simple proof of the following theorem proved first by F. Bruhat and J. Tits in case k is a discretely valuated complete field with perfect residue field and then in general by G. Rousseau in his thesis (Orsay, 1977).

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THEOREM (BTR). — *$G(k)$ is bounded if and only if G is anisotropic over k.*

Remark. — Thus in case k is a non-discrete locally compact field, $G(k)$ is compact if and only if G is anisotropic over k.

We shall also give a simple proof of the following (unpublished) theorem of J. Tits:

THEOREM (T). — *Let G be semi-simple and almost k-simple. Then any proper open subgroup of $G(k)^+$ is bounded.*

Acknowledgment. — The proof, given below, of Theorem (BTR) is based on a suggestion of G. A. Margulis that it should be possible to use the following lemma (Lemma 1) to prove Theorem (BTR). I had originally used the lemma to give a simple proof of Theorem (T). The comments of J. Tits on an earlier version have lead to further simplifications in the proofs of both the theorems. I thank Margulis and Tits heartily.

LEMMA 1. — *Let H be a subgroup of $G(k)$ which is dense in G in the Zariski topology. Assume that H is unbounded. Then there is an element h of H which has an eigenvalue α with $v(\alpha)<0$.*

Proof. — Let:

$$V=V_0\supset V_1\supset \ldots \supset V_r\supset V_{r+1}=\{0\},$$

be a flag of G-invariant vector subspaces (not necessarily defined over k) such that for $0\leqslant i\leqslant r$, the natural representation ρ_i of G on $W_i=V_i/V_{i+1}$ is absolutely irreducible. Let $\rho(=\oplus\ _i)$ be the natural representation of G on $\oplus W_i$; ρ is defined over a finite galois extension of k. The kernel of ρ is obviously a unipotent normal subgroup of G, and as G is reductive, we conclude that ρ is faithful. Now, as H is a unbounded subgroup of $G(k)$, $\rho(H(k))$ is unbounded, and hence there is a non-negative integer a, $a\leqslant r$, such that $\rho_a(H(k))$ is unbounded.

Now assume, if possible, that the eigenvalues of all the elements of H lie in the local ring of the valuation on $\mathscr{K}$. Then, the trace form of ρ_a, restricted to H, also takes values in the local ring of the valuation (this ring is bounded!). But since W_a is an absolutely irreducible G-module, and since H is dense in G in the Zariski topology, $\rho_a(H)$ spans $\operatorname{End}(W_a)$. So, in view the non-degeneracy of the trace form, we conclude that $\rho_a(H)$ is bounded (*see* TITS [5], Lemma 2.2). This is a contradiction, which proves the lemma.

Proof of Theorem (BTR). — If T is a one-dimensional k-split torus, then $T(k)$ is isomorphic to k^* and hence it is unbounded. This implies that if G is isotropic over k, then $G(k)$ is unbounded. We shall now assume that $G(k)$ is unbounded and prove the converse.

It is well known that $G(k)$ is dense in G in the Zariski topology ([1], 18.3), hence, according to the preceding lemma, there is an element $g \in G(k)$ which has an eigenvalue α with $v(\alpha) \neq 0$. Now, in case k is of positive characteristic, after replacing g by a suitable power, we shall assume that g is semi-simple. In case k is of characteristic zero, let $g = u.s = s.u$ be the Jordan decomposition of g, with u (resp. s) unipotent (resp. semi-simple). Then $u, s \in G(k)$, and the eigenvalues of g are the same as that of s. Thus we may (and we shall), after replacing g by s, again assume that g is semi-simple.

Now there is a maximal torus S in G defined over k, such that $g \in S(k)$. (*See* Borel-Tits [2], Proposition 10.3 and Theorem 2.14a; note that according to Theorem 11.10 of [1], g is contained in a maximal torus of G.) Since any absolutely irreducible representation of a torus is 1-dimensional, there is a character χ of S, χ defined over a finite galois extension $\mathfrak{K}$ of k, such that $\chi(g) = \alpha$. Let $m = [\mathfrak{K}: k]$. Then:

$$v((\textstyle\sum_{\gamma \in \mathrm{Gal}(\mathfrak{K}/k)} {}^{\gamma}\chi)(g)) = mv(\chi(g)) = mv(\alpha) \neq 0.$$

Thus the character $\sum_{\gamma \in \mathrm{Gal}(\mathfrak{K}/k)} {}^{\gamma}\chi$ is non-trivial. On the other hand, it is obviously defined over k. Thus S admits a non-trivial character defined over k, and hence it contains a non-trivial k-split torus. This proves that in case $G(k)$ is unbounded, G is isotropic over k.

We shall now assume that G is semi-simple and almost k-simple.

