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THE UNIVERSAL ORDINARY DISTRIBUTION

BY

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RÉSUMÉ. — Soit k un entier positif. Soit U^k le groupe abélien libre sur $\mathbf{Q}^k/\mathbf{Z}^k$, modulo le sous-groupe des relations de « distribution », définies plus loin. On appelle U^k la distribution ordinaire universelle de dimension k . Nous développons les propriétés fondamentales de U^k , qui trouvent des applications dans la théorie des nombres algébriques et la théorie des fonctions modulaires. Soit $U^k(N)$ le sous-module engendré par l'image de $(1/N)\mathbf{Z}^k/\mathbf{Z}^k$ dans U^k . Notons par $Z_k^*(N)$ l'ensemble des éléments primitifs d'ordre N dans $(1/N)\mathbf{Z}^k/\mathbf{Z}^k$. Parmi d'autres résultats, nous montrons que $U^k(N)$ est un $\mathbf{Z}$ -module libre de rang égal à la cardinalité de $Z_k^*(N)$. De plus, $U^k(N) \otimes \mathbf{Q}$ est isomorphe (comme $GL_k(\mathbf{Z}/N\mathbf{Z})$ -module) au $\mathbf{Q}$ -espace vectoriel libre sur l'ensemble $Z_k^*(N)$. On développe également la théorie des distributions de Bernoulli, ainsi qu'un autre modèle pour la distribution universelle ayant sa source dans un travail récent de Sinnott.

ABSTRACT. — Let k be a positive integer. Let U^k be the free abelian group on $\mathbf{Q}^k/\mathbf{Z}^k$ modulo the group of distribution relations (defined below). We call U^k the universal ordinary distribution of dimension k . We work out some of the basic structure theory for U^k , having applications in algebraic number theory and the theory of modular functions. Let $U^k(N)$ be the submodule generated by the image of $(1/N)\mathbf{Z}^k/\mathbf{Z}^k$ in U^k . Let $Z_k^*(N)$ denote the set of elements primitive of order N in $(1/N)\mathbf{Z}^k/\mathbf{Z}^k$. Then among other things, we show that $U^k(N)$ is a free $\mathbf{Z}$ -module of rank equal to the cardinality of $Z_k^*(N)$. Furthermore $U^k(N) \otimes \mathbf{Q}$ is isomorphic to the free $\mathbf{Q}$ -vector space on the set $Z_k^*(N)$, as a $GL_k(\mathbf{Z}/N\mathbf{Z})$ -module. The theory of Bernoulli distributions is also developed as well as another model for the universal distribution having its source in recent work of Sinnott.

Let k be a positive integer. Let M be the abelian group $(\mathbf{Q}/\mathbf{Z})^k$. Let f be a function from M to an abelian group A which satisfies the identity

$$\sum_{Nb=m} f(b) = f(m),$$

for all $m \in M$ and all positive integers N . We say then that f is an ordinary distribution from M to A .

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Given M there is an abelian group U_M and a map

$$\delta: M \rightarrow U_M,$$

which is the universal distribution for M . In other words, if $f: M \rightarrow A$ is a distribution, there exists a homomorphism $f_*: U_M \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\delta} & U_M \\ & \searrow f & \swarrow f_* \\ & A & \end{array}$$

It is obvious how to construct U_M . One simply takes the free abelian group on M , modulo the distribution relations.

The motivation for this study comes from the fact that ordinary distributions arise naturally in number theory when $k = 1$, and in the theory of modular forms when $k = 2$. Let $\mathbf{Q}^{ab}$ be the maximal abelian extension of $\mathbf{Q}$, which by Kronecker's Theorem is generated by all roots of unity. Let $A = (\mathbf{Q}^{ab})^*/\mathbf{Q}^*$, so A is an abelian group under multiplication. We define

$$f: \mathbf{Q}/\mathbf{Z} \rightarrow A,$$

by $f(0) = 1$, and for $x \in \mathbf{Q}/\mathbf{Z}$, $x \neq 0$, we let $f(x) = 1 - e^{2\pi i x}$. The identity

$$\prod_{i=1}^N (1 - \zeta^i X) = 1 - X^N,$$

shows that f is an ordinary distribution (see [B]).

Another distribution when $k = 1$ comes from the Bernoulli polynomial $\mathbf{B}_1(X) = X - (1/2)$. If $x \in \mathbf{R}$, we let $\langle x \rangle$ be the unique number such that

$$0 \leq \langle x \rangle < 1 \quad \text{and} \quad x \equiv \langle x \rangle \pmod{\mathbf{Z}}.$$

Let $A = \mathbf{Q}$, and put $f(x) = \mathbf{B}_1(\langle x \rangle)$. Then f is an ordinary distribution from $\mathbf{Q}/\mathbf{Z}$ to $\mathbf{Q}$. MAZUR uses this to obtain a measure theoretic approach to p -adic L -functions (see [M]).

A third distribution when $k = 1$ comes from the p -adic gamma function which has been used by GROSS and KOBLITZ to prove a version of Deligne's conjecture for periods of Fermat surfaces (see [G]).

In the case $k = 2$, the Siegel functions generate a natural ordinary distribution. Let $a = (a_1, a_2) \in \mathbf{Q}^2$ and $a \notin \mathbf{Z}^2$.

Define g_a by the q -expansion

$$g_a = -q_\tau^{(1/2) B_2(a_1)} e^{2\pi i a_2 (a_1 - 1)/2} (1 - q_z) \prod_{n=1}^{\infty} (1 - q_\tau^n q_z) (1 - q_\tau^n / q_z),$$

where $z = a_1 \tau + a_2$, $q_\tau = e^{2\pi i \tau}$, and $B_2(X) = X^2 - X + (1/6)$ (see [L], p. 251).

It is easy to see that if $a \equiv a' \pmod{\mathbf{Z}^2}$, then $g_a = g_{a'}$ modulo constants. Let A be the group generated by the functions g_a modulo constants. Then we define a map

$$g: (\mathbf{Q}/\mathbf{Z})^2 \rightarrow A,$$

by $g(a) = g_a$, which is well-defined by the above remark. It is then easy to check that g is an ordinary distribution.

Let U^k denote the universal distribution associated with $(\mathbf{Q}/\mathbf{Z})^k$. Then U^k is naturally a $GL_k(\hat{\mathbf{Z}})$ -module, where $\hat{\mathbf{Z}}$ is the completion of $\mathbf{Z}$ under the ideal topology, and

$$\hat{\mathbf{Z}} \approx \prod_p \mathbf{Z}_p.$$

The module structure is derived from the natural action of $GL_k(\hat{\mathbf{Z}})$ on $(\mathbf{Q}/\mathbf{Z})^k$, and the fact that $GL_k(\hat{\mathbf{Z}})$ takes the group of distribution relations into itself.

