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Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits


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Let $k$ be a field with a non-trivial non-archimedean valuation $v$. We shall assume that the valuation $v$ has a (up to equivalence) unique extension to any finite field extension of $k$, or, equivalently, $k$ is henselian for $v$ (i.e. the Hensel's lemma holds in $k$ with respect to $v$). We fix an algebraic closure $\mathcal{K}$ of $k$ and shall denote the unique valuation on it, which extends the given valuation on $k$, again by $v$. Let $K$ be the separable closure of $k$ in $\mathcal{K}$; the extended valuation on $K$ is obviously invariant under the Galois group $\text{Gal}(K/k)$.

Let $V$ be a finite dimensional $k$-vector space. Let $G$ be a connected reductive $k$-subgroup of $\text{SL}(V)$. For any extension $L$ of $k$ contained in $\mathcal{K}$, let $G(L)$ be the group of $L$-rational points of $G$ endowed with the Hausdorff topology and the bornology induced by the valuation on $L$. Let $G(k)^+$ be the normal subgroup of $G(k)$ generated by the $k$-rational points of the unipotent radicals of parabolic $k$-subgroups of $G$.

$G$ is said to be isotropic over $k$ if $G$ contains a non-trivial $k$-split torus, and $k$-anisotropic (or anisotropic over $k$) otherwise.

The object of this note is to give a simple proof of the following theorem proved first by F. Bruhat and J. Tits in case $k$ is a discretely valuated complete field with perfect residue field and then in general by G. Rousseau in his thesis (Orsay, 1977).
THEOREM (BTR). — $G(k)$ is bounded if and only if $G$ is anisotropic over $k$.

Remark. — Thus in case $k$ is a non-discrete locally compact field, $G(k)$ is compact if and only if $G$ is anisotropic over $k$.

We shall also give a simple proof of the following (unpublished) theorem of J. Tits:

THEOREM (T). — Let $G$ be semi-simple and almost $k$-simple. Then any proper open subgroup of $G(k)^+$ is bounded.

Acknowledgment. — The proof, given below, of Theorem (BTR) is based on a suggestion of G. A. Margulis that it should be possible to use the following lemma (Lemma 1) to prove Theorem (BTR). I had originally used the lemma to give a simple proof of Theorem (T). The comments of J. Tits on an earlier version have lead to further simplifications in the proofs of both the theorems. I thank Margulis and Tits heartily.

LEMMA 1. — Let $H$ be a subgroup of $G(k)$ which is dense in $G$ in the Zariski topology. Assume that $H$ is unbounded. Then there is an element $h$ of $H$ which has an eigenvalue $\alpha$ with $v(\alpha) < 0$.

Proof. — Let:

$$V = V_0 \supset V_1 \supset \ldots \supset V_r \supset V_{r+1} = \{0\},$$

be a flag of $G$-invariant vector subspaces (not necessarily defined over $k$) such that for $0 \leq i \leq r$, the natural representation $\rho_i$ of $G$ on $W_i = V_i / V_{i+1}$ is absolutely irreducible. Let $\rho(= \bigoplus \rho_i)$ be the natural representation of $G$ on $\bigoplus W_i$; $\rho$ is defined over a finite galois extension of $k$. The kernel of $\rho$ is obviously a unipotent normal subgroup of $G$, and as $G$ is reductive, we conclude that $\rho$ is faithful. Now, as $H$ is a unbounded subgroup of $G(k)$, $\rho(H(k))$ is unbounded, and hence there is a non-negative integer $a, a \leq r$, such that $\rho_a(H(k))$ is unbounded.

Now assume, if possible, that the eigenvalues of all the elements of $H$ lie in the local ring of the valuation on $K$. Then, the trace form of $\rho_a$, restricted to $H$, also takes values in the local ring of the valuation (this ring is bounded!). But since $W_a$ is an absolutely irreducible $G$-module, and since $H$ is dense in $G$ in the Zariski topology, $\rho_a(H)$ spans $\text{End}(W_a)$. So, in view the non-degeneracy of the trace form, we conclude that $\rho_a(H)$ is bounded (see Tits [5], Lemma 2.2). This is a contradiction, which proves the lemma.
Proof of Theorem (BTR). — If $T$ is a one-dimensional $k$-split torus, then $T(k)$ is isomorphic to $k^*$ and hence it is unbounded. This implies that if $G$ is isotropic over $k$, then $G(k)$ is unbounded. We shall now assume that $G(k)$ is unbounded and prove the converse.

