BULLETIN DE LA S. M. F.

ROBERT F. COLEMAN

The dilogarithm and the norm residue symbol

Bulletin de la S. M. F., tome 109 (1981), p. 373-402
<http://www.numdam.org/item?id=BSMF_1981__109__373_0>

© Bulletin de la S. M. F., 1981, tous droits réservés.

L'accès aux archives de la revue « Bulletin de la S. M. F. » (http://smf.emath.fr/Publications/Bulletin/Presentation.html) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Bull. Soc. math. France
109, 1981, p. 373-402

THE DILOGARITHM AND THE NORM RESIDUE SYMBOL

BY

ROBERT F. COLEMAN (*)

ABSTRACT. — This paper contains a complete formula for the Hilbert Norm Residue Symbol, for cyclotomic extensions of $\mathbb{Q}_p$, which is of similar shape to Iwasawa's incomplete formula [I]. It is proven, essentially by verifying the Steinberg Identity which in this context is a consequence of p-adic analytic properties of the dilogarithm series. The author is also able to deduce the non-degeneracy of the symbol directly from the formula.

RÉSUMÉ. — Dans ce papier il y a une formule complète pour le symbole local de Hilbert, qui a une forme similaire de celle d'IWASAWA [I] (qui n'est pas complète). On prouve cette formule en démontrant l'identité de Steinberg qui est une conséquence des properties p-adique de la série dilogarithme. L'auteur déduit, aussi, la « non-dégénérescence » du symbole directement de la formule.

I. Introduction

In this paper we apply our analytic theory of the norm [C] and the p-adic analytic properties of the dilogarithm series to the study of the norm residue symbol. In particular we give a complete formula for the norm residue symbol of exponent p^n attached to a cyclotomic extension of $\mathbb{Q}_p$ containing the p^n-th roots of unity, and we give an analytic proof of the non-degeneracy of this symbol. In contrast to earlier results, the formula obtained takes the same form for the prime two as for the odd primes. The dilogarithm series has begun to play a role in geometry and algebra particularly in the work of BLOCH [B], MILNOR and THURSTON [T].

We attempt to make more visible the relationship of our formula for the norm residue symbol to norm residues, by relating the quality of being a norm

(*) Texte reçu le 3 janvier 1980, révisé le 19 mars 1981.

R. F. COLEMAN, Harvard University, Department of Mathematics, 1, Oxford Street, Cambridge M.A. 02138, U.S.A.

© Gauthier-Villars

to that of being an exact differential. We are thus led to the study of a class of differential equations whose solutions may be expressed in terms of the logarithm and the dilogarithm.

Let p be a rational prime and let $\mathbb{Q}_p$ be the field of p-adic numbers. Let μ_m denote the group of m-th roots of unity in $\mathbb{C}_p$, the completion of a fixed algebraic closure of $\mathbb{Q}_p$. Let H be a fixed finite unramified extension of $\mathbb{Q}_p$, and let $H_n = H(\mu_{p^{n+1}})$. We let $(\ ,\ )_n$ denote the Hilbert norm residue symbol on H_n with respect to $\mu_{p^{n+1}}$. Recall that this symbol is defined by the equation:

$$(a, b)_n = \sigma_b(\alpha)/\alpha,$$

where $a, b \in H_n^*$, $\alpha \in \mathbb{C}_p$ is any solution of $\alpha^{p^{n+1}} = a$ and σ_b denotes the image of b in Gal $(H_n(\alpha)/H_n)$ under the Artin map. It follows that $(\ ,\ )_n$ is a bilinear pairing from $H_n^* \times H_n^*$ into $\mu_{p^{n+1}}$ which satisfies $(a, -a)_n = 1$ and so is skew-symmetric.

Let $\mathcal{O}$ and $\mathcal{O}_n$ denote the rings of integers in H and H_n respectively. Let $\mathcal{P}_n$ denote the maximal ideal in $\mathcal{O}_n$. We will now give a rough idea of the formula and a sketch of its proof. (The precise statement is in the next section.)

The law takes the following shape: For each $n \geqslant -1$ we will define a linear functional $\int_n$ on $\mathcal{O}[[T]]$ (the ring of Taylor series over $\mathcal{O}$). Let ζ be a primitive p^{n+1}-st root of unity. Then to each $a \in \mathcal{O}_n^*$ and $b \in H_n^*$ we associate an element $\omega_\zeta(a, b)$ of $\mathcal{O}[[T]]$ (there is a lot of choice involved here). The element $\omega_\zeta(a, b)$ takes the form $\operatorname{Log}(f).D\operatorname{Log}(g)$ where $D = (1-T)\,d/dT$ and $f, g \in \mathcal{O}((T))^*$ (invertible elements in the Laurent series over $\mathcal{O}$) such that $f(1-\zeta) = a$ and $g(1-\zeta) = b$. (This, of course, is very imprecise since $\operatorname{Log}(f)\,D\operatorname{Log}(g)$ is in general not defined let alone in $\mathcal{O}[[T]]$.) We show:

$$(a, b)_n = \zeta^{\int_n \omega_\zeta(a, b)}. \tag{1}$$

(Actually, there will be an error term unless $f \in 1 + T^2\mathcal{O}[[T]]$.)

We will prove (1) by first showing (in Section III) that the right-hand side depends only on a and b (and not on the choice of ζ, f or g) and gives a well defined pairing $[\ ,\ ]_n : \mathcal{O}_n^* \times H_n^* \to \mu_{p^{n+1}}$. We will show in Section IV that:

$$[\zeta, a]_n = (\zeta, a)_n, \tag{2}$$

for $a \in H_n^*$ and in Section V that:

$$(3) \qquad [a, b]_n = 1,$$

when b is a norm from the subgroup of $H_n(\alpha)^*$ generated by $1-\alpha$ and $\mathcal{O}[\alpha]^*$ ($\alpha^{p^{n+1}} = a$) which we shall call A_α. This subgroup coincides with $H_n(\alpha)^*$ only when $1-a$ is a parameter of H_n. However (3) is enough to imply:

$$(4) \qquad [b, 1-b]_n = 1$$

for $b \in \mathcal{O}_n^*$ ($b \neq 1$). We then show that the extension of $[\ ,\ ]_n$ to $H_n^* \times H_n^*$ by skew symmetry satisfies (4) for all $b \in H_n^*$ ($b \neq 1$). This is sufficient to prove (1) since it is well known that properties (2) and (4) characterize $(\ ,\ )_n$.

The proof of (3) is based on the relationship of $\int_n$ with D. We will show (in Section V) that:

$$\int_n Df \equiv 0 \mod p^{n+1},$$

when $f \in \mathcal{O}[[T]]$. Essentially, we show, when b is a norm from A_α, that $\omega_\zeta(a, b)$ is sufficiently close to an element in $D\mathcal{O}[[T]]$ to imply (3). We are thereby led (in Section VI) to the study of the differential equations:

$$(5) \qquad Dh = \operatorname{Log}(1-f)\, D \operatorname{Log}(g \circ f)$$

for $f, g \in \mathcal{O}((T))^*$. If we were only interested in proving (4) we could take $g(T) = T$. In this case the solution of (5) is

$$\operatorname{Dilog}(f).$$

In Appendix I, we show how one can see the non-degeneracy of $(\ ,\ )_n$ directly from our explicit formula. In the process we recover Iwasawa's law and construct a Kummer generator in $\mathcal{O}_n^*$ for the unramified extension of degree p^{n+1} over H_n.

In Appendix II, we collect together and reformulate some of the results we obtained in the course of proof of (1) about the ring $\mathcal{O}((T))$. These results seem to underlie (1), but are not consequences of (1).

We note that most of the results in this paper have some generalization to Lubin-Tate formal groups of arbitrary height. In particular it seems quite likely that the analogue for a Lubin-Tate group of our explicit formula will give a complete formula for the symbol studied by WILES [W]. Unfortunately, the generalizations of the estimates obtained in Section VI are not strong enough to imply the analogue of (3) when the height is greater than one.

Other authors, SEN [S], BRUCKNER [B], and VOSTOKOV [V] have recently given formulae for the Hilbert Norm Residue Symbol attached to an arbitrary local field of characteristic zero. The novelty of our approach is the connection we make with the dilogarithm series, and the application of our theory of canonical power series in [C]. The point of view taken in [B] is the closest among the above authors to that taken here, although the formula in [B] is a residue formula while ours is an integral formula. It is quite likely that a unified approach will soon be found. Further off in the program of making local class field theory completely explicit are formulae exhibiting the compatibility between the Lubin-Tate laws over different base fields.

II. Statement of Theorem

In general we will maintain the notations of [C], specializing to the case $K=\mathbf{Q}_p, \mathfrak{F}(X, Y)=X+Y-XY$. Then $[a](T)=1-(1-T)^a$ for $a\in\mathbf{Z}_p$ and:

$$\lambda(T)=-\operatorname{Log}(1-T)=\sum_{n=1}^{\infty}\frac{T^n}{n}.$$

We let H denote a fixed finite extension of $\mathbf{Q}_p$, $\mathcal{O}$ the ring of integers of H, φ the Frobenius element of the Galois group of $H/\mathbf{Q}_p$ acting on $\mathcal{O}((T))$ coefficient-wise, I the ring $\mathcal{O}[[T]]$, and $H((T))_1$ the ring of power series which converge on the open unit ball in $\mathbf{C}_p$. Let $D=(1-T)\,d/dt$ be the intrinsic derivation with respect to $\mathfrak{F}$.