We shall determine exactly the structure of U^k as a group, and we show that U^k is free. We also give a canonical system of free generators. We also determine precisely the structure of $\mathbf{Q} \otimes U^k$ as a $GL_k(\hat{\mathbf{Z}})$ -module.

In the present paper, we first produce free generators for $U^k(N)$.

Next we give two examples of universal distributions on $(\mathbf{Q}/\mathbf{Z})^k$. The first one arises from a classical construction of Stickelberger elements, and the second is related to some ideas of SINNOTT [S].

In the paper which immediately follows the present one, we calculate the cohomology groups of U^k as a module over the group $\{\pm \text{id}\}$. We are motivated to do this for the following reasons. First in the case when $k = 2$, we have shown in [K 2] that the calculation of the unit group in the modular function field involves the determination of $H^0(\pm \text{id}, U^2)$. More precisely, it had been shown earlier that the Siegel units given above have rank equal to the rank of the full set of modular units. In [K 2], we show that the group of units modulo the Siegel units is in fact a $\mathbf{Z}/2$ $\mathbf{Z}$ -vector space, which injects naturally into $H^0(\pm \text{id}, U^2)$, and which maps onto $H^0(\pm \text{id}, U^2(N))$ when N is odd.

In [S], SINNOTT calculates the index of the Stickelberger ideal for composite N . He finds that this index equals the odd part of the class number of the cyclotomic field of conductor N times half the square root of the order of $H^1(\pm \text{id}, U^1(N))$. He also calculates the index of the units in the cyclotomic field of conductor N modulo the circular units, and finds this index to be the even part of the class number times a power of 2, again closely related to the order of $H^0(\pm \text{id}, U^1(N))$.

The group $H^0(\pm \text{id}, U^1(N))$ also represents an obstruction in [6] to getting the field of definition predicted by DELIGNE for certain periods of Fermat surfaces, thus providing another motivation for its study.

1. Free generators for $U^k(N)$

In this section, we show how to produce a set of free generators for $U^k(N)$. Let $F^k(N)$ be the free abelian group

$$\frac{1}{N} \mathbf{Z}^k / \mathbf{Z}^k = \left(\frac{1}{N} \mathbf{Z} / \mathbf{Z} \right)^k.$$

We let F^k be the free abelian group on $\mathbf{Q}^k / \mathbf{Z}^k$, so we have an injection for each positive integer N :

$$(1.1) \quad 0 \rightarrow F^k(N) \rightarrow F^k.$$

Let D^k be the subgroup of F^k generated by the distribution relations. Set

$$D_N^k = F^k(N) \cap D^k.$$

We define a related group $D^k(N)$ as the group generated by elements

$$(1.2) \quad \sum_{Mb=a} (b) - (a), \quad \text{with } M \mid N, \text{ and } a \in \frac{M}{N} \mathbf{Z}^k / \mathbf{Z}^k.$$

Thus $D^k(N) \subset D_N^k$. We will show that, in fact, $D^k(N) = D_N^k$. We define $U^k(N) = F^k(N) / D_N^k$, so we have a surjective map

$$(1.3) \quad F^k(N) / D^k(N) \rightarrow U^k(N) \rightarrow 0.$$

For $M \mid N$ define

$$(1.4) \quad Z_k^*(M) = \left\{ x \in \frac{1}{M} \mathbf{Z}^k / \mathbf{Z}^k \text{ such that } x \text{ has order } M \right\}.$$

Thus $Z_k^*(M)$ is the set of primitive elements in $(1/M)\mathbf{Z}^k/\mathbf{Z}^k$. When k is fixed, we often omit the subscript k and write simply $Z^*(M)$.

We now exhibit explicit generators for $F^k(N)/D^k(N)$ whose cardinality is equal $Z_k^*(N)$. We shall see later that $U^k(N)$ has free rank at least equal to this cardinality, so we can conclude that our generators are free, that therefore

$$D^k(N) = D_N^k,$$

and finally that $U^k(N)$ is a free $\mathbf{Z}$ -module of rank $|Z_k^*(N)|$.

Let $N = \prod p^{n(p)}$ be the prime power decomposition of N . We say that M is an admissible divisor of N if $(M, N/M) = 1$. This means that if p divides M , then $p^{n(p)}$ divides M . Using k -tuples to describe elements of $\mathbf{Q}^k/\mathbf{Z}^k$, we denote by $e(N)$ the element

$$(1.5) \quad e(N) = \left(\frac{1}{N}, 0, \dots, 0 \right).$$

Note that $Z^*(M)$ is naturally the direct product of the sets

$$Z^*(p^{n(p)}), \quad \text{where } p \mid M.$$

Let $Z^*(M, p)$ be the subset of elements of $Z^*(M)$ with p -component equal to $e(p^{n(p)})$. Let

$$(1.6) \quad T^*(M) = Z^*(M) - \bigcup_{p \mid M} Z^*(M, p) \quad \text{if } M \neq 1, \\ T^*(1) = \{0\}.$$

Then let

$$(1.7) \quad T(N) = \bigcup_M T^*(M) \text{ where } M \text{ is admissible, } M \mid N.$$

The Theorem we wish to prove is the following.

THEOREM 1.8.

- (i) The cardinality of $T(N)$ is $|Z_k^*(N)|$.
- (ii) $T(N)$ is a free basis for $U^k(N)$.

From (i) and the lower bound, we shall obtain for the rank of $U^k(N)$, to prove (ii), it suffices to prove the following Proposition.

PROPOSITION 1.9. — $T(N)$ generates $F(N)/D(N)$.

We see immediately that

$$T(N) = \prod_{p \mid N} [Z^*(p^{n(p)}) - \{e(p^{n(p)})\} \cup \{0\}].$$

Thus the first part of the Theorem follows immediately.

We now prove Proposition 1.9. Set $A(N) = F(N)/D(N)$. Let $B(N)$ be the group generated by the image of $T(N)$ in $A(N)$. We say that an element $t \in (1/N) \mathbb{Z}^k / \mathbb{Z}^k$ is *available* if the image of (t) in $F(N)/D(N)$ belongs to $B(N)$. We must show that each element of $(1/N) \mathbb{Z}^k / \mathbb{Z}^k$ is available. We induct on the number of prime factors of N . Suppose first that $N = p^n$, where p is prime. The only element of $Z^*(M)$ which is not available is $e(p^n)$. But since by the distribution relations

$$\sum_{t \in Z^*(N)} (t) = 0,$$

we see that $e(p^n)$ is available. If $M \mid N$ and $M \neq 1$, and $t \in Z^*(M)$, then

$$t = \sum_{s \in Z^*(N), (N/M)s=t} (s),$$

so t is available. Finally (0) is available, because $(0) \in T(N)$. Thus the Proposition is proved in the prime power case.

Let $A'(N)$ be the group generated by the elements (t) for $t \in Z^*(M)$, where M is admissible.

LEMMA 1.10. $A(N) = A'(N)$.