Notation. — For $g \in G(k)$, let $\mathscr{P}_g$ be the subset of $G(K)$ consisting of those x in $G(K)$ for which the sequence $\{g^i x g^{-i}\}_{i>0}$ is contained in a bounded subset of $G(K)$, and let $\mathscr{U}_g$ be the subset consisting of those x in $G(K)$ for which the sequence $\{g^i x g^{-i}\}_{i>0}$ converges to the identity. It is obvious that $\mathscr{P}_g$ is a subgroup of $G(K)$, and $\mathscr{U}_g$ is a normal subgroup of $\mathscr{P}_g$.

We let $\mathscr{P}_g^-$ denote $\mathscr{P}_{g^{-1}}$ and $\mathscr{U}_g^-$ denote $\mathscr{U}_{g^{-1}}$.

In the sequel we shall denote the adjoint representation of an algebraic group on its Lie algebra by Ad.

LEMMA 2. — *Let t be an element of $G(k)$ such that* Ad t *has an eigenvalue* α *with* $v(\alpha)\neq 0$. *Then:*

(i) $\mathscr{P}_t$ *is the group of K-rational points of a proper parabolic k-subgroup* P_t *of G and* $\mathscr{U}_t$ *is the group of K-rational points of the unipotent radical* U_t *of* P_t;

(ii) P_t^- $(:=P_{t^{-1}})$ *is opposed to* P_t.

Proof. — Since for any integer $n>0$, $\mathscr{P}_{t^n}=\mathscr{P}_t$ and $\mathscr{U}_{t^n}=\mathscr{U}_t$, in case k is of positive characteristic, after replacing t by a suitable (positive) power of t, we shall assume that t is semi-simple. In case k is of characteristic zero, let $t=u.s=s.u$ be the Jordan decomposition of t with u (resp. s) unipotent (resp. semi-simple). Then s, $u\in G(k)$ and the eigenvalues of Ad t are the same as that of Ad s. Since the cyclic group generated by a unipotent element is bounded, we see easily that $\mathscr{P}_t=\mathscr{P}_s$ and $\mathscr{U}_t=\mathscr{U}_s$. Thus we may (and we shall) assume, after replacing t by s, that t is semi-simple.

Now there is a maximal torus T of G, defined over K, such that $t\in T(K)$. Since K is separably closed, any K-torus splits over K. Let $\mathfrak{t}+\sum_{\varphi\in\Phi}\mathfrak{g}^\varphi$ be the root space decomposition of the Lie algebra $\mathfrak{g}$ of G with respect to T; where $\mathfrak{t}$ is the Lie algebra of T and Φ is the set of roots. According to BOURBAKI [4], Chapitre VI, paragraph 1, Proposition 22, there is an ordering on Φ such that the subset $\{\varphi\mid\varphi\in\Phi,\ v(\varphi(t))>0\}$ is contained in the set Φ^+ of roots positive with respect to this ordering; let $\Delta\subset\Phi$ be the set of simple roots.

For a subset Θ of Δ, let T_Θ be the identity component of $\bigcap_{\theta\in\Theta}\operatorname{Ker}\theta$ and let M_Θ be the centralizer of T_Θ in G. Let $\mathfrak{u}_\Theta=\sum_{\varphi\in\Phi^+-\langle\Theta\rangle}\mathfrak{g}^\varphi$ (resp. $\mathfrak{u}_\Theta^-=\sum_{\varphi\in\Phi^+-\langle\Theta\rangle}\mathfrak{g}^{-\varphi}$), and U_Θ (resp. U_Θ^-) be the connected unipotent K-subgroup of G, normalized by T, and with Lie algebra $\mathfrak{u}_\Theta$ (resp. $\mathfrak{u}_\Theta^-$). Let $P_\Theta=M_\Theta.U_\Theta$ and $P_\Theta^-=M_\Theta.U_\Theta^-$. Then P_Θ and P_Θ^- are opposed parabolic K-subgroups of G, and if $\Theta\neq\Delta$, these subgroups are proper. Moreover, U_Θ (resp. U_Θ^-) is the unipotent radical of P_Θ (resp. P_Θ^-).

Now let $\Pi=\{\delta\in\Delta\mid v(\delta(t))=0\}$. Then since Ad t has an eigenvalue α with $v(\alpha)\neq 0$, Π is a proper subset of Δ. It is obvious that $P_\Pi(K)\subset\mathscr{P}_t$, $P_\Pi^-(K)\subset\mathscr{P}_t^-$ and $U_\Pi(K)\subset\mathscr{U}_t$. Since $\mathscr{P}_t$ contains $P_\Pi(K)$, *it equals* $P_\Theta(K)$ for a subset Θ of Δ, containing Π. But since the action of Ad t on $\mathfrak{u}_\Pi^-(K)$ is "expanding", we conclude at once that $\Theta=\Pi$ and hence, $P_\Pi(K)=\mathscr{P}_t$. A similar argument shows that $P_\Pi^-(K)=\mathscr{P}_t^-$. We set $P_t=P_\Pi$ and $P_t^-=P_\Pi^-$.