For each $n\geqslant -1$ we define a continuous linear functional on $H((T))_1$ by setting:

$$\int_n f=p^{-(n+1)}\sum_{u\in\mathfrak{F}_n} f(u).$$

(Recall that $\mathfrak{F}_n$ is the group of division points on $\mathfrak{F}$ of order p^{n+1}, so that $\mathfrak{F}_n=\{1-\zeta : \zeta\in\mu_{p^{n+1}}\}$.) This linear functional maps I into $\mathcal{O}$. Let $\mathcal{M}^{(n)}$ denote the subgroup of $\mathcal{M}$ (recall, $\mathcal{M}=\mathcal{O}((T))^{\times}$) consisting of all elements g such that:

$$g([p^{n-i}](u))=\varphi^{n-i}N_{n,\,i}(g(u))$$

for any generator u of the cyclic $\mathbb{Z}_p$-module $\mathfrak{F}_n$ where $N_{n,i}$ denotes the norm from $H_n = H(\mathfrak{F}_n)$ to H_i. Equivalently, $\mathcal{M}^{(n)}$ is the set of all $g \in \mathcal{M}$ satisfying:

$$(1) \qquad \frac{\varphi^{-1}\mathcal{N}(g)(T)}{g(T)} = 1 - p\frac{[p^n](T)}{T}.k(T),$$

for some $k \in \mathcal{O}[[T]]$, by Corollary 12 (ii), and Lemma 13 (ii) of [C]. (Recall that $\mathcal{N}$ is the norm operator defined in Section IV of [C].) With g and k as in (1) we define:

$$\gamma_n(g) = k(0).$$

By Theorem 15 of [C], for every $a \in H_n^*$ and each generator u of $\mathfrak{F}_n$, there exists an $f \in \mathcal{M}^{(n)}$ such that:

$$f(u) = a.$$

If u is a generator of $\mathfrak{F}_n$ and $\zeta \in \mu_{p^{n+1}}$ we define $\mathrm{Ind}_u \zeta$ to be the integer modulo $p^{n+1}\mathbb{Z}$ such that:

$$(1-u)^{\mathrm{Ind}_u \zeta} = \zeta.$$

Finally, let $T_{H/\mathbb{Q}_p}$ denote the trace map from H to $\mathbb{Q}_p$.

We may now state our formula.

THEOREM 1. — *Let $f \in 1 + TI$, $g \in \mathcal{M}^{(n)}$ and u be any generator of $\mathfrak{F}_n$. Then:*

$$(2) \quad \mathrm{Ind}_u(f(u), g(u))_n$$
$$= T_{H/\mathbb{Q}_p}\left[f'(0)\gamma_n(g) + \int_{n} \frac{1}{p}\mathrm{Log}\left(\frac{\varphi f \circ [p]}{f^p}\right).D\,\mathrm{Log}(g)\right] \mathrm{mod}\, p^{n+1}\mathbb{Z}_p.$$

Remarks. — First we note that the integral makes sense because $D\,\mathrm{Log}(g) = D(g)/g$ has at most a simple pole while $\mathrm{Log}(\varphi f \circ [p]/f^p)$ vanishes at the origin. Second, it follows from the preceding remarks that the right-hand side of (2) does indeed lie in $\mathbb{Z}_p$. Third, it is clear that (2) gives a formula for $(a, b)_n$ when $a \in 1 + \mathcal{P}_n$ and $b \in H_n^*$. But this is the whole story as the pairing is skew symmetric and is trivial on roots of unity of order prime to p. At the end of Section III we will explain how this formula may be extended to the case where $f(0)$ is an arbitrary element of $\mathcal{O}^*$.

Fundamental identities. — Recall (from [C]) that $\mathcal{N}$ and $\mathcal{S}$ are operators on $\mathcal{M}$ and $H((T))_1$ (Laurent series with finite poles which converge on the unit ball in $\mathbb{C}_p$) respectively. They are characterized by the identities:

(3) $$\mathcal{N}(f)([p](T))=\prod_{u\in\mathfrak{F}_0} f(T[+]u),$$

(4) $$\mathcal{S}(g)([p](T))=\sum_{u\in\mathfrak{F}_0} g(T[+]u).$$

for $f\in\mathcal{M}$ and $g\in H((T))_1$. For g and f as above and $h\in H[[T]]_1$, we deduce the following formulae:

(5) $$N_{n,\,n-1}(f(v))=\mathcal{N}(f)([p](v)),$$

(6) $$N_0(f(u))=\left(\frac{\mathcal{N}(f)([p])}{f}\right)\Bigg|_0,$$

(7) $$T_{n,\,n-1}(g(v))=\mathcal{S}(g)([p](v)),$$

(8) $$T_0(g(u))=(\mathcal{S}(g)([p])-g)|_0,$$

where $n\geqslant 1$, where $N_0=N_{H_0/H}$ $T_0=\mathrm{Tr}_{H_0/H}$, v generates $\mathfrak{F}_n$, $n\geqslant 1$ and u generates $\mathfrak{F}_0$. Of crucial importance for us are the identities:

(9) $$\mathcal{S}(D(g))=p\,D\,\mathcal{S}(g)$$

(10) $$\mathcal{S}(\delta(f))=p\,\delta\,\mathcal{N}(f),$$

with f and g as above. These identities are consequences of (3) and (4) and the invariance of D with respect to the group law. For example let us prove (9); we have:

$$p(D\,\mathcal{S}(f))([p](T))=D(\mathcal{S}(f)([p](T)))=D\sum_{u\in\mathfrak{F}_0} f(T[+]u)$$
$$=\sum_{u\in\mathfrak{F}_0}(Df)(T[+]u)=\mathcal{S}(Df)([p](T)).$$

Q.E.D.

The proof of (10) is similar.

We also recall the following congruences, Lemmas 6 and 13 of [C]. Let $f\in I$ and $g\in\mathcal{M}$ then:

(11) $$\sum_{u\in\mathfrak{F}_n} g(u)=p^{n+1}\int_n f\equiv 0 \bmod p^{n+1}\mathcal{O}.$$

(12) $$\frac{\mathcal{N}^n(f)}{\varphi\,\mathcal{N}^{(n-1)}(f)}\equiv 1 \bmod p^{n+1} I.$$

In the remainder of this paper we shall make use of the following notation:

If F is a field with a discrete valuation, $b\in F$ ($b\neq 0$) and π is a parameter of F, then $\mathrm{ord}_\pi(b)$ is defined to be the integer k such that $\pi^{-k}b$ has zero valuation. In this paper, F will either be a finite extension of $\mathbb{Q}_p$ or $\mathbb{C}_p((T))$ (Laurent series over $\mathbb{C}_p$ with finite poles). For $f \in \mathbb{C}_p((T))$ we let $\mathrm{Res}_0(f)$ denote the coefficient of T^{-1} in the expansion of f.

III. The Explicit Pairing

In this section we show that the right-hand side of (II (2)) defines a pairing from $(1+\mathscr{P}_n) \times H_n^\times$ into $\mu_{p^{n+1}}$. For $f \in m$ (the maximal ideal in I) we set:

$$\Theta(f) = \lambda(f) - \frac{\varphi\lambda(f \circ [p])}{p}$$

and for $g \in \mathscr{M}$ we set:

$$\delta g = \frac{Dg}{g}.$$

Then Θ and δ determine continuous homomorphisms from $\mathfrak{F}(m)$ and m, respectively, into I and $T^{-1} I$, respectively. Now if $f \in TI$, $g \in \mathscr{M}^{(n)}$ we set:

$$\langle f, g \rangle_n = f'(0)\gamma_n(g).$$

It is not difficult to check that $\langle \ , \ \rangle_n$ induces a bilinear, bicontinuous pairing from:

$$\mathfrak{F}(T.I) \times \mathscr{M}^{(n)} \text{ into } \mathcal{O}/p^{n+1}\mathcal{O}.$$

Now let:

$$(f, g)'_n = T_{H/\mathbb{Q}_p}\left(\langle f, g \rangle_n + \int_n \Theta(f).\delta g\right),$$

for $f \in T.I$, $g \in \mathscr{M}^{(n)}$. Clearly $(\ ,\ )'_n$ induces a bilinear, bicontinous pairing:

$$\mathfrak{F}(T.I) \times \mathscr{M}^{(n)} \to \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

We must prove:

$$\mathrm{Ind}_u(f(u), g(u))_n = (1-f, g)'_n \bmod p^{n+1}\mathbb{Z}_p,$$

for $f \in 1+T.I$, $g \in \mathscr{M}^{(n)}$.

PROPOSITION 3. — *Let u be any generator of $\mathfrak{F}_n$. Then the value of $(f, g)'_n$ modulo p^{n+1} depends only on $f(u)$ and $g(u)$.*

Proof. — By bilinearity, it suffices to show that $(f, g)'_n \equiv 0 \bmod p^{n+1}$ when either (i) $g(u)=1$ or (ii) $f(u)=0$.

Case (*i*). — $g(u)=1$. Since $g\in\mathcal{M}^{(n)}$ it follows that:

$$g(T)=1+\frac{[p^{n+1}](T)}{T}.h(T) \qquad \text{for some} \quad h\in I, \tag{1}$$

and then, by a straightforward computation:

$$\delta g(u)=p^{n+1}.\frac{h(u)}{u} \qquad \text{for} \quad u\in\mathfrak{F}'_n,$$

If $r=\Theta(f)$, we have:

$$\int_n r.\delta g=\sum_{u\in\mathfrak{F}'_n}\frac{r(u).h(u)}{u}\equiv -r'(0).h(0) \bmod p^{n+1},$$

using (II(11)).