Proof. — Let M be an admissible divisor of N . Let E be a positive integer dividing M and having the same prime factors as M . Then

$$(t) = \sum_{s \in Z^*(M), (M/E)s=t} (s).$$

This proves the Lemma.

So by the Lemma, we must show that $B(N) = A'(N)$. By induction we know that M is admissible divisor of N , $M \neq N$, then

$$A'(M) = B(M) \subset B(N).$$

Hence it suffices to show that if $t \in Z^*(N)$, then $t \in B(N)$. Set

$$W(N) = \bigcup_{p \mid N} Z^*(N, p).$$

We must show that if $t \in W(N)$ then $t \in B(N)$, since these are the elements excised from $Z^*(N)$. Given $t \in W(N)$, let $V(t)$ be the set of primes p such that $t \in Z^*(N, p)$. Set

$$v(t) = |V(t)|.$$

We induct on $v(t)$ to show that each t is available. Define

$$(1.11) \quad W_i = \{t \in W(N) \text{ such that } v(t) = i\}.$$

Then

$$W(N) = \coprod_{i=1}^s W_i$$

is the disjoint union. We make use of the following Lemma.

LEMMA 1.12. — Given $t \in W(N)$ and p such that $t \in Z^*(N, p)$. Set

$$Y = \{y \in Z^*(N) \text{ such that } p^{n(p)}y = p^{n(p)}t\}.$$

Suppose that $Y - \{t\}$ is available. Then t is available.

Proof. — Let Y_p be the set of $p^{n(p)-1}$ multiples of elements of Y . Then the distribution relations show that

$$Y_p - \{p^{n(p)-1}t\},$$

is available. Write $N = p^{n(p)}M$. By induction, $z = p^{n(p)}t$ is available, and so is $w \in Z^*(M)$ for which $pw = z$. Now by the distribution relations,

$$\sum_{s \in Y_p} (s) + (w) = z.$$

We conclude that $p^{n(p)-1}t$ is available. But then using the obvious distribution relation, we see that t is available.

Suppose now that $t \in W_1$. There is a unique p such that $t \in Z^*(N, p)$. We claim that if $y \in W(N)$ and $p^{n(p)}y = p^{n(p)}t$, then $y = t$. To see this, since $p^{n(p)}y = p^{n(p)}t$, we may write the partial fraction decomposition

$$t = \sum_{q|N, q \neq p} a(q) + e(p^{n(p)}), \quad y = \sum_{q|N, q \neq p} a(q) + a(p).$$

Since $t \in W_1$, we have $a(q) \neq e(q^{n(q)})$ for $q \neq p$. So if $y \in W(N)$, then $a(p) = e(p^{n(p)})$ implies $y = t$. Applying Lemma 1.12, we conclude that W_1 is available.

Suppose $t \in W^r$, and by induction that W_s is available for $s < r$. Choose p such that $t \in Z^*(N, p)$. Let Y be as in Lemma 1.12. Then

$$Y \cap \coprod_{i \geq r} W_i = t.$$

Indeed, suppose y is in the intersection on the left hand side. We have

$$t = \sum_{q \neq p} a(q) + e(p^{n(p)}) \quad \text{and} \quad y = \sum_{q \neq p} a(q) + a(p).$$

Thus $v(y) \leq v(t)$, with equality if, and only if, $a(p) = e(p^{n(p)})$, which implies that $y = t$, as claimed.

Applying Lemma 1.12 shows that W^r is available, and concludes the proof of Proposition 1.9 and Theorem 1.8.

2. The Cartan group

Let k be a positive integer. Given a prime number p , there is a unique unramified extension of $\mathbf{Q}_p$ of degree k , which we denote by $\mathbf{Q}_p^k$. We denote the integers of $\mathbf{Q}_p^k$ by $\mathfrak{o}_p^k$, or also $\mathfrak{o}_p$. The units $\mathfrak{o}_p^*$ form a group $C_p^k = C_p$, which is the non-split unramified Cartan group of degree k associated with the prime p , the Cartan group for short. Given a positive integer n , we define

$$(2.1) \quad \mathfrak{o}^k(p^n) = \mathfrak{o}_p^k / p^n \mathfrak{o}_p^k.$$

We set

$$(2.2) \quad C^k(p^n) = \mathfrak{o}_p^* / (1 + p^n \mathfrak{o}_p^k),$$

and we call $C^k(p^n)$ the non-split Cartan group of level p^n . We have

$$(2.3) \quad C^k(p^n) = (\mathfrak{o}^k(p^n))^*.$$

Let N be a positive integer,

$$N = \prod_{p|N} p^{n(p)}$$

Set

$$(2.4) \quad \mathfrak{o}^k(N) = \prod_{p|N} \mathfrak{o}^k(p^{n(p)}) \quad \text{and} \quad C^k(N) = \mathfrak{o}_N^* = \prod_{p|N} C^k(p^{n(p)}).$$

The groups $C^k(N)$ clearly form a projective system and we denote by C^k the projective limit, which is the non-split Cartan group of degree k . Clearly,

$$(2.5) \quad C^k = \prod_p C_p^k.$$

If k is fixed in the course of a discussion, we will often omit the superscript k .

There is a natural isomorphism

$$\mathfrak{o}^k(N) \approx \prod_{p|N} \frac{1}{N} \mathfrak{o}_p^k / \mathfrak{o}_p^k,$$

as an $\mathfrak{o}^k(N)$ -module, with 1 going to the element with coordinate $1/N$ at each prime p under the natural map. As a group,

$$\prod_{p|N} \frac{1}{N} \mathfrak{o}_p^k / \mathfrak{o}_p^k \text{ is isomorphic to } \left(\frac{1}{N} \mathbf{Z}/\mathbf{Z} \right)^k.$$

The group $C^k(N)$ corresponds under this isomorphism to the primitive elements of $((1/N) \mathbf{Z}/\mathbf{Z})^k$, i. e. to $Z_k^*(N)$. In particular, the order of $C^k(N)$ equals the number of primitive elements of $((1/N) \mathbf{Z}/\mathbf{Z})^k$, and we may consider that $C^k(N)$ acts simply transitively on $Z_k^*(N)$.