It is well known that $G(k)$ is dense in $G$ in the Zariski topology ([1], 18.3), hence, according to the preceding lemma, there is an element $g \in G(k)$ which has an eigenvalue $\alpha$ with $v(\alpha) \neq 0$. Now, in case $k$ is of positive characteristic, after replacing $g$ by a suitable power, we shall assume that $g$ is semi-simple. In case $k$ is of characteristic zero, let $g = u.s = s.u$ be the Jordan decomposition of $g$, with $u$ (resp. $s$) unipotent (resp. semi-simple). Then $u, s \in G(k)$, and the eigenvalues of $g$ are the same as that of $s$. Thus we may (and we shall), after replacing $g$ by $s$, again assume that $g$ is semi-simple.

Now there is a maximal torus $S$ in $G$ defined over $k$, such that $g \in S(k)$. (See Borel-Tits [2], Proposition 10.3 and Theorem 2.14 a; note that according to Theorem 11.10 of [1], $g$ is contained in a maximal torus of $G$.) Since any absolutely irreducible representation of a torus is 1-dimensional, there is a character $\chi$ of $S$, $\chi$ defined over a finite galois extension $\mathfrak{R}$ of $k$, such that $\chi(g) = \alpha$. Let $m = [\mathfrak{R} : k]$. Then:

$$v((\sum_{7 \in \text{Gal}(\mathfrak{R}/k)} \gamma \chi)(g)) = m v(\chi(g)) = m v(\alpha) \neq 0.$$  

Thus the character $\sum_{7 \in \text{Gal}(\mathfrak{R}/k)} \gamma \chi$ is non-trivial. On the other hand, it is obviously defined over $k$. Thus $S$ admits a non-trivial character defined over $k$, and hence it contains a non-trivial $k$-split torus. This proves that in case $G(k)$ is unbounded, $G$ is isotropic over $k$.

We shall now assume that $G$ is semi-simple and almost $k$-simple.

Notation. — For $g \in G(k)$, let $\mathcal{P}_g$ be the subset of $G(K)$ consisting of those $x$ in $G(K)$ for which the sequence $\{g^i x g^{-i}\}_{i \geq 0}$ is contained in a bounded subset of $G(K)$, and let $\mathcal{U}_g$ be the subset consisting of those $x$ in $G(K)$ for which the sequence $\{g^i x g^{-i}\}_{i \geq 0}$ converges to the identity. It is obvious that $\mathcal{P}_g$ is a subgroup of $G(K)$, and $\mathcal{U}_g$ is a normal subgroup of $\mathcal{P}_g$.

We let $\mathcal{P}_g^{-1}$ denote $\mathcal{P}_g^{-1}$ and $\mathcal{U}_g^{-1}$ denote $\mathcal{U}_g^{-1}$.

In the sequel we shall denote the adjoint representation of an algebraic group on its Lie algebra by $\text{Ad}$. 

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LEMMA 2. — Let \( t \) be an element of \( G(k) \) such that \( \text{Ad} \ t \) has an eigenvalue \( a \) with \( \nu(a) \neq 0 \). Then:

(i) \( \mathcal{P}_t \) is the group of \( K \)-rational points of a proper parabolic \( k \)-subgroup \( P_t \) of \( G \) and \( \mathcal{U}_t \) is the group of \( K \)-rational points of the unipotent radical \( U_t \) of \( P_t \);

(ii) \( P_t^- (:= P_{r_t}) \) is opposed to \( P_r \).

Proof. — Since for any integer \( n > 0 \), \( \mathcal{P}_r^+ = \mathcal{P}_t \) and \( \mathcal{U}_r = \mathcal{U}_t \), in case \( k \) is of positive characteristic, after replacing \( t \) by a suitable (positive) power of \( t \), we shall assume that \( t \) is semi-simple. In case \( k \) is of characteristic zero, let \( t = u.s = s.u \) be the Jordan decomposition of \( t \) with \( u \) (resp. \( s \)) unipotent (resp. semi-simple). Then \( s, u \in G(k) \) and the eigenvalues of \( \text{Ad} t \) are the same as that of \( \text{Ad} s \). Since the cyclic group generated by a unipotent element is bounded, we see easily that \( \mathcal{P}_t^+ = \mathcal{P}_s^+ \) and \( \mathcal{U}_t = \mathcal{U}_s^+ \). Thus we may (and we shall) assume, after replacing \( t \) by \( s \), that \( t \) is semi-simple.