On the other hand, it follows from (1) that:

$$\mathcal{N}(g)(0)=\prod_{u\in\mathfrak{F}_0} g(u)=g(0) \qquad \text{and} \qquad \gamma_n(g)=(1-\varphi^{-1})h(0).$$

Thus:

$$\frac{\varphi^{-1}\mathcal{N}(g)(0)}{g(0)}\equiv 1-p^{n+1}(1-\varphi^{-1})h(0) \bmod p^{2(n+1)},$$

$$\langle f, g\rangle_n\equiv f'(0).(1-\varphi^{-1})\,h(0) \bmod p^{n+1}.$$

Now clearly, $r'(0)=(1-\varphi)f'(0)$ so that:

$$\langle f, g\rangle_n\equiv r'(0).h(0) \bmod ((1-\varphi)\mathcal{O}+p^{n+1}\,\mathcal{O}).$$

This concludes the proof of Case (i).

Case (ii). — $f(u)=0$. It follows that:

$$f=\frac{[p^{n+1}]}{[p^n]}.r$$

for some $r\in T.I$. Also as $g\in\mathcal{M}^{(n)}$:

$$t\overset{\text{def}}{=}\frac{\varphi^{-1}\mathcal{N}(g)}{g}=1+\frac{p[p^n](T)}{T}w(T)$$

for some $w\in I$. We see that:

$$f'(0)=pr'(0),$$

mod p^{n+1}; so:

$$\langle f, g\rangle_n \equiv -pr'(0)\,w(0) \bmod p^{n+1}. \tag{3}$$

On the other hand:

$$\int_n \Theta(f).\delta d = \frac{1}{p^{n+1}}(B+C),$$

where:

$$B = \mathrm{Res}_0\, T^{-1}(\Theta(f).\delta g)(T),$$
$$C = \textstyle\sum_{u\in\mathfrak{F}'_n}(\Theta(f).\delta g)(u).$$

Let $s = p^{-1}\lambda(pr) \in T.I.$ Now:

$$B = \left(\frac{d}{dT}\Theta(f)\right)(0).\mathrm{Res}_0\,\delta g,$$

so, as $(d/dT\,\Theta(f))(0) \in (1-\varphi)\mathcal{O}$ we have:

$$T_{H/\mathbf{Q}_p}(B) = 0.$$

As for C : Let $u_i = [p^{n-i}](u)$, $-1 \leqslant i \leqslant n$, so that:

$$C = \textstyle\sum_{i=0}^{n} T_i((\Theta(f).\delta g)(u_i)).$$

Now:

$$\Theta(f)(u_i) = \begin{cases} -\varphi\, s(u_n), & i=n, \\ ps(u_i) - \varphi\, s(u_{i-1}), & 0 \leqslant i \leqslant n, \end{cases}$$

so that after re-arranging the terms in C we have:

$$C \equiv -\textstyle\sum_{i=0}^{n-1}(T_{i+1}(s(u_i).\varphi^{-1}\delta g(u_{i+1})) - T_i(s(u_i).p\,\delta g(u_i))) \bmod (1-\varphi)\mathcal{O},$$

as $s(0) = 0$. But:

$$T_{i+1,\,i}(\varphi^{-1}\delta g(u_{i+1})) = p\,\delta g(u_i) + p\,\delta t(u_i)$$

using the identity $\mathcal{S}\,\delta g = p\,\delta\mathcal{N}(g)$ and the definition of t above. Therefore:

$$C \equiv -p\textstyle\sum_{u\in\mathfrak{F}_{n-1}} s(u).\delta t(u) \bmod (1-\varphi)\mathcal{O}.$$

Thus:

$$\int_n \Theta(f).\delta g \equiv -\frac{1}{p^n}\textstyle\sum_{u\in\mathfrak{F}_{n-1}} s(u).\delta t(u) \bmod (1-\varphi)\mathcal{O}.$$

Now, we see as above that:

$$\delta t(u) = p^{n+1}\frac{w(u)}{u} \qquad \text{for} \quad u \in \mathfrak{F}'_n$$

and so, using (II(11)):

$$\int_n \Theta(f).\delta g \equiv -p\sum_{u\in\mathfrak{F}'_{n-1}} \frac{s(u).w(u)}{u}$$
$$\equiv ps'(0).w(0) \equiv pr'(0).w(0) \bmod (p^{n+1}\mathcal{O} + (1-\varphi)\mathcal{O}).$$

But, comparing this with (3) we see that $(f, g)'_n \equiv 0 \bmod p^{n+1}$ in this case as well, and so we have proven Proposition 2.

COROLLARY 4. — *There exists a unique pairing:*

$$[\ ,\ ]_n : \mathcal{O}_n^\times \times H_n^\times \to \mu_{p^{n+1}},$$

such that for $f \in \mathfrak{F}(T.I)$, $g \in \mathcal{M}^{(n)}$:

(4) $$\mathrm{Ind}_u [1 - f(u), g(u)]_n \equiv (f, g)'_n \bmod p^{n+1}$$

for any u which generates $\mathfrak{F}_n$.

Proof. — By the previous proposition, if we fix u, (4) defines a pairing ($[\varepsilon, \alpha]_n$ must equal 1 for $\varepsilon \in V$) and it is not hard to check that this definition does not depend on the choice of u. (Recall that V denotes the group of roots of unity in H of order prime to p.)

Recall that $(\ ,\ )_n$ is the Norm Residue Symbol defined in the introduction. It is well known that $(\ ,\ )_n$ is a bilinear pairing characterized by the properties:

(5) $$\mathrm{Ind}_u (1 - u, a)_n = p^{-(n+1)}\left(1 - \frac{N(a)}{p^e}\right) \overset{\text{def}}{=} d_n(a),$$

(6) $$(b, 1-b)_n = 1,$$

where $a, b \in H_n^*$, $N(a)$ is the norm of a from H_n^* to $\mathbb{Q}_p$, $e = \mathrm{ord}_p(N(a))$ and u is a generator of $\mathfrak{F}_n$. In the next two sections we will show that $[\ ,\ ]_n$ satisfies (5) for $a \in H_n^*$ and (6) for $b \in \mathcal{O}_n^*$. This suffices to prove our reciprocity formula because of the next Lemma.

LEMMA 5. — *If* (,) *is a pairing from* $\mathcal{O}_n^* \times H_n^*$ *into some abelian group which satisfies:*

(7) $$(a, 1-a)=0,$$

for $a \in \mathcal{O}_n^*$ $(a \neq 1)$. *Then* (,) *can be extended to a pairing from* $H_n^* \times H_n^*$ *which satisfies* (7) *for all* $a \in H_n^*$ $(a \neq 1)$.

Proof. — Let π be a fixed parameter of H_n. Then we define $(\ ,\ )^{\sim}$ on $H_n^* \times H_n^*$ by bilinearity (over $\mathbb{Z}$) and:

(i) $(\pi, -\pi)^{\sim}=0$;

(ii) $(\pi, a)^{\sim}=-(a, \pi)$;

(iii) $(b, c)^{\sim}=(b, c)$,

for $a, b \in \mathcal{O}_n^*$, $c \in H_n^*$. Now using only bilinearity we deduce:

(8) $$(a, 1-a)^{\sim}+(a^{-1}, 1-a^{-1})^{\sim}=(a, -a)^{\sim},$$

(9) $$(a, b)^{\sim}+(a, -a)^{\sim}+(b, -b)^{\sim}+(b, a)^{\sim}=(ab, -ab).$$

It follows from this and (7) that:

(10) $$(a, -a)^{\sim}=0,$$

(11) $$(b, c)^{\sim}+(c, b)^{\sim}=0,$$

for $a, b, c, \in \mathcal{O}_n^*$. But this together with (i) and (ii) imply that (11) also holds for b and c arbitrary in H_n^*. Now (11), (9) and (i) imply that (10) holds for a arbitrary in H_n^*. This and (8) imply:

(12) $$(a, 1-a)^{\sim}+(a^{-1}, 1-a^{-1})^{\sim}=0,$$

for arbitrary $a \in H_n^*$ $(a \neq 1)$. But if $a \notin \mathcal{O}_n^*$ then $1-a$ or $1-a^{-1}$ lies in $\mathcal{O}_n^*$ and so (11) implies that one of the terms on the left in (12) is zero. This implies that (7) holds for $a \in H_n^*$ and so we have our Lemma.

Remark. — One can extend the pairing $(\ ,\ )_n$ to $\mathfrak{F}(m) \times \mathcal{M}^{(n)}$ as follows. First extend $\int_n$ to $T^{-1}.I$ by setting:

$$\int_n T^{-1}=\frac{1}{2}.$$

Second, extend $\langle\ ,\ \rangle_n$ to $\mathfrak{F}(m) \times \mathcal{M}^{(n)}$ by setting:

$$\langle f, g \rangle_n=\begin{cases} \delta(f-1)(0)\gamma_n(g) & \text{if } p \neq 2, \\ (\delta(f-1)(0)+2^{n-1}.f(0)\gamma_n(g)) & \text{if } p=2. \end{cases}$$

Finally, set:

$$(f, g)'_n = T_{H/\mathbf{Q}_p}\left(\langle f, g\rangle_n + \int_n \Theta(f).\delta g\right)$$

as before. One can then use the same methods as in the proof of Proposition 3 to prove:

(13) $$\operatorname{Ind}_u (f(u), g(u))_n \equiv (1-f, g)'_n \bmod p^{n+1}.$$

We will not need this extended formula and we omit the proof. We note that from (13) the classical formulas for $p=2$, $n=0$ can be deduced.