Let L be a field and fix k . Denote by $L^k(N)$ the free L -vector space generated by the primitive elements of $((1/N) \mathbf{Z}/\mathbf{Z})^k$. Then $L^k(N)$ is a module over $GL_k(\hat{\mathbf{Z}})$, which factors through $GL_k(\mathbf{Z}/N\mathbf{Z})$. Let $M \mid N$. Then we have an injection

$$(2.6) \quad 0 \rightarrow L^k(M) \xrightarrow{i} L^k(N)$$

as $GL_k(\hat{\mathbf{Z}})$ -modules, which is defined as follows. If x is a primitive element of $((1/M) \mathbf{Z}/\mathbf{Z})^k$ set

$$i(x) = \sum_{(N/M)y=x} (y),$$

where the sum is taken over primitive elements y in $((1/N) \mathbf{Z}/\mathbf{Z})^k$ such that $(N/M)y = x$. The map is clearly a $GL_k(\hat{\mathbf{Z}})$ -morphism. We wish to identify this map with a map on the Cartan group rings. Let $L[C^k(N)]$ be the group ring of $C^k(N)$. If $M \mid N$, we have an injection of C^k -modules

$$(2.7) \quad 0 \rightarrow L[C^k(M)] \xrightarrow{i} L[C^k(N)],$$

given by

$$i(x) = \sum_{y \equiv x \pmod{M}} (y),$$

where $x \in C^k(M)$ and $y \in C^k(N)$. Maps (2.6) and (2.7) are identical under the isomorphism of $\mathfrak{o}^k(N)$ with $\prod_{p \mid N} (1/N) \mathfrak{o}_p^k / \mathfrak{o}_p^k$.

Denote by $L \langle C^k \rangle$ the injective limit of $L[C^k(N)]$. We shall construct ordinary distributions from $(\mathbf{Q}/\mathbf{Z})^k$ to $L \langle C^k \rangle$, i. e. homomorphisms from U^k to $L \langle C^k \rangle$. Let

$$\varphi: \mathbf{Q}/\mathbf{Z} \rightarrow L$$

be a map such that if $M \mid N$ and $a \in (1/M) \mathbf{Z}/\mathbf{Z}$, then

$$(2.8) \quad N^{k-1} \sum_{(N/M), b=a} \varphi(b) = M^{k-1} \varphi(a).$$

(This may be called a distribution of weight $k-1$.) We shall construct an associated distribution Φ with values in $L \langle C^k \rangle$, i. e. a map

$$\Phi: U^k \rightarrow L \langle C^k \rangle$$

as follows. Let

$$\lambda: \prod_p \mathfrak{o}_p^k \rightarrow \prod_p \mathbf{Z}_p$$

be a surjective homomorphism. Then we have naturally derived surjective homomorphisms

$$\lambda_N: \mathfrak{o}^k(N) \rightarrow \mathbf{Z}/N\mathbf{Z}$$

which satisfy an obvious consistency property. If $x \in (1/N) \mathbf{Z}^k/\mathbf{Z}^k$, we may consider $Nx \in \mathfrak{o}^k(N)$ as seen above. Set

$$(2.9) \quad \Phi(x) = \sum_{c \in C^k(N)} \varphi\left(\frac{1}{N} \lambda_N(cNx)\right) c^{-1}.$$

We must first check that this map is consistent and then that the distribution relations are satisfied. So suppose $x \in (1/M) \mathbf{Z}^k/\mathbf{Z}^k$, where $M \mid N$. We must show

$$(2.10) \quad \sum_{c \in C^k(N)} \varphi\left(\frac{1}{N} \lambda_N(cNx)\right) c^{-1} = \sum_{d \in C^k(M)} \varphi\left(\frac{1}{M} \lambda_M(dMx)\right) d^{-1}.$$

as elements of the injective limit.

To see this, fix d , and let $\{c\}$ be such that $c \bmod M = d$. Then in the group ring,

$$d^{-1} = \sum_{c \bmod M = d} c^{-1}.$$

Now $\lambda_N(cNx) = \lambda_M(cNx)$, considering cNx as an element of $\mathfrak{o}^k(M)$, and

$$\lambda_M(cNx) = \frac{N}{M} \lambda_M(cMx) = \frac{N}{M} \lambda_M(dMx),$$

which proves (2.10). We now show that the distribution relations are satisfied. Let $x \in (1/M) \mathbf{Z}^k/\mathbf{Z}^k$. We wish to show that if $M \mid N$, then

$$(2.11) \quad \sum_{(N/M)y=x} \Phi(y) = \Phi(x).$$

But

$$\sum_{(N/M)y=x} \Phi(y) = \sum_{(N/M)y=x} \sum_{c \in C^k(N)} \varphi\left(\frac{1}{N} \lambda_N(cNy)\right) c^{-1}.$$

Now

$$\frac{N}{M} \cdot \frac{1}{N} \lambda_N(cNy) = \frac{1}{N} \lambda_N(cNx) \bmod \mathbf{Z}.$$

Furthermore λ_N maps $\mathfrak{o}^k(N)$ onto $\mathbf{Z}/N\mathbf{Z}$. If $z \in (1/N) \mathbf{Z}/\mathbf{Z}$ is such that

$$(N/M)_I z = \frac{1}{N} \lambda_N(cNx),$$

then there exists y such that $(1/N) \lambda_N(c N y) = z$. It is clear from the elementary divisor Theorem that the number of such y is equal to $(N/M)^{k-1}$. So

$$\begin{aligned} \sum_{(N/M)y=x} \Phi(y) &= \sum_{c \in C^k(N)} (N/M)^{k-1} \sum_{(N/M)z=(1/N)\lambda_N(cNx)} \varphi(z) c^{-1} \\ &= \sum_{c \in C^k(N)} \varphi\left(\frac{1}{N} \lambda_N(cNx)\right) c^{-1} \quad \text{by (2.8)} \\ &= \Phi(x). \end{aligned}$$

In the next section we exhibit functions φ satisfying (2.8).

3. Bernoulli distributions

The following relation is equivalent to (2.8).

$$(3.1) \quad N^{k-1} \sum_{Nb=a} \varphi(b) = \varphi(a).$$

simply by replacing N/M by N . We consider the special case when $L = \mathbf{R}$ is the field of real numbers. We also consider functions φ which satisfy (3.1) for $a \in \mathbf{R}/\mathbf{Z}$. Choosing $t \in (0, 1)$, we may rewrite (3.1) as

$$(3.2) \quad N^{k-1} \sum_{r=0}^{N-1} \varphi\left(\frac{t}{N} + \frac{r}{N}\right) = \varphi(t),$$

PROPOSITION 3.3. — *Let $\varphi(t)$ be of class $C^{(k+1)}$ on $(0, 1)$, and assume that φ satisfies (3.2) for all positive integers N . Then there is a constant α such that*

$$\varphi(t) = \alpha \mathbf{B}_k(t),$$

where $\mathbf{B}_k(X)$ is the k -th Bernoulli polynomial.