Now there is a maximal torus \( T \) of \( G \), defined over \( K \), such that \( t \in T(K) \). Since \( K \) is separably closed, any \( T \)-torus splits over \( K \). Let \( \mathfrak{g} = \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha \) be the root space decomposition of the Lie algebra \( \mathfrak{g} \) of \( G \) with respect to \( T \); where \( t \) is the Lie algebra of \( T \) and \( \Phi \) is the set of roots. According to BOURBAKI [4], Chapitre VI, paragraph 1, Proposition 22, there is an ordering on \( \Phi \) such that the subset \( \{ \varphi \mid \varphi \in \Phi, \nu(\varphi(t)) > 0 \} \) is contained in the set \( \Phi^+ \) of roots positive with respect to this ordering; let \( \Delta \subset \Phi \) be the set of simple roots.

For a subset \( \Theta \) of \( \Delta \), let \( T_{\Theta} \) be the identity component of \( \cap_{\alpha \in \Theta} \text{Ker} \theta \) and let \( M_{\Theta} \) be the centralizer of \( T_{\Theta} \) in \( G \). Let \( \mathfrak{u}_{\Theta} = \sum_{\alpha \in \Theta} \mathfrak{g}_\alpha \) (resp. \( \mathfrak{u}_{\Theta}^- = \sum_{\alpha \in \Theta} \mathfrak{g}_\alpha^- \)), and \( U_{\Theta} \) (resp. \( U_{\Theta}^- \)) be the connected unipotent \( K \)-subgroup of \( G \), normalized by \( T \), and with Lie algebra \( \mathfrak{u}_{\Theta} \) (resp. \( \mathfrak{u}_{\Theta}^- \)). Let \( P_{\Theta} = M_{\Theta} U_{\Theta} \) and \( P_{\Theta}^- = M_{\Theta} U_{\Theta}^- \). Then \( P_{\Theta} \) and \( P_{\Theta}^- \) are opposed parabolic \( K \)-subgroups of \( G \), and if \( \Theta \neq \Delta \), these subgroups are proper. Moreover, \( U_{\Theta} \) (resp. \( U_{\Theta}^- \)) is the unipotent radical of \( P_{\Theta} \) (resp. \( P_{\Theta}^- \)).

Now let \( \Pi = \{ \delta \in \Delta \mid \nu(\delta(t)) = 0 \} \). Then since \( \text{Ad} t \) has an eigenvalue \( \alpha \) with \( \nu(\alpha) \neq 0 \), \( \Pi \) is a proper subset of \( \Delta \). It is obvious that \( P_{\Pi}^- (K) \subset \mathcal{P}_t \), \( P_{\Pi}^- (K) \subset \mathcal{P}_r \) and \( U_{\Pi}^- (K) \subset \mathcal{U}_t \). Since \( \mathcal{P}_t \) contains \( P_{\Pi}^- (K) \) it equals \( P_{\Pi}^- (K) \) for a subset \( \Theta \) of \( \Delta \), containing \( \Pi \). But since the action of \( \text{Ad} t \) on \( \mathfrak{u}_{\Pi}^- (K) \) is "expanding", we conclude at once that \( \Theta = \Pi \) and hence, \( P_{\Pi}^- (K) = \mathcal{P}_t \). A similar argument shows that \( P_{\Pi}^- (K) = \mathcal{P}_r^- \). We set \( P_r = P_{\Pi}^- \) and \( P_r^- = P_{\Pi}^- \).