IV. Computation of $[\zeta, b]_n$

In this section we show that $[\ ,\ ]_n$ satisfies (III(5)), i.e.:

PROPOSITION 6. — *If $b \in H_n^\times$ and u is a primitive element of $\mathfrak{F}_n$:*

$$\operatorname{Ind}_u [1-u, b]_n \equiv d_n(b) \bmod p^{n+1},$$

where $d_n(b)$ is as in (III(5)).

Proof. — Let $f(T)=T$ and g be an element of $\mathcal{M}^{(n)}$ such that $g(u)=b$. Then by definition:

$$\operatorname{Ind}_u [1-u, b]_n \equiv (f, g)'_n \bmod p^{n+1}$$

but $\Theta(f)=0$ and $f(0)=0$ so that:

$$(f, g)'_n = T_{H/\mathbf{Q}_p}(\langle f, g\rangle_n) = T_{H/\mathbf{Q}_p}(\gamma_n(g)).$$

But, by definition:

$$\gamma_n(g) = p^{-(n+1)}(1-t(0))$$

where $t=(\varphi^{-1}\mathcal{N}(g))/g$. Now as:

$$N_{H/\mathbf{Q}_p}(t(0)) \equiv 1 - T_{H/\mathbf{Q}_p}(p^{n+1}\gamma_n(g)) \bmod p^{2(n+1)}$$

$(\gamma_n(g) \in \mathcal{O})$, the proposition will follow the following claim:

Claim:

$$N_{H/\mathbf{Q}_p}(t(0)) = \frac{1}{p^e} N_{H_n/\mathbf{Q}_p}(b),$$

where $e=\operatorname{ord}_p(N_{H_n/\mathbf{Q}_p}(b))$. But, if $u_0=[p^n](u)$:

$$N_{H_n/H_0}(b) = \varphi^{-n} g(u_0) \quad \text{since} \quad g \in \mathcal{M}^{(n)}.$$

Also:

$$N_0(g(u_0))=\left.\frac{\mathcal{N}(g)\circ[p]}{g}\right|_0=\left.\frac{\varphi g\circ[p]}{g}\right|_0 .t(0).$$

But:

$$\left.\frac{\varphi g\circ[p]}{g}\right|_0=a^{\varphi-1}.p^r,$$

where $a\,T^r$ is the leading term of g. Thus:

$$N_{H_n/\mathbf{Q}_p}(b)=N_{H/\mathbf{Q}_p}(N_{H_0/H}(\varphi^{-n}g(u_0)))$$
$$=N_{H/\mathbf{Q}_p}(a^{\varphi-1}.p^r.t(0))=N_{H/\mathbf{Q}_p}(p^r).N_{H/\mathbf{Q}_p}(t(0)).$$

But, $N_{H/\mathbf{Q}_p}(p^r)=p^e$ since $t(0)\in\mathcal{O}^*$, and we have our claim.

V. Norms and Exactness

Let n be a fixed non-negative integer, u a fixed generator of $\mathfrak{F}_n$ and a a fixed element of $1+\mathcal{P}_n$. Let α be any solution of $\alpha^{p^{n+1}}=a$ in $\mathbf{C}_p$, and let A_α be as in the introduction. Then A_α is a subgroup of $H_n(\alpha)^*$ and A_α equals $H_n(\alpha)^*$ when $\mathrm{ord}_u(1-a)=1$. Let N_α denote the norm from $H_n(\alpha)$ to H_n. In this section we will prove:

(1) $$[a, b]_n=1,$$

for $b\in N_\alpha(A_\alpha)$. As a consequence of (1) we have, for $c\in\mathcal{O}_n^*$;

(2) $$[c, 1-c]_n=1.$$

Indeed, if $c=\varepsilon c'$ where $\varepsilon\in V$ and $c'\in 1+\mathcal{P}_n$, and we set $\eta=\varphi^{-(n+1)}(\varepsilon)$, then $1-c=\prod_\alpha(1-\eta\alpha)$ where the product is taken over all $\alpha\in\mathcal{C}_p$ satisfying $\alpha^{p^{n+1}}=c'$. Therefore $1-c$ is in the subgroup of H_n^* generated by $\{N_\alpha(1-\eta\alpha)\ ;\ \alpha^{p^{n+1}}=c'\}$, and $1-\eta\alpha\in A_\alpha$. Thus we will have completed the proof of the explicit reciprocity law once we have established (1).

The proof of (1) will be based on the following result:

PROPOSITION 7. — *If $f\in I$ then:*

$$\int_n Df\equiv 0 \ mod\ p^{n+1}.$$

Proof. — First set $\overline{\mathcal{S}}=p^{-1}\mathcal{S}$ so that if $g\in I$, $\overline{\mathcal{S}}(g)$ is in I by Lemma 6 of [1], and:

(3) $$\int_n g=\overline{\mathcal{S}}^{n+1}(g)(0).$$

Proposition 7 follows immediately from (II (9)).

In view of this proposition, we see that (1) would follow if we could choose power series f, $g\in\mathcal{M}$ such that:

(i) $$g\in\mathcal{M}^{(n)},$$

(ii) $$1-f(u)=a, \qquad g(u)=b,$$

(iii) $$\Theta(f).\delta g\in D\mathcal{O}[[T]],$$

(iv) $$\langle f, g\rangle_n\equiv 0 \bmod p^{n+1}.$$

Unfortunately, we are not able to do this and must proceed in a somewhat roundabout way to achieve essentially the same ends.

Proof of (1). — We may assume that $a\notin(\mathcal{O}_n^*)^p$ for otherwise either there is an $a'\in\mathcal{O}_n^*$, $a'\notin(\mathcal{O}_n^*)^p$ such that $a'^{p^i}=a$, $i\geqslant 1$ or $a=1$. The latter case is trivial and in the former case:

$$[a, b]_n=[a', b^{p^i}]_n,$$

and if $b\in N_\alpha(A_\alpha)$ it follows that $b^{p^i}\in N_{\alpha'}(A_{\alpha'})$ where $\alpha'^{p^i}=\alpha$.

Now, from the definition of A_α we see that there exists an $r\in\mathcal{M}$ such that $N_\alpha(r(1-\alpha))=b$. On the other hand, since $a\in(\mathcal{O}_n^*)^p$,

$$N_\alpha(r(1-\alpha))=\prod\nolimits_{\zeta\in\mu_{p^{n+1}}}r(1-\zeta\alpha)=\mathcal{N}^{n+1}(r)(1-a).$$

But from (II (12)) we have:

$$\varphi\,\mathcal{N}^{n+1}(r)\equiv\mathcal{N}^{n+2}(r) \bmod p^{n+2}\mathcal{O}[[T]].$$

Hence we are reduced to proving:

(5) $$[a, g(1-a)]_n=1,$$

whenever $\mathcal{N}(g)\equiv\varphi\, g \bmod p^{n+2}$. Now the set of all g's in $\mathcal{M}$ satisfying this condition makes up a group containing T. Therefore, we may assume $\mathrm{ord}_T(g)=1$. By Theorem 15 of [1] there is an $h\in\mathcal{M}^{(n)}$ such that $h(u)=g(1-a)$, so as $\mathrm{ord}_T(g)=1$ and the leading coefficient of g is a unit, it follows that there exists an $j \in \mathcal{M}$ such that $g \circ j = h$ and $j(u) = 1 - a$. It follows that $\mathrm{ord}_u(1-a) = \mathrm{ord}_T j = \mathrm{ord}_T h \geqslant 1$. Thus (5) will follow from (III (4)) and;

PROPOSITION 8. — *If f, $g \in \mathcal{M}$, $\mathcal{N}(g) \equiv \varphi g \bmod p^{n+2}$, $\mathrm{ord}_T(g) = 1$, $\mathrm{ord}_T(f) \geqslant 1$ and $g \circ f \in \mathcal{M}^{(n)}$ then:*

(6) $$(f, g \circ f)'_n \equiv 0 \bmod p^{n+1}.$$

Proof. — We will compute the left-hand side of (6). Since $\mathrm{ord}_T(f) \geqslant 1$, $\{f, g \circ f\}_n \equiv 0 \bmod p^{n+1}$ and:

(7) $$\langle f, g \circ f \rangle_n = p^{-(n+1)} f'(0)(1 - t(0)),$$

where $t = \varphi^{-1} \mathcal{N}(g \circ f)/g \circ f$. It remains to evaluate:

$$B = \int_n \Theta(f).\delta(g \circ f).$$

We need;

PROPOSITION 9. — *If f, g, r are elements of $\mathcal{M}$ such that $\mathrm{ord}_T(f) \geqslant 1$, $g'(0) \in \mathcal{O}^*$ and $\mathcal{N}(g) \circ r = \mathcal{N}(g \circ f)$, then:*

(8) $$\overline{\mathcal{S}}(\Theta(f).\delta(g \circ f)) = \frac{\lambda(r[-]\varphi f)}{p}.\delta \mathcal{N}(g \circ f) + \frac{p}{2} \varphi D^2 \Theta(f) + p\, Dh,$$

for some $h \in \mathcal{O}[[T]]$.

We relegate the proof of this proposition to the next section.

PROPOSITION 10. — *With f, g, r as in Proposition 9, we have:*

$$\int_n \Theta(f).\delta(g \circ f) \equiv \frac{1}{p} \int_{n-1} \lambda(r[-]\varphi f).\delta \mathcal{N}(g \circ f)$$

modulo $p^{n+1}\mathcal{O} + (1-\varphi)\mathcal{O}$.