Proof. — The polynomial $\mathbf{B}_k(X)$ is defined by the series

$$\frac{ue^{uX}}{e^u - 1} = \sum \mathbf{B}_k(X) \frac{u^k}{k!}.$$

It is a classical fact that $\mathbf{B}_k(t)$ satisfies (3.2), and can easily be shown from the above definition (see for instance [L], p. 230). If φ satisfies (3.2) for a certain integer k , then φ' satisfies (3.2) for $k-1$. By induction, $\varphi'(t) = \alpha \mathbf{B}_k(t)$, so φ' is uniquely determined up to an additive constant. Since $\mathbf{B}_k(t)$ satisfies (3.2), and since for any number $c \neq 0$ the function

$\mathbf{B}_k(t) + c$ does not satisfy (3.2), the Proposition follows if we prove it for $k = 1$. Differentiating twice, we get

$$N^{-2} \sum_{r=0}^{N-1} \varphi''\left(\frac{t}{N} + \frac{r}{N}\right) = \varphi''(t).$$

Since φ'' is bounded on $[0, 1]$ by assumption, letting $N \rightarrow \infty$ we conclude that $\varphi''(t) = 0$ for all t . So φ'' is linear. Since $\mathbf{B}_1(t)$ satisfies (3.2) while $\mathbf{B}_1(t) + c$ does not, the Proposition is proved.

We wish to find a function φ such that the associated distribution Φ gives an isomorphism from U^k to its image. The Bernoulli distribution does not accomplish this since the polynomial $\mathbf{B}_k(t)$ is odd (resp. even) as k is odd (resp. even) under the map $x \mapsto 1 - x$. As we have seen in [K 1], $\mathbf{B}_k(t)$ essentially yields the universal even or odd distribution, depending on the parity of k . Proposition 3.2 says that we must loosen the smoothness conditions on φ to accomplish this.

Let L now be the field of complex numbers $\mathbf{C}$. Let φ be an L^2 -function from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{C}$. I am indebted to D. ROHRICH for the following Lemma.

LEMMA 3.4. — *Let $\varphi \in L^2([0, 1])$, and suppose φ satisfies (3.2) for all positive integers N . Let $a(N)$ be the N -th Fourier coefficient of φ . Then:*

$$a_0 = 0, \quad a(N) = \frac{a(1)}{N^k}, \quad a(-N) = \frac{a(-1)}{N^k}$$

for all integers $N > 0$.

Proof. — For $c \in \mathbf{Z}$, set

$$\hat{\varphi}(c) = \int_0^1 \varphi(t) e^{-2\pi i c t} dt.$$

By (3.2) we have

$$\begin{aligned} \hat{\varphi}(c) &= N^{k-1} \int_0^1 \sum_{j=0}^{N-1} \varphi\left(\frac{t}{N} + \frac{j}{N}\right) e^{-2\pi i c t} dt \\ &= N^k \int_0^1 \sum_{j=0}^{N-1} \varphi\left(\frac{t}{N} + \frac{j}{N}\right) e^{-2\pi i c t} d\left(\frac{t}{N}\right). \end{aligned}$$

Set $t' = t/N$. Then

$$\begin{aligned} \hat{\varphi}(c) &= N^k \int_0^1 \sum_{j=0}^{N-1} \varphi\left(t' + \frac{j}{N}\right) e^{-2\pi i c N t'} dt' \\ &= N^k \sum_{j=0}^{N-1} \int_0^{1/N} \varphi\left(t' + \frac{j}{N}\right) e^{-2\pi i c N t'} dt' \\ &= N^k \int_0^1 \varphi(t') e^{-2\pi i c N t'} dt'. \end{aligned}$$

Putting $c = 0$, $N \neq 1$, we see that $a(0) = 0$. Putting $c = 1$, we get $a(N) = a(1)/N^k$. Putting $c = -1$, we get $a(-N) = a(-1)/N^k$, which proves the Lemma.

So the family of L^2 -functions satisfying (3.2) for fixed k is essentially one-dimensional. It is easy to see that

$$(3.5) \quad \mathbf{B}_k(t) = \frac{(-1)^{(k/2)-1} k!}{(2\pi)^k} \sum_{n=1}^{\infty} \left(\frac{e^{2\pi i n t}}{n^k} + \frac{e^{-2\pi i n t}}{n^k} \right) \quad \text{if } k \text{ is even,}$$

$$\mathbf{B}_k(t) = \frac{(-1)^{(k+1)/2} k!}{i(2\pi)^k} \sum_{n=1}^{\infty} \left(\frac{e^{2\pi i n t}}{n^k} - \frac{e^{-2\pi i n t}}{n^k} \right) \quad \text{if } k \text{ is odd.}$$

Let

$$(3.6) \quad G_k(t) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n t}}{n^k} \quad \text{for } k > 1,$$

$$G_1(t) = \log(1 - e^{2\pi i t}),$$

where the log is the principal branch, and $0 < t < 1$. If $k > 1$, we check easily that $G_k(t)$ satisfies (3.2), namely

$$\begin{aligned} \sum_{j=0}^{N-1} G_k\left(\frac{t}{N} + \frac{j}{N}\right) &= \sum_{j=0}^{N-1} \sum_{n=1}^{\infty} \frac{e^{2\pi i (t/N + j/N) n}}{n^k} \\ &= \sum_{n=1}^{\infty} \frac{e^{2\pi i t n/N}}{N^k} \sum_{j=0}^{N-1} e^{2\pi i j n/N} \\ &= \sum_{N \mid n} N \frac{e^{2\pi i t n/N}}{n^k} \\ &= \frac{1}{N^{k-1}} \sum_{n=1}^{\infty} \frac{e^{2\pi i n t}}{n^k} = \frac{1}{N^{k-1}} G_k(t). \end{aligned}$$

For $k = 1$, we have the representation

$$(3.7) \quad G_1(t) = \log(1 - e^{2\pi i t}) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n t}}{n},$$

which is valid by the Abel summation formula for $0 < t < 1$. By considering $\exp G_1(t) = 1 - e^{2\pi i t}$, it is easy to see that $G_1(t)$ satisfies (3.2) for $0 < t < 1$. The corresponding relation to that for $t = 0$ is

$$(3.8) \quad \sum_{r=0}^{N-1} G_1\left(\frac{r}{N}\right) = \log N.$$

So $G_1(t)$ will not strictly produce a distribution. We might say it produces a modified distribution. We may however produce a function φ satisfying (3.1) for each $a \in \mathbf{Q}/\mathbf{Z}$ by choosing $u \in \hat{\mathbf{Z}}^*$, setting $\varphi(0)$ to any arbitrary value, and

$$(3.9) \quad \varphi(a) = G_1(ua) - G_1(a) \quad \text{for } a \in \mathbf{Q}/\mathbf{Z}, \quad a \neq 0.$$

Using (3.9) one can then construct the universal distribution for $k = 1$ from the function $G_1(t)$. We shall leave the details to the reader, and give the Theorem here only for $k > 1$.