To prove the second assertion of (i) we need to show that $U_\Pi(K)=\mathscr{U}_t$. For this purpose we observe that $U_\Pi(K)\subset\mathscr{U}_t$ and since

$\mathscr{P}_t = P_\Pi(K) = M_\Pi(K) . U_\Pi(K)$:

$$\mathscr{U}_t = (M_\Pi(K) \cap \mathscr{U}_t) . U_\Pi(K);$$

also $\mathscr{U}_t$ and hence $M_\Pi(K) \cap \mathscr{U}_t$ are normalized by $T(K)$. We now note that the Lie algebra of M_Π is $\mathfrak{t} + \sum_{\varphi \in \pm \langle \Pi \rangle} \mathfrak{g}^\varphi$, and $v(\varphi(t)) = 0$ for $\varphi \in \langle \Pi \rangle$. From these observations it is evident that $M_\Pi(K) \cap \mathscr{U}_t$ is trivial, and hence, $\mathscr{U}_t = U_\Pi(K)$.

Now to complete the proof of the lemma it only remains to show that both P_t and P_t^- are defined over k. But this is obvious, from the Galois criteria, in view of the fact that $\mathscr{P}_t$ and $\mathscr{P}_t^-$ are stable under $\Gamma = \mathrm{Gal}(K/k)$ since t is a k-rational element, and $\mathscr{P}_t (= P_t(K))$ is dense in P_t, whereas $\mathscr{P}_t^- (= P_t^-(K))$ is dense in P_t^- in the Zariski topology.

LEMMA 3. — *Let $t \in G(k)$ be such that* Ad t *has an eigenvalue α with $v(\alpha) \neq 0$. Then $\mathscr{U}_t(k) (:= \mathscr{U}_t \cap G(k))$ and $\mathscr{U}_t^-(k) (:= \mathscr{U}_t^- \cap G(k))$ together generate $G(k)^+$.*

Proof. — According to the preceding lemma, $\mathscr{U}_t(k)$ and $\mathscr{U}_t^-(k)$ are the groups of k-rational points of the unipotent radicals of two opposed proper parabolic k-subgroups of G. Hence, according to BOREL-TITS [3], Proposition 6.2 v, $\mathscr{U}_t(k)$ and $\mathscr{U}_t^-(k)$ together generate $G(k)^+$.

Proof of Theorem (T). — Let $\mathscr{G}$ be the adjoint group of G and $\pi: G \to \mathscr{G}$ be the natural (central) isogeny. Then since π is a *finite* morphism, the induced map $G(k) \to \mathscr{G}(k)$ is a proper map, i. e., the inverse image of a bounded subset of $\mathscr{G}(k)$ is bounded.

Now let H be an unbounded open subgroup of $G(k)^+$. Then, clearly, H is dense in G in the Zariski topology and hence, $\pi(H)$ is an unbounded Zariski-dense subgroup of $\mathscr{G}(k)$. Now Lemma 1 (applied to $\pi(H) (\subset \mathscr{G}(k))$) implies that there is an element h of H such that Ad h has an eigenvalue α with $v(\alpha) \neq 0$.

Let $\mathscr{U}_h(k) = \mathscr{U}_h \cap G(k)$ and $\mathscr{U}_h^-(k) = \mathscr{U}_h^- \cap G(k)$. Then as H is open in $G(k)^+$, $H \cap \mathscr{U}_h(k)$ is an open subgroup of $\mathscr{U}_h(k)$, and obviously

$$\bigcup_{n>0} h^n (H \cap \mathscr{U}_h^-(k)) h^{-n} = \mathscr{U}_h^-(k).$$

Thus H contains both $\mathscr{U}_h(k)$ and $\mathscr{U}_h^-(k)$. But according to Lemma 3, $\mathscr{U}_h(k)$ and $\mathscr{U}_h^-(k)$ together generate $G(k)^+$. Therefore, $H = G(k)^+$. This proves the theorem.

REFERENCES

[1] BOREL (A.). — *Linear algebraic groups*, W. A. BENJAMIN, éd., New York, 1969.

[2] BOREL (A.) and TITS (J.). — GROUPES RÉDUCTIFS, *Publ. Math. I.H.E.S.* No., 27, 1965, p. 55-150.

[3] BOREL (A.) and TITS (J.). — Homomorphisms « abstraits » de groupes algébriques simples, *Ann. Math.*, Vol. 97, 1973, p. 499-571.

[4] BOURBAKI (N.). — *Groupes et Algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[5] TITS (J.). — Free subgroups in Linear Groups, *J. Alg.*, Vol. 20, 1972, p. 250-270.