Proof. — For $p > 2$, this is immediate from (3), (8) and Proposition 7. For $p = 2$, we need only show:

$$\overline{\mathcal{S}}^n(\varphi D^2 \Theta(f))|_0 \equiv 0 \bmod 2^{n+1}.$$

But this equals:

$$2^{2n} \varphi D^2 \overline{\mathcal{S}}^n(\Theta(f))|_0,$$

so as $\Theta(f) \in \mathcal{O}[[T]]$, this is congruent to zero modulo 2^{n+1} as long as $n > 0$. If $n=0$, then:

$$D^2(\Theta(f))|_0 = \Theta(f)'|_0 - (\Theta(f))''|_0 \equiv (1-\varphi) f'(0) \bmod 2,$$

since:

$$\Theta(f)''(0) \equiv 0 \bmod 2,$$
$$D(\lambda \circ f)(0) = D(2^{-1} \lambda(f([2])))(0) = f'(0) \qquad (f(0)=0).$$

Corollary 10 now follows.

Now suppose f and g are as in Proposition 8. Then as $\mathrm{ord}_T(\mathcal{N}(g)) = \mathrm{ord}_T(g) = 1$, there exists an $r \in \mathcal{M}$ such that $\mathcal{N}(g) \circ r = \mathcal{N}(g \circ f)$. Then our hypothesis on g implies that there is an $h \in \mathcal{O}[[T]]$ such that $\mathrm{ord}_T(h) \geqslant 1$ and:

$$(9) \qquad \varphi g \circ r + p^{n+2} h = \varphi(g \circ f).\varphi t.$$

(Recall $t = \varphi^{-1} \mathcal{N}(g \circ f)/g \circ f$.) Applying g^* (the composition inverse to φg) to both sides of (9) and expanding the left-hand side in $p^{n+2} h$ we deduce:

$$(10) \qquad r + p^{n+2} h/\varphi g'(r) \equiv g^*(\varphi(g \circ f).\varphi t) \bmod p^{2(n+2)}.$$

Now $t(v) = 1$ for $v \in \mathfrak{F}_{n-1}$, $v \neq 0$, as $g \circ f \in \mathcal{M}^{(n)}$ so that if we evaluate (10) at v for $v \in \mathfrak{F}_{n-1}$ (both sides vanish at zero) we get:

$$r(v) + p^{n+2} h(v)/\varphi g'(r(v)) \equiv \varphi f(v) \bmod p^{2(n+2)}.$$

Since $r[-]\varphi f = (r - \varphi f)/(1 - \varphi f)$ and $\lambda(x + p^k y) \equiv \lambda(x) \bmod p^k$ for $|x| < 1$ and k a positive integer, we deduce (setting $s = h/\varphi g'(r)$):

$$(11) \quad (\lambda(r[-]\varphi f).\delta \mathcal{N}(g \circ f))(v)$$
$$\equiv \left(\lambda\left(\frac{p^{n+2} s}{1 - \varphi f}\right).\delta \mathcal{N}(g \circ f)\right)(v) \bmod p^{2n+3},$$

for $v \in \mathfrak{F}_{n-1} - \{0\}$. Differentiating both sides of (9) and evaluating at zero gives us:

$$r'(0) + p^{n+2} h'(0)/\varphi g'(0) = \varphi(f'(0).t(0)),$$

since $r(0)=h(0)=g(0)=f(0)=0$ and $g'(0)\neq 0$. Therefore:

$$\begin{aligned}\lambda(r[-]\varphi f)'(0)&=r'(0)-\varphi f'(0)\\&=\varphi(f'(0)(t(0)-1))-p^{n+2}h'(0)/\varphi g'(0)\\&=-p^{n+1}\varphi\langle f, g\circ f\rangle_n-\lambda\left(\frac{p^{n+2}s}{1-\varphi f}\right)'(0),\end{aligned}$$

using (7). So as $\delta\mathcal{N}(g\circ f)$ has a simple pole with residue $\mathrm{ord}_T(f)$, we deduce from Proposition 10, (11) and (12) that:

$$B\equiv p^{-1}\int_{n-1}\lambda\left(\frac{p^{n+2}s}{1-\varphi f}\right).\delta\mathcal{N}(g\circ f)-\mathrm{ord}_T(f).\varphi\langle f, g\circ f\rangle_n,$$

modulo $p^{n+1}\mathcal{O}+(1-\varphi)H$. But $\lambda(p^{n+2}s/(1-\varphi f))\in p^{n+2}T\mathcal{O}[[T]]$ so we have:

$$B\equiv-\mathrm{ord}_T(f).\varphi\langle f, g\circ f\rangle_n\equiv-\varphi\langle f, g\circ f\rangle_n \text{ modulo } p^{n+1}\mathcal{O}+(1-\varphi)H,$$

since $\mathrm{ord}_T(f)>1$ if and only if $f'(0)=0$ if and only if $\langle f, g\circ f\rangle_n=0$. Now Proposition 8 follows immediately and (1) is proven.

VI. The Dilogarithm and Related Functions

In this section we will prove the following theorem and deduce Proposition 9 from it. (Recall that $\overline{\mathcal{S}}=p^{-1}\mathcal{S}$.)

THEOREM 11. – *Let $g\in\mathcal{M}$ and h any solution in $H[[T]]$, of:*

(1) $$Dh=\lambda\delta g.$$

Let f, r be elements of $\mathcal{M}$ such that $\mathrm{ord}_T(f)$ and $\mathrm{ord}_T(r)$ are positive, $\mathcal{N}(g)\circ r=\mathcal{N}(g\circ f)$ and $\varphi f\equiv r \bmod p\mathcal{O}[[T]]$. Then if $p>2$:

$$\overline{\mathcal{S}}(h\circ f)-\overline{\mathcal{S}}(h)\circ r\equiv 0 \bmod \mathcal{O}[[T]]$$

and if $p=2$ and $\mathcal{N}(g)'\in I^{x}$:

$$\overline{\mathcal{S}}(h\circ f)-\overline{\mathcal{S}}(h)r\equiv\frac{1}{2}\varphi D\Theta(f) \bmod \mathcal{O}[[T]].$$

Remark. – When $g=T$, we may take h to be the dilogarithm:

$$h(T)=\sum_1^\infty\frac{T^n}{n^2}.$$

Then in fact:

$$\overline{\mathcal{S}}(h)=\frac{h}{p^2}.$$

Before we begin the proof of Theorem 11, we will need a Lemma.

Let X and Y be variables. If $f\in\mathcal{O}[[X, Y]]$, $f(0, 0)=1$ we set:

$$\operatorname{Log} f=-\sum_{n=1}^{\infty}\frac{(1-f)^n}{n}.$$

Log f is then an element of $H[[X, Y]]$ which converges for X, $Y\in B\,(=\{a\in\mathbb{C}_p : |a|<1\})$. Therefore, if we set:

$$t_h(X, Y)=h(X(1+Y))-h(X)-\lambda(X).\operatorname{Log}\left(\frac{g(X(1+Y))}{g(X)}\right),$$

then $t_h(X, Y)$ is an element of $H[[X, Y]]$ which converges for $X, Y\in B$. Let $M_p=|p|^{(p-1)^{-1}}$.

LEMMA 12. — *If $x, y\in B$ then:*

$$|t_h(x, y)-e_h(x, y)|\leqslant\begin{cases}|y| & \text{if } |y|\leqslant M_p,\\ |y^2| & \text{if } |y|<|p|,\end{cases}$$

where:

$$e_h(X, Y)=\begin{cases}0 & \text{if } p>2,\\ \left(\dfrac{g'(X)}{g(X)}\cdot\dfrac{(XY)^2}{2(1-X)}\right)-\dfrac{1}{2}\left(\dfrac{g'(X)}{g(X)}\cdot\dfrac{(XY)^2}{2(1-X)}\right)^2 & \text{if } p=2.\end{cases}$$

Proof. — Let $\{h_i(X)\}_{i=1}^{\infty}$ be the elements of $H[[X]]$ such that:

$$\sum_{i=1}^{\infty} h_i(X)\,Y^i=t_h(X, Y)$$

$(t_h(X, 0)=0)$. Then differentiating both sides with respect to Y and using (1) we get:

$$(2)\quad \sum_{i=1}^{\infty} ih_i(X)\,Y^{i-1}=(\lambda(X(1+Y))-\lambda(X)).\frac{X\,g'(X(1+Y))}{g(X(1+Y))}$$

$$=\lambda\left(\frac{XY}{1-X}\right).\frac{X\,g'(X(1+Y))}{g(X(1+Y))}.$$

Since:

$$\lambda(T)=\sum_{n=1}^{\infty}\frac{T^n}{n},$$

we see easily from (2) that:

$$h_n(X) \in \frac{1}{n} p^{-m(n)} \mathcal{O}[[T]],$$

where $m(n) = \max_{1 \leqslant i < n} \operatorname{ord}_p(i)$. An easy estimate shows that:

$$\left| \frac{1}{n} p^{-m(n)} y^n \right| \leqslant |y|^i, \tag{3}$$

when $|y| \leqslant M_p$ and $i=1$, or when $|y| \leqslant |p|$, $i=2$, $p \geqslant 3$ and $n \geqslant 2$. Since (2) implies $h_1 = 0$, we have the Lemma for $p \geqslant 3$ and we are reduced to the case $p=2$, $|y| \leqslant |2|$, and $i=2$. As (3) holds for $n=3$ or $n > 4$, we need only estimate h_2 and h_4. Clearly from (2):

$$h_2(X) = \frac{X^2}{2(1-X)} \cdot \frac{g'(X)}{g(X)}$$

expanding the right-hand side of (2) in Y we deduce:

$$\begin{aligned} h_4(X) &\equiv \frac{1}{2}\left(\frac{X^2}{2(1-X)}\right)^2 \cdot \frac{d}{dX}\left(\frac{g'(X)}{g(X)}\right) \\ &\equiv -\frac{1}{2}\left(\frac{X^2}{2(1-X)} \cdot \frac{g'(X)}{g(X)}\right)^2 \bmod \frac{1}{4}\mathcal{O}[[X]], \end{aligned}$$

as $(d^2/dX^2)\mathcal{O}((X)) \subseteq 2\mathcal{O}((X))$. The Lemma follows from this and the fact that:

$$\left| \frac{1}{4} y^4 \right| \leqslant |y|^2 \quad \text{if} \quad |y| \leqslant |2|.$$

A few general comments will be useful before we begin the proof of Theorem 11.