THEOREM 3.10. — *Let Φ_k be the distribution associated with the function $G_k(t)$ for $k > 1$. Then:*

(i) *The map Φ_k gives an isomorphism of $U^k(N)$ with its image, as $C^k(N)$ -modules.*

(ii) *Let $\Phi_k(N)$ be the image of $U^k(N)$ under Φ_k . Then*

$$\Phi_k(N) \otimes \mathbf{C} = \mathbf{C}[C^k(N)].$$

Proof. — Since $U^k(N)$ has a set of generators $T^k(N)$ of cardinality $|Z_k^*(N)| = |C^k(N)|$, it suffices to prove that $\Phi_k(N)$ has free rank $|C^k(N)|$, and thus it suffices to prove (ii). This is equivalent to showing that for each character χ of $C^k(N)$, the χ -component of $\Phi_k(N) \otimes \mathbf{C}$ is non-trivial, since $\Phi_k(N)$ is a $C^k(N)$ -module by construction. By (2.10), the χ -component is

$$(3.11) \quad S(\Phi, \chi) = \sum_{c \in C^k(N)} \bar{\chi}(c) \varphi\left(\frac{1}{N} \lambda_N(cNx)\right).$$

But from [KL], we find that for each χ , the sum $S(\Phi, \chi)$ is non-zero if, and only if, for each character ψ of $(\mathbf{Z}/N\mathbf{Z})^*$, we have

$$(3.12) \quad \sum_{c \in (\mathbf{Z}/f\mathbf{Z})^*} \psi(c) \varphi\left(\frac{c}{f}\right) \neq 0,$$

where f is the conductor of ψ . In our case, $k > 1$, we have

$$\varphi\left(\frac{c}{f}\right) = \sum_{n=1}^{\infty} \frac{e^{2\pi i cn/f}}{n^k}.$$

Hence

$$\begin{aligned} \sum_{c \in (\mathbf{Z}/f\mathbf{Z})^*} \psi(c) \varphi\left(\frac{c}{f}\right) &= \sum_{c \in (\mathbf{Z}/f\mathbf{Z})^*} \psi(c) \sum_{n=1}^{\infty} \frac{e^{2\pi i cn/f}}{n^k} \\ &= \sum_{n=1}^{\infty} \sum_c \frac{\psi(c) e^{2\pi i cn/f}}{n^k}. \end{aligned}$$

By standard properties of Gauss sums, we know that

$$\sum_c \psi(c) e^{2\pi i c n / f} \begin{cases} = 0 & \text{if } (n, f) \neq 1, \\ \neq 0 & \text{if } (n, f) = 1. \end{cases}$$

so

$$\sum_{n=1}^{\infty} \sum_c \frac{\psi(c) e^{2\pi i c n / f}}{n^k} = S(\chi) \sum_{(n, f)=1} \frac{\bar{\psi}(n)}{n^k}$$

where

$$S(\psi) = \sum_c \psi(c) e^{2\pi i c / f} \quad \text{and} \quad \sum_{(n, f)=1} \frac{\bar{\psi}(n)}{n^k} = L(k, \bar{\psi}) \neq 0,$$

by the product expression for the L -series. This proves the Theorem.

4. The rational distribution

In this section, we present another model for the universal distribution, taking its values in $\mathbf{Q} \langle C^k \rangle$, and which we therefore call the rational distribution. For the case $k = 1$, the image of the distribution appears in SINNOTT [S], although it is not identified as such.

We will define maps

$$r^k(N) : \frac{1}{N} \mathbf{Z}^k / \mathbf{Z}^k \rightarrow \mathbf{Q}[C^k(N)],$$

which will be $GL_k(\hat{\mathbf{Z}})$ -morphisms, after a choice of basis for $\mathfrak{o}^k(N)$. We have a natural bijection

$$\frac{1}{N} \mathfrak{o}^k(N) / \mathfrak{o}^k(N) \rightarrow \mathfrak{o}^k(N) / N \mathfrak{o}^k(N),$$

obtained by $x \mapsto Nx$. A choice of basis identifies $\mathfrak{o}^k(N)$ with $\mathbf{Z}^k / N \mathbf{Z}^k$, which then becomes a $C^k(N)$ -module.

Let $a \in (1/N) \mathbf{Z}^k / \mathbf{Z}^k$. Let $f(a)$ be the order of a in $(1/N) \mathbf{Z}^k / \mathbf{Z}^k$. Define

$$(4.1) \quad \frac{1}{N} X(a) = \left\{ x \in \frac{1}{N} \mathbf{Z}^k / \mathbf{Z}^k \text{ such that } x \right. \\ \left. \text{is primitive and } (N/f(a))x = a \right\}.$$

In terms of the Cartan group, we can then write

$$(4.2) \quad X(a) = \{c \in C^k(N) \text{ such that} \\ (N/f(a))c_p \equiv (Na)_p \bmod p^{n(p)}\},$$

where $(Na)_p$ is the p -th coordinate of Na , for $p \mid N$.

Let $N = \prod q^{n(q)}$. For $p \mid N$ define the set $X_p(N)$ by

$$(4.3) \quad X_p(N) = \{c = (c_q)_q \in C^k(N) \text{ such that if } q \neq p \\ \text{then } c_q \equiv p^{-1} \bmod q^{n(q)} \mathfrak{o}_q\}.$$

If X is a subset of $C^k(N)$, we define

$$(4.4) \quad s(X) = \sum_{x \in X} (x).$$

We now define $r^k(N) = r(N)$ by

$$(4.5) \quad r(N)(a) = s(X(a)) \sum_{p \mid f(a)} \left(1 - \frac{s(X_p(N))}{|X_p(N)|}\right),$$

so that $r(N)(a) \in \mathbf{Q}[C(N)]$. We first show that $r(N)$ is a $GL_k(N)$ -map.

PROPOSITION 4.6. — Let $\gamma \in GL_k(N)$. Then

$$r(N)(\gamma a) = \gamma(r(N)(a)).$$

Proof. — It is clear that $f(\gamma a) = f(a)$. Set

$$\varepsilon = \prod_{p \mid f(a)} \left(1 - \frac{s(X_p(N))}{|X_p(N)|}\right).$$

Multiplication by ε is an element of $\text{End}(\mathbf{Q}^k(N))$. Then

$$r(N)(a) = \varepsilon(s(X(a))) \quad \text{and} \quad r(N)(\gamma a) = \varepsilon(s(X(\gamma a))).$$

From (4.1), it is clear that $s(X(\gamma a)) = \gamma s(X(a))$. So it suffices to show that

$$\gamma \varepsilon = \varepsilon \gamma \quad \text{for all } \gamma \in GL_k(N).$$

Let

$$(4.7) \quad \varepsilon_p(N) = 1 - \frac{s(X_p(N))}{|X_p(N)|}.$$

It then suffices to show that $\gamma \varepsilon_p(N) = \varepsilon_p(N) \gamma$, or that

$$s(X_p(N))\gamma(c) = \gamma(s(X_p(N))c) \quad \text{for } c \in C^k(N).$$

Now $s(X_p(N))c = s(X)$, where

$$X = \{x \in C^k(N) \text{ such that } x_q = p^{-1}c_q \text{ for all } q \neq p\},$$

$$\gamma X = \{x \in C^k(N) \text{ such that } x_q = p^{-1}(\gamma c_q) \text{ for all } q \neq p\}.$$

Thus $\gamma \varepsilon_p(N) = \varepsilon_p(N) \gamma$, and the Proposition follows.