To prove the second assertion of (i) we need to show that \( U_{\Pi}^- (K) = \mathcal{U}_t \). For this purpose we observe that \( U_{\Pi}^- (K) \subset \mathcal{U}_t \), and since
\[ \mathcal{P}_\tau = P_\Pi(K) = M_\Pi(K) \cdot U_\Pi(K); \]
\[ \mathcal{U}_\tau = (M_\Pi(K) \cap \mathcal{U}_\tau) \cdot U_\Pi(K); \]
also \( \mathcal{U}_\tau \), and hence \( M_\Pi(K) \cap \mathcal{U}_\tau \), are normalized by \( T(K) \). We now note that the Lie algebra of \( M_\Pi \) is \( t + \sum_{\mathfrak{g}_1} \mathfrak{g}_1^* \), and \( \nu(\varphi(t)) = 0 \) for \( \varphi \in (\Pi) \). From these observations it is evident that \( M_\Pi(K) \cap \mathcal{U}_\tau \) is trivial, and hence, \( \mathcal{U}_\tau = U_\Pi(K) \).

Now to complete the proof of the lemma it only remains to show that both \( P_\sigma \) and \( P_\sigma^- \) are defined over \( k \). But this is obvious, from the Galois criteria, in view of the fact that \( \mathcal{P}_\tau \) and \( \mathcal{P}_\tau^- \) are stable under \( \Gamma = \text{Gal}(K/k) \) since \( t \) is a \( k \)-rational element, and \( \mathcal{P}_\tau(=P_\Pi(K)) \) is dense in \( P_\Pi \), whereas \( \mathcal{P}_\tau^- (=P_\Pi(K)) \) is dense in \( P_\Pi^- \) in the Zariski topology.

**Lemma 3.** — Let \( t \in G(k) \) be such that \( \text{Ad} \ t \) has an eigenvalue \( \alpha \) with \( \nu(\alpha) \neq 0 \). Then \( \mathcal{U}_\tau(k) (\mathcal{U}_\tau \cap G(k)) \) and \( \mathcal{U}_\tau^-(k) (\mathcal{U}_\tau^- \cap G(k)) \) together generate \( G(k)^+ \).

**Proof.** — According to the preceding lemma, \( \mathcal{U}_\tau(k) \) and \( \mathcal{U}_\tau^-(k) \) are the groups of \( k \)-rational points of the unipotent radicals of two opposed proper parabolic \( k \)-subgroups of \( G \). Hence, according to Borel-Tits [3], Proposition 6.2, \( \mathcal{U}_\tau(k) \) and \( \mathcal{U}_\tau^-(k) \) together generate \( G(k)^+ \).

**Proof of Theorem (T).** — Let \( \mathcal{G} \) be the adjoint group of \( G \) and \( \pi: G \to \mathcal{G} \) be the natural (central) isogeny. Then since \( \pi \) is a finite morphism, the induced map \( G(k) \to \mathcal{G}(k) \) is a proper map, i.e., the inverse image of a bounded subset of \( \mathcal{G}(k) \) is bounded.

Now let \( H \) be an unbounded open subgroup of \( G(k)^+ \). Then, clearly, \( H \) is dense in \( G \) in the Zariski topology and hence, \( \pi(H) \) is an unbounded Zariski-dense subgroup of \( \mathcal{G}(k) \). Now Lemma 1 (applied to \( \pi(H) \) (\( \subset \mathcal{G}(k) \)) implies that there is an element \( h \) of \( H \) such that \( \text{Ad} \ h \) has an eigenvalue \( \alpha \) with \( \nu(\alpha) \neq 0 \).

Let \( \mathcal{U}_h(k) = \mathcal{U}_h \cap G(k) \) and \( \mathcal{U}_h^-(k) = \mathcal{U}_h^- \cap G(k) \). Then as \( H \) is open in \( G(k)^+ \), \( H \cap \mathcal{U}_h(k) \) is an open subgroup of \( \mathcal{U}_h(k) \), and obviously
\[ \bigcup_{n>0} h^n(H \cap \mathcal{U}_h^-(k)) h^{-n} = \mathcal{U}_h^-(k). \]
Thus \( H \) contains both \( \mathcal{U}_h(k) \) and \( \mathcal{U}_h^-(k) \). But according to Lemma 3, \( \mathcal{U}_h(k) \) and \( \mathcal{U}_h^-(k) \) together generate \( G(k)^+ \). Therefore, \( H = G(k)^+ \). This proves the theorem.
REFERENCES