For each real number r, $0 \leqslant r < 1$, let S_r denote the ring of analytic functions on $A(r) = \{a \in \mathbb{C}_p : r \leqslant |a| < 1\}$. That is, S_r is the set of all functions on $A(r)$ which have Laurent expansions around zero. Let S_r^0 denote the subring of S_r consisting of all $w \in S_r$ such that:

$$\operatorname{Lim}_{S \to 1, S > r} \operatorname{Sup}_{|T| = S} |w(T)| \leqslant 1.$$

Then if $1 > r' > r$, restriction maps S_r into $S_{r'}$ and S_r^0 into $S_{r'}^0$. We denote the direct limit of the S_r by S and of the S_r^0 by S^0. Under the natural map, S_r injects into S and we identify it with its image. If $w_1, w_2 \in S$ we write $w_1 \sim w_2$ whenever w_1 and w_2 are in the same coset of S^0.

This notion is useful to us because of the following fact.

Fact 13. — *If* $w \in H\,[[T]]_1$ *then* $w \in \mathcal{O}\,[[T]]$ *if and only if the image of* w *in* S *lies in* S^0.

This is an immediate consequence of the p-adic maximum principle and the fact that an element of $H\,[[T]]_1$ is bounded on B by the supremum of the absolute values of its coefficients. In fact;

Fact 14. — *If* $w \in S_a$ *for some* $0 \leqslant a < 1$, *then* $\mathrm{Lim}_{S \to 1, S > a} \sup_{|T|=S} |w(T)|$ *coincides with the supremum of the absolute values of the coefficients of the Laurent expansion of* w.

Let:

$$F = \overline{\mathcal{S}}(h \circ f) - \overline{\mathcal{S}}(h) \circ r.$$

Fact 13 implies that Theorem 11 is equivalent with the statement that (7) and (8) hold:

$$F \sim 0 \qquad \text{for} \quad p \geqslant 3, \tag{7}$$

$$F \sim \frac{1}{2}\,\varphi\, D\Theta(f) \qquad \text{for} \quad p=2, \qquad \mathrm{ord}_T(g)=1. \tag{8}$$

Proof of Theorem 12. — For $u \in \mathfrak{F}_0$ let:

$$X_u(T) = f(T)\,[+]\,u \qquad \text{and} \qquad Y_u(T) = \frac{f(T\,[+]\,u)}{f(T\,[+]\,u)} - 1.$$

Then for s close enough to 1 ($s < 1$), X_u and Y_u are elements of S_s and satisfy $X_u(T) < 1$ and $Y_u(T) < 1$ for $T \in A(s)$. It follows that:

$$G_u = t_h(X_u, Y_u),$$

is an element of S. It is evident that $\mathrm{Lim}_{S \to 1} \mathrm{Sup}_{|T|=S} |Y_u(T)| \leqslant M_p$ since $|u| = M_p$. It is also evident that $|e_h(x, y)| \leqslant |y|$ for $|y| \leqslant M_p$. Therefore, Lemma 11 implies:

$$p^{(1-p)^{-1}}\, G_u \sim 0 \tag{9}$$

(for any $(p-1)$ *st* root of p). Let $G = \sum_u G_u$, then (9) implies:

$$p^{(1-p)^{-1}}\, G \sim 0. \tag{10}$$

But, by (II (4)):

$$G = \mathcal{S}(h \circ f) \circ [p] - \mathcal{S}(h) \circ [p] \circ f - \lambda(f)\,\mathrm{Log}\left(\frac{\mathcal{N}(g \circ f) \circ [p]}{\mathcal{N}(g) \circ [p] \circ f}\right) \tag{11}$$

and so as a Laurent series, G has coefficients in H. Hence Fact 14 implies that (10) can be improved to:

$$p^{-1} G \sim 0. \tag{12}$$

Now let:

$$X = [p] \circ f \quad \text{and} \quad Y = \frac{r \circ [p]}{[p] \circ f} - 1.$$

Then for s close enough to 1, X and Y are elements of S_s and satisfy $|X(T)| < 1$ and $|Y(T)| < 1$, for $T \in A(s)$. Let $k = p^2 \overline{\mathscr{S}}(h)$, then $Dk = \lambda\delta \mathscr{N} g$ using (II, (10)). Thus:

$$J(T) = t_k(X(T), Y(T)) \quad \text{and} \quad E(T) = e_k(X(T), Y(T)),$$

are elements of S. It is evident that $p^{-1} Y(T) \sim 0$ so that Lemma 11 implies:

$$p^{-2} J \sim p^{-2} E, \tag{13}$$

but:

$$J = p\, \overline{\mathscr{S}}(h) \circ r \circ [p] - p\, \overline{\mathscr{S}}(h) \circ [p] \circ f - p\, \lambda(f) \operatorname{Log}\left(\frac{\mathscr{N}(g \circ f) \circ [p]}{\mathscr{N}(g) \circ [p] \circ f}\right),$$

so using (5) and (11) we see that:

$$p^{-1} G - p^{-2} J = F \circ [p] \sim p^{-2} E.$$

Now since $[p]$ maps B onto itself, (7) is immediate and we have proven the first part of the Theorem.

We now assume $p=2$, and $\mathscr{N}(g) \in I^*$. We must estimate $(1/4) E$ in this case. Let:

$$s = \mathscr{N}(g \circ f) \circ [2] - \mathscr{N}(g) \circ [2] \circ f,$$

and $t = \mathscr{N}(g) - \mathscr{N}(g)(0)$, then:

$$t \circ [2] \circ f + s = t \circ r \circ [2].$$

Let w denote the composition inverse to t. Then applying w to both sides of this equation and expanding in s we deduce:

$$r \circ [2] \equiv [2] \circ f + s.w'(t \circ [2](f)) \equiv [2] \circ f + s/t'([2](f)) \bmod 4.$$

On the other hand, using $t([2](T)) = g(T).g(2-T)$ we have:

$$s = g(f)(g(f(i+2(1-i))) - g(f+2(1-f)),$$

where $i(T) = T$. Expanding the last two terms in $2(1-i)$ and $2(1-f)$ respectively, we deduce:

$$s \equiv 2g(f)g'(f)((1-i)f' - (1-f)) \bmod 4.$$

With X and Y as above, we conclude:

$$(14) \qquad XY \equiv r \circ [2] - [2] \circ f \equiv 2g(f)\frac{g'(f)}{\mathcal{N}(g)'([2](f))}(Df - (1-f)),$$

modulo 4. Let:

$$N = \frac{\mathcal{N}(g)'(X)}{\mathcal{N}(g)(X)} \cdot \frac{(XY)^2}{2(1-X)}.$$

Then:

$$E = N - \frac{1}{2}N^2.$$

On the other hand (14) implies:

$$\frac{1}{2}\left(\frac{1}{2}N\right)^i \sim \frac{1}{2}h_1^i h_2^i\left(\left(\frac{(Df)^2}{1-[2]\circ f}\right)^i - h_3^i\right),$$

where $i = 1$ or 2 and:

$$h_1 = \frac{g(f)^2}{\mathcal{N}(g)([2](f))}, \qquad h_2 = \frac{g'(f)^2}{\mathcal{N}(g)'([2](f))},$$

$$h_3 = \frac{(1-f)^2}{1-[2](f)}.$$

Now it is easy to see that:

$$\frac{1}{2}h_1 \sim \frac{1}{2}h_2 \sim \frac{1}{2}h_3 \sim \frac{1}{2},$$

while:

$$\frac{1}{2}\left(\frac{(Df)^2}{1-[2]f}\right)^i \sim \frac{1}{2}(\delta f)^i, \qquad i = 1, 2.$$

On the other hand (14) also implies:

$$\frac{1}{2}N \sim \frac{1}{4}N^2 \sim 0.$$

It follows that:

$$\frac{1}{4}N \sim \frac{1}{2}((\delta f)^2 - 1),$$

$$\frac{1}{8}N^2 \sim \frac{1}{2}((\delta f)^4 - 1),$$

using the elementary fact that the hypotheses $(1/2)\,h \sim 1/2$ and $m \sim 0$ imply $(1/2)\,hm \sim (1/2)\,m$.

Thus:

$$\frac{1}{4}E \sim \frac{1}{2}((\delta f)^2 - (\delta f)^4)$$
$$\sim \frac{1}{2}\varphi(\delta(f)\circ[2] - \varphi\delta(f)\circ[4]) \sim \frac{1}{2}\varphi(D\Theta(f))\circ[2]$$

and this completes the proof of the Theorem.