Next we show that the maps $\gamma(N)$ are compatible with the injective limits. If $M \mid N$, we let $i: \mathbf{Q}[C(M)] \rightarrow \mathbf{Q}[C(N)]$ be the map of (2.7):

PROPOSITION 4.8. — *If $M \mid N$, the following diagram commutes.*

$$\begin{array}{ccc} \frac{1}{M} \mathbf{Z}^k / \mathbf{Z}^k & \xrightarrow{r(M)} & \mathbf{Q}[C^k(M)] \\ \downarrow & & \downarrow \\ \frac{1}{N} \mathbf{Z}^k / \mathbf{Z}^k & \xrightarrow{r(N)} & \mathbf{Q}[C^k(N)]. \end{array}$$

Proof. — Let $c_1, c_2 \in C^k(M)$. Set

$$N = \prod_p p^{n(p)} \quad \text{and} \quad M = \prod_p p^{m(p)}.$$

Then

$$(4.9) \quad i(c_1 c_2) = \frac{C(M)}{C(N)} i(c_1) i(c_2).$$

If g is the number of distinct prime factors of $f(a)$, we have

$$i(r(M)(a)) = \left(\frac{|C(M)|}{|C(N)|} \right)^g i(s(X_M(a)) \prod_{p \mid f(a)} i\left(1 - \frac{s(X_p(M))}{|X_p(M)|}\right).$$

Now $i(s(X_M(a))) = s(X_N(a))$ because

$$(M/f(a))x_p \equiv (Ma)_p \pmod{p^{m(p)}},$$

is equivalent with

$$(N/f(a))x_p \equiv (Na)_p \pmod{p^{n(p)}}.$$

But $i(1) = \sum (c)$, where the sum is taken for $c \equiv 1 \pmod{M}$,

$$i(s(X_p(M))) = s(X_p(N)) i(1) \frac{|X_p(M)|}{|X_p(N)|}.$$

So

$$\prod_{p \mid f(a)} i\left(1 - \frac{s(X_p(M))}{|X_p(M)|}\right) = \prod_{p \mid f(a)} i(1) \left(1 - \frac{s(X_p(N))}{|X_p(N)|}\right).$$

But

$$s(X_N(a))i(1) = |i(1)|s(X_N(a)) = \frac{|C(N)|}{|C(M)|}s(X_N(a)),$$

which proves the Proposition.

PROPOSITION 4.10. — *The maps $r(N)$ define a distribution. Precisely, let $f(a) = N$ and $M \mid N$. Then*

$$\sum_{Mb=Ma} r(N)(b) = r(N)(Ma).$$

Proof. — By induction we may assume that $M = q$ is prime. We distinguish the cases $q \mid (N/q)$ and $q \nmid (N/q)$.

First suppose $q \mid (N/q)$. In this case, each b such that $qb = qa$ is primitive, i. e. has order N . So

$$\sum_b r(N)(b) = \sum_b s(X(b)) \prod_{p \mid N} \left(1 - \frac{s(X_p)}{|X_p|}\right).$$

Set $a' = qa$. Since $q \mid (N/q)$, it follows that $p \mid f(a')$ if, and only if, $p \mid N$. So

$$r(N)(a') = s(X(qa)) \prod_{p \mid N} \left(1 - \frac{s(X_p)}{|X_p|}\right).$$

So we need only show that

$$X(qa) = \bigcup_{qb=qa} X(b).$$

Since b is primitive, we have $X(b) = \{Nb\}$, and

$$c \in X(qa) \text{ if and only if } c_p = Na_p \text{ for } p \neq q \text{ and } qc_q = qNa_q.$$

But this is equivalent to $qc/N = qa$. So

$$\sum_b s(X(b)) = s(X(qa)),$$

and the Proposition is proved in this case.

Next suppose $q \nmid (N/q)$. If $qb = qa$ then $b_p = a_p$ for $p \neq q$ and $q(b_q - a_q) \in \mathfrak{o}_q$. Since $q \nmid (N/q)$, it follows that $qa_q \in \mathfrak{o}_q$. Define $\bar{b}$ to be the element such that $\bar{b}_q = 0$, and if $q \neq p$, then $\bar{b}_q = a_q$. Thus if $qb = qa$ and b does not equal $\bar{b}$, we see that $b_q \notin \mathfrak{o}_q$, and that $f(b) = f(a) = N$. Now

$$\sum_{qb=qa} r(N)(b) = \sum_{b \neq \bar{b}} r(N)(b) + r(N)(\bar{b}).$$

If $b \neq \bar{b}$ we have

$$r(N)(b) = (Nb) \prod_{p \mid N} \left(1 - \frac{s(X_p)}{|X_p|}\right).$$

Therefore

$$\begin{aligned}\sum_{b \neq \bar{b}} r(N)(b) &= (\sum_{b \neq \bar{b}} \bar{b}(N b)) \prod_{p|N} \left(1 - \frac{s(X_p)}{|X_p|}\right) \\ &= (\sum_{b \neq \bar{b}} \bar{b}(N b)) \prod_{p \neq q} \left(1 - \frac{s(X_p)}{|X_p|}\right) \\ &\quad - \frac{1}{|X_q|} (\sum_{b \neq \bar{b}} s(X_q)(N b)) \prod_{b \neq \bar{b}} \left(1 - \frac{s(X_p)}{|X_p|}\right).\end{aligned}$$

Set $a' = qa$. Then $f(a') = f(\bar{b}) = N/q$. Furthermore,

$$\begin{aligned}X(\bar{b}) &= \{c \in C(N) \text{ such that } qc_p = Na_p \bmod p^{n(p)} \text{ if } p \neq q \\ &\quad \text{and } qc_q = 0 \bmod q\}\end{aligned}$$

$$r(N)(\bar{b}) = s(X(\bar{b})) \prod_{p \neq q} \left(1 - \frac{s(X_p)}{|X_p|}\right).$$

Now from the above,

$$s(X_q) \sum_{b \neq \bar{b}} \bar{b}(N b) = |X_q| s(X(\bar{b})).$$

So

$$\frac{1}{|X_q|} (\sum_{b \neq \bar{b}} s(X_q)(N b)) \prod_{p \neq q} \left(1 - \frac{s(X_p)}{|X_p|}\right) = r(N)(\bar{b}),$$

and thus

$$\sum_{qa=q\bar{b}} r(N)(b) = (\sum_{b \neq \bar{b}} \bar{b}(N b)) \prod_{p \neq q} \left(1 - \frac{s(X_p)}{|X_p|}\right).$$

It is immediate from the Definitions that

$$s(X(a')) = \sum_{b \neq \bar{b}} \bar{b}(N b).$$

Since

$$r(N)(a') = s(X(a')) \prod_{p \neq q} \left(1 - \frac{s(X_p)}{|X_p|}\right),$$

The Proposition is proved in this case also.