Proof of Proposition 9. — Let F be as above. Then, on the one hand,

$$pDF = \overline{\mathscr{S}}(D(h\circ f)) - p^{-1}D(k\circ r)$$
$$= \overline{\mathscr{S}}(\lambda(f).\delta(g\circ f)) - p^{-1}\lambda(r).\delta(\mathscr{N}(g)\circ r)$$
$$= \overline{\mathscr{S}}(\lambda(f).\delta(g\circ f)) - p^{-1}\lambda(r).\delta\,\mathscr{N}(g\circ f).$$

While on the other hand:

$$\overline{\mathscr{S}}(\Theta(f).\delta(g\circ f)) = \overline{\mathscr{S}}(\lambda(f).\delta(g\circ f)) - p^{-1}\,\overline{\mathscr{S}}(\varphi\lambda(f([p]).\delta(g\circ f))$$
$$= \overline{\mathscr{S}}(\lambda(f).\delta(g\circ f)) - p^{-1}\varphi\lambda(f).\delta\,\mathscr{N}(g\circ f).$$

Thus Proposition 9 follows immediately from Theorem 11.

Appendix I: The Iwasawa Law and Non-degeneracy.

In this section we deduce IWASAWA's law [I] from ours and use it together with our results in $[C_1]$ to prove directly from the explicit formula that $[\alpha, \beta]_n = 1$ for all $\beta \in H_n^\times$ if and only if $\alpha \in (\mathscr{O}_n^\times)^{p^{n+1}}$. We also compute the right kernel. We extend $\int_n$ to power series which converge on the open unit ball in $\mathbb{C}_p$ in the obvious way.

LEMMA 13. — *If $f \in T\mathcal{O}_H[[T]]$, $g \in \mathcal{M}$, $t = \varphi^{-1}\mathcal{N}(g)/g$ and u is any generator of $\mathfrak{F}_n$, then:*

$$T_{H/\mathbf{Q}_p}\int_n \Theta(f).\delta g = \frac{1}{p^{n+1}} T_{H_n/\mathbf{Q}_p}(\lambda(f(u)).\delta g(u)) - \frac{1}{p} T_{H/\mathbf{Q}_p}\int_{n-1} \Theta(f).\delta t.$$

Proof:

$$\int_n \Theta(f).\delta g = \left(\overline{\mathcal{S}}^{n+1}(\lambda(f).\delta g) - \frac{1}{p}\overline{\mathcal{S}}^{n}(\varphi\lambda(f).\mathcal{S}\delta g)\right)\Big|_0$$

$$= \left(\overline{\mathcal{S}}^{n+1}(\lambda(f).\delta g) - \frac{1}{p}\varphi\overline{\mathcal{S}}^{n}(\lambda(f).\delta g) - \frac{1}{p}\varphi\overline{\mathcal{S}}^{n}(\lambda(f).\delta t)\right)\Big|_0$$

$$= \frac{1}{p^{n+1}}\left(\textstyle\sum_{u\in\mathfrak{F}_n}\lambda(f(u)).\delta g(u) - \varphi\sum_{v\in\mathfrak{F}_{n-1}}\lambda(f(v)).\delta g(v)\right)$$

$$-\frac{\varphi}{p}\int_{n-1}\lambda(f).\delta t.$$

The Lemma now follows.

PROPOSITION 14. — *If $f \in T\mathcal{O}_H[[T]]$, $g \in \mathcal{M}^{(2n+1)}$ and u is any generator of $\mathfrak{F}_n$, we have:*

$$(f, g)'_n \equiv \frac{1}{p^{n+1}} T_{H_n/\mathbf{Q}_p}(\lambda(f(u)).\delta g(u)) \bmod p^{n+1}.$$

Proof. — Since $g \in \mathcal{M}^{(2n+1)}$,

$$\frac{\varphi^{-1}\mathcal{N}(g)}{g}\Big|_0 \equiv 1 \bmod p^{2(n+1)}$$

so:

$$\langle f, g\rangle_n \equiv \{f, g\}_n \equiv 0 \bmod p^{n+1}.$$

On the other hand, if:

$$t = \frac{\varphi^{-1}\mathcal{N}(g)}{g} = 1 + \frac{p[p^{2n+1}](T)}{T}h(T),$$

$$u_i = [p^{n-i}](u), \qquad 0 \leqslant i \leqslant n,$$

then:

$$\delta t(u_i) = p^{2(n+1)}\frac{h(u_i)}{u_i}, \qquad 0 \leqslant i \leqslant n-1.$$

Therefore, as $\lambda(f(0)) = 0$:

$$\frac{1}{p}\int_{n-1}\Theta(f).\delta t = p^{n+1}\textstyle\sum_{i=0}^{n-1} T_i\left(\frac{\lambda(f(u_i)).h(u_i)}{u_i}\right),$$

where T_i is the trace from H_i to H. But, by Theorem 25 of [1]:

$$\frac{\lambda(f(u_i)).h(u_i)}{u_i}\in p^{-i}\mathcal{O}_i$$

and we know that $T_i(\mathcal{O}_i)=p^i\mathcal{O}_H$. The proposition now follows immediately from the preceding Lemma.

COROLLARY 15 (IWASAWA). — *If* $\alpha\in 1+\mathcal{P}_n$, $g\in\mathcal{M}^{(2n+1)}$ *and* u *is any generator of* $\mathfrak{F}_n$ *then:*

$$\operatorname{Ind}_u(\alpha, g(u))_n=T_{H_n/\mathbb{Q}_p}\left(\lambda(\alpha).\frac{\delta g(u)}{p^{n+1}}\right).$$

Proof. — This follows immediately from (II(2)) and the preceding Proposition.

Recall, $G_\infty=\operatorname{Gal}(H_\infty/H)$.

If A is a G_∞-module we let $A^{(1)}=A\bigotimes_{\mathbb{Z}_p}T_{\mathfrak{F}}$ where $T_{\mathfrak{F}}$ is the Tate module of $\mathfrak{F}$. If $A\subseteq H_n$ we identify $A^{(1)}$ with a submodule of $H_n^{(1)}$ in the natural way; $H_n^{(1)}$ is, of course, an H_n-module. Also, we define $T_{H_n/\mathbb{Q}_p}:H_n^{(1)}\to\mathbb{Q}_p^{(1)}$ by setting $T_{H_n/\mathbb{Q}_p}(\alpha\otimes u)=T_{H_n/\mathbb{Q}_p}(\alpha)\otimes u$. If $\beta\in\mathbb{Z}_p^{(1)}=T_{\mathfrak{F}}$ we let β_n denote the image of β in $\mathfrak{F}_n$ and we identify $\mathfrak{F}_n$ with $\mu_{p^{n+1}}$ under the natural isomorphism $u\mapsto 1-u$. Let:

$$\mathcal{X}'_n=N_{2n+1,n}H^\times_{2n+1}$$

and let:

$$\mathcal{X}_n=\{\beta\in H_n: T_{H_n/\mathbb{Q}_p}(\lambda(\alpha)\beta)\in\mathbb{Z}_p \text{ for all } \alpha\in\mathcal{P}_n\}.$$

We may now state:

COROLLARY 16. — *There exists a unique* G_∞*-homomorphism:*

$$\psi_n:\ X'_n\to\mathcal{X}_n^{(1)}/p^{n+1}\mathcal{X}_n^{(1)}$$

such that, for $\alpha\in 1+\mathcal{P}_n$, $\beta\in X'_n$,

$$[\alpha,\beta]_n=T_{H_n/\mathbb{Q}_p}(\lambda(1-\alpha).\tilde{\psi}_n(\beta))_n,$$

where $\tilde{\psi}_n(\beta)$ *is any element of* $\mathcal{X}_n^{(1)}$ *in the class of* $\psi_n(\beta)$ *mod* $p^{n+1}\mathcal{X}_n^{(1)}$.

Proof. — This follows immediately from Propositions 2 and 14. Indeed, if $\beta\in\mathcal{X}'_n$ and u is a generator of $\mathfrak{F}_n$, let $g\in\mathcal{M}^{(2n+1)}$ such that $g(u)=\beta$. Then we may set:

$$\psi_n(\beta)=\frac{\delta g(u)}{p^{n+1}}\otimes u \bmod p^{n+1}\mathfrak{X}_n^{(1)},$$

it is easy to see that this definition does not depend on the choice of u.

We proved the following result in [C₁].

THEOREM 17. — *The map ψ_n is a surjection.*

We extend $[\ ,\ ]_n$ to all of $H_n^\times\times H_n^\times$ by the relation $[-\alpha,\alpha]_n=1$. We will now use Theorem 12 to prove:

THEOREM 18. — *If $\alpha\in H_n^\times$ then $[\alpha,\beta]_n=1$ for all $\beta\in H_n^\times$ if and only if $\alpha\in(H_n^\times)^{p^{n+1}}$.*

Proof.— First suppose $\alpha\in\mathcal{O}_n^\times$. Then by Corollary 16 and Theorem 17 $\lambda(1-\alpha)\in p^{n+1}\lambda(\mathcal{P}_n)$. It follows that:

$$\alpha=\zeta\gamma^{p^{n+1}},$$

for some $\zeta\in\mu_{p^{n+1}}$ and $\gamma\in\mathcal{O}_n^\times$. But then if $\beta\in\mathcal{O}_n^\times$:

$$1=[\alpha,\beta]_n=[\zeta,\beta]_n=\zeta^{d_n(\beta)},$$

by Proposition 6. But we have given a direct proof in [C₁] that there exists a $\beta\in\mathcal{O}_n^\times$ such that $d_n(\beta)=1$. Thus $\zeta=1$ and $\alpha\in(\mathcal{O}_n^\times)^{p^{n+1}}$.

Now suppose α arbitrary in $H_n^\times$. We may write $\alpha=u^k\gamma$, where u is a generator of $\mathfrak{F}_n$, k is an integer and $\gamma\in\mathcal{O}_n^\times$. In view of what we have already proven, it suffices to find an element $\iota_u\in\mathcal{O}_n^\times$ such that:

$$[\iota_u,\gamma]_n=1 \quad \text{for all } \gamma\in\mathcal{O}_n^\times$$

and:

$$[\iota_u,u]_n=1-u.$$

To this end let $u_i=[p^{n-i}](u)$ for $-1\leqslant i\leqslant n$ and let:

$$\iota_{u,\varepsilon}=\exp\left(p^{n+1}\sum_{i=0}^{n}\frac{\varepsilon^{p^i}u_{n-i}}{p^i}\right) \qquad \text{for } \varepsilon\in V,$$

where:

$$\exp(T)=\sum_{k=0}^{\infty}\frac{T^k}{k!}.$$

LEMMA 19:

(i) $[\iota_{u,\varepsilon},\gamma]_n=1$ *for all* $\gamma\in\mathcal{O}_n^\times$;

(ii) $[\iota_{u,\varepsilon},u]_n=(1-u)\,T_{H/\mathbf{Q}_p}(\varepsilon)$.

Proof. — Let $h(T)=(p^{n+1}+[p^{n+1}](T)).(T-1)+p^{n+1}$ then $h(T)\in T^2\mathcal{O}_H[[T]]$, $h(u_i)=p^{n+1}u_i$ for $-1\leqslant i\leqslant n$. Let:

$$f=\exp\left(\sum_{i=0}^{\infty}\frac{\varepsilon^{p^i}h([p^i])}{p^i}\right).$$

Then by Theorem 25 of [C₂] $f(T)\in 1+T^2\mathcal{O}_H[[T]]$ and it is easy to see that:

$$f(u)=\iota_{u,\varepsilon}$$

and also that:

$$\Theta(1-f)=-\varepsilon h.$$

It follows that, for $g\in\mathcal{M}^{(n)}$:

$$(f,g)_n=T_{H/\mathbb{Q}_p}\int_n -\varepsilon h.\delta g=-T_{H/\mathbb{Q}_p}\left(\sum_{u\in\mathfrak{F}_n'}u.\delta g(u)\right).$$

Thus $(f,g)_n\equiv 0 \bmod p^{n+1}$ if $g\in\mathcal{O}_H[[T]]^\times$, and (i) follows. If $g(T)=T$ then:

$$(f,g)_n=T_{H/\mathbb{Q}_p}(-\varepsilon\sum_{u\in\mathfrak{F}_n'}(1-u))=T_{H/\mathbb{Q}_p}(\varepsilon).$$

Thus we have (ii). This completes the proof of Lemma 19 and so of Proposition 18.

Remark. — If $H'\supseteq H$ are unramified extensions of $\mathbb{Q}_p$, $\alpha\in H$ and $\beta\in H'$ then one can show directly from the formula that (in an obvious notation):

$$[\alpha,\beta]_{H',n}=[\alpha,N_{H'/H}(\beta)]_{H,n}.$$

We easily deduce from this:

THEOREM 20 (IWASAWA). — *If $T_{H'/\mathbb{Q}_p}(\varepsilon)$ is a unit in $\mathbb{Z}_p$ then the field $H_n((\iota_{u,\varepsilon})^{p^{-(n+1)}})$ is the unramified extension of H_n of degree p^{n+1}.*

Appendix II: The Generic Situation

In the course of proving our explicit law, we proved more about $\mathcal{O}_H[[T]]$ than we ever used. In this section, we reformulate some of these results in terms of a certain pairing on $\mathcal{O}_H((T))^\times$. For simplicity, we suppose that p is odd.

Let:

$$(\ ,\ ):\ \mathcal{O}_H[[T]]^\times\times\mathcal{O}_H((T))^\times\to\mathcal{O}_H((T))$$

denote the bilinear pairing determined by the conditions:

$$(f, g) = \Theta(1-f).\delta g,$$
$$(\varepsilon, g) = 0$$

for $f \in 1+\mathcal{M}$, $g \in \mathcal{O}_H((T))^x$ and $\varepsilon \in V$. Then we have:

THEOREM 21. — *If* $f \in \mathcal{M}^\infty$ *and* $1-f \in \mathcal{O}[[T]]^*$:

$$(1-f, f) \in D\mathcal{O}_H[[T]].$$

Before we give the proof of Theorem 21, we will need the following Lemma about D.

Let $\varepsilon_0 = \{ f \in \mathcal{O}_H[[T]] : \overline{\mathcal{S}} f = 0 \}$.

LEMMA 22. — $D\mathcal{E}_0 = \mathcal{E}_0$.

Proof. — Let:

$$A = \mathcal{O}_H[[T]] + \left\{ \frac{1}{p} f \circ [p] : f \in H[[T]],\ Df \in \mathcal{O}_H[[T]] \right\}.$$

It is clear that:

$$DA = D\mathcal{O}_H[[T]] + \mathcal{O}_H[[[p]]], \tag{1}$$

so that, in particular DA is a closed subset of $\mathcal{O}_H[[T]]$. We claim $DA = \mathcal{O}_H[[T]]$. It is sufficient to show $(d/dT)A = \mathcal{O}_H[[T]]$. But clearly,

$$T^{k-1} \in \frac{d}{dT} A$$

for k prime to p; and for $j>0$ in $\mathbb{Z}$:

$$\frac{d}{dT}\left(\frac{[p]^j}{j}\right) \in \frac{d}{dT} A.$$

But:

$$\frac{d}{dT}\left(\frac{[p]^j}{j}\right) = T^{jp-1} + \text{lower terms in } \mathcal{O}_H[T].$$

Since DA is closed our claim now follows.

Now let Λ be the operator on $\mathcal{O}_H[[T]]$ defined by:

$$\Lambda:\ f \to f - \overline{\mathcal{S}}(f) \circ [p].$$

Then Λ is a projector from $\mathcal{O}_H[[T]]$ onto $\mathcal{E}_0$ whose kernel is $\mathcal{O}_H[[[p]]]$. Also:

$$\Lambda Df = D\Lambda f. \tag{2}$$

Therefore, if $f \in \mathcal{E}_0$ by (1) we may write:

$$f = Dg + h \circ [p]$$

for some g and h in $\mathcal{O}_H[[T]]$. Applying (2) to this we get:

$$f = \Lambda f = \Lambda Dg + \Lambda h \circ [p] = D\Lambda g,$$

so that $f \in D\mathcal{E}_0$ and our Lemma is proven.

COROLLARY 23. — *If* $\overline{\mathcal{S}}(f) \in pD\mathcal{O}_H[[T]]$ *then* $f \in D\mathcal{O}_H[[T]]$.

Proof. — Using Λ we may write:

$$f = g + h \circ [p],$$

for some $g \in \mathcal{E}_0$ and $h \in \mathcal{O}_H[[T]]$. Then $\overline{\mathcal{S}}(f) = h$, so if $Dr = g$ and $pDs = h$ it follows that:

$$f = D(r + s \circ [p]).$$

Proof of Theorem 21. — Suppose $f \in \mathcal{M}^{\infty}$ and $1 - f \in \mathcal{O}[[T]]^*$. The $\operatorname{ord}_T(f) \geqslant 0$, and there exists an $\varepsilon \in V$ such that $f(0) \equiv 1 - \varepsilon \bmod p$. Let:

$$g(T) = 1 - \varepsilon(1 - T)$$

and:

$$w = 1 - \varepsilon^{-1}(1 - f).$$

Also let h be the solution of equation (VI(1)) with the above g such that:

$$\overline{\mathcal{S}}(h) = \frac{h}{p^2}$$

($\mathcal{N}(g) = \varphi\, g$ so such an h exists.) Then $g(w) = f$ and:

$$\overline{\mathcal{S}}((1-f, f)) = \overline{\mathcal{S}}(\lambda(w).\delta f) - \frac{\varphi(\lambda(w).\delta f)}{p}$$

$$= \overline{\mathcal{S}}(\lambda(w).\delta(g \circ w)) - \frac{\varphi\lambda(w).\delta(g \circ w)}{p} = pD\left(\overline{\mathcal{S}}(h \circ w) - \frac{h \circ \varphi w}{p^2}\right).$$

So as $\mathcal{N}(g) \circ \varphi w = \varphi g \circ \varphi w = \mathcal{N}(g \circ w)$, Theorem 21 follows from Theorem 11, and Corollary 23.

REFERENCE

[B_1] BLOCH (S.). — Applications of the dilogarithm function in algebraic *K*-theory and algebraic geometry, *Intl. Symp. on Algebraic Geometry*, Kyoto, 1977, p. 103-114.

[B_2] BLOCH (S.). — *Higher regulators, algebraic K-theory and zeta-functions of elliptic curves* (preprint).

BRUCKNER (H.). — *Explizetes rezirozitätsgesetz und anwendangen*, Vorlesuugen aus dem Fachbereich Mathematik der Universität Essen, 1979.

[C] COLEMAN (R.). — Division values in local fields, *Inventiones math.*, t. 53, 1979, p. 91-116.

[C_1] COLEMAN (R.). — *On the arithmetic of Lubin-Tate division towers* (to appear).

[I] IWASAWA (K.). — Explicit formulas for the norm residue symbol, *J. Math. Soc. Japan*, 1968, p. 151-164.

[S] SEN (S.). — *On explicit reciprocity laws*, Crelle Band, t. 313, 1980, p. 1-26.

[T] THURSTEN (W.). — *The geometry and topology of three-manifolds*, Princeton, Princeton University Press, 1980, Chapt. 7.

[V] VOSTOKOV, *Izvestia number*, t. 42, 1978, p. 6.

[W] WILES (A.). — Higher explicit reciprocity laws, *Ann. of Math.*, t. 107, 1978, p. 235-254.