THEOREM 4.11. — *The maps $r(N)$ define the universal distribution, and we have*

$$r(N) U(N) \otimes \mathbf{Q} = \mathbf{Q}[C(N)].$$

Proof. — Let $V(N) = r(N) U(N)$ and $V_{\mathbf{Q}}(N) = \mathbf{Q} \otimes V(N)$. Then $V(N)$ is a $C(N)$ -module. Recall that a divisor M of N is called admissible

if $(M, N/M) = 1$. Then the distribution relations show that $V(N)$ is generated as a $\mathbf{Z}$ -module by $r(N)b$, where $f(b) = M$, M admissible. Since for any element $c \in C(N)$ we have

$$cr(N)(b) = r(N)(cb)^{\dagger}$$

we conclude that the elements $r(N)(1/M)$ with admissible divisors M of N generate $V(N)$ as a $C(N)$ -module. Here $1/M$ means the element with p -component $1/M$ for each $p \mid N$,

$$\frac{1}{M} \in \frac{1}{p^{n(p)}} \mathfrak{o}_p / \mathfrak{o}_p.$$

Put $R(N) = \mathbf{Z}[C(N)]$ and $R_{\mathbf{Q}}(N) = \mathbf{Q}[C(N)]$. Let

$$(4.12) \quad V_p = s(X(p^{n(p)}/N))R(N) + \left(1 - \frac{s(X_p)}{|X_p|}\right)R(N).$$

Then we claim that

$$V_p \otimes \mathbf{Q} = R_{\mathbf{Q}}(N).$$

We first see that

$$|X_p| = |X_p| \left(1 - \frac{s(X_p)}{|X_p|}\right) + s(X_p),$$

so it suffices to show that

$$s(X_p) \in s(X(p^{n(p)}/N))R(N).$$

Let $\lambda \in C(N)$ be such that $\lambda_q \equiv p^{-1} \pmod{q^{n(q)}}$ for $q \neq p$. Now

$$X(p^{n(p)}/N) = \{c \in C(N) \text{ such that } c_q \equiv 1 \pmod{q^{n(q)}} \text{ for } q \neq p\}.$$

So $\lambda X(p^{n(p)}/N) = X_p$ and $s(X_p) \in s(X(p^{n(p)}/N))R(N)$. Thus the following Proposition will prove the Theorem.

PROPOSITION 4.13. — $V(N) = \prod_{p \mid N} V_p$.

Proof. — The proof here follows as in SINNOTT [S]. Set

$$V_N = \prod_{p \mid N} V_p, \quad \text{and} \quad \bar{N} = \prod_{p \mid N} p.$$

Also let

$$\gamma_p = 1 - \frac{s(X_p)}{|X_p|}.$$

As an $R(N)$ -module, V_N is generated by the following elements:

$$\prod_{q \mid (\bar{N}/\bar{M})} s\left(X\left(\frac{q^{n(q)}}{N}\right)\right) \prod_{p \mid \bar{M}} \gamma_p \quad \text{for all divisors } \bar{M} \mid \bar{N}.$$

But it is easy to see that

$$\prod_{q \mid (\bar{N}/\bar{M})} s\left(X\left(\frac{q^{n(q)}}{N}\right)\right) = s\left(X\left(\frac{1}{M}\right)\right)$$

where $M = \prod_{p \mid \bar{M}} p^{n(q)}$ and M is admissible. Moreover,

$$\prod_{p \mid \bar{M}} \gamma_p = \prod_{p \mid M} \gamma_p,$$

which proves the Proposition and thus also Theorem 4.11.

We now summarize our knowledge about $U^k(N)$.

THEOREM 4.14.

- (i) $U^k(N)$ is a free $\mathbf{Z}$ -module of rank $|C^k(N)|$.
- (ii) $U^k(N) \otimes \mathbf{Q}$ is isomorphic to the free $\mathbf{Q}$ -vector space on $Z_k^*(N)$ as a $GL_k(\mathbf{Z}/N\mathbf{Z})$ -module.
- (iii) $D^k(N) = D_N^k$.
- (iv) The map $\Phi_k(N)$, $k \geq 2$, is an isomorphism of $U^k(N)$ with its image as a $C^k(N)$ -module.
- (v) The map $r_k(N)$, $k \geq 1$, is an isomorphism of $U^k(N)$ with its image, as a $GL_k(\mathbf{Z}/N\mathbf{Z})$ -module.

Finally, we draw some conclusion about the universal even and odd ordinary distribution. We define U_+^k to be the quotient module of U^k obtained from the relations

$$(x) - (-x) = 0, \quad x \in \mathbf{Q}^k/\mathbf{Z}^k.$$

We define U_-^k to be the quotient module of U^k obtained from the relations

$$(x) + (-x) = 0, \quad x \in \mathbf{Q}^k/\mathbf{Z}^k.$$

Let $U_+^k(N)$ (resp. $U_-^k(N)$) be the groups generated by the image of $(1/N)\mathbf{Z}^k/\mathbf{Z}^k$ in U_+^k (resp. U_-^k). We have the following Corollary.

COROLLARY 4.15.

- (i) $U_+^k(N) \otimes \mathbf{Q}$ has rank equal to $(1/2) |C^k(N)|$.
 $U_-^k(N)$ has rank equal to $(1/2) |C^k(N)|$ if $N > 2$.
- (ii) If $N = 2$, then rank $U_+^k(N) \otimes \mathbf{Q} = 2^k - 1$, and

$$\text{rank } U_-^k(N) \otimes \mathbf{Q} = 0.$$

If $N = 1$, then

$$\text{rank } U_+^k(N) \otimes \mathbf{Q} = 1 \quad \text{and} \quad \text{rank } U_-^k(N) \otimes \mathbf{Q} = 0.$$

(iii) *We have isomorphisms as $\mathbb{Z}/2$ $\mathbb{Z}$ -vector spaces:*

$$U_+^k(\text{torsion}) \approx H^1(\pm \text{id}, U^k),$$

$$H_-^k(\text{torsion}) \approx H^0(\pm \text{id}, U^k).$$

Proof. — Statements (i) and (ii) follow directly from Theorem 4.17 (ii). To prove (iii) we use Theorem 4.17 (i). Suppose

$$\bar{u} \in U_+^k(N), \quad n\bar{u} = 0.$$

Let u belong to $U^k(N)$ such that the image of u in $U_+^k(N)$ is $\bar{u}$. Then, since nu is a relation, we have

$$n((u) + (-u)) = 0,$$

where $-u$ represents the action of -1 as an element of $GL_k(\hat{\mathbb{Z}})$ on u . Since $U^k(N)$ is torsion free, we must have $(u) + (-u) = 0$, or $u \in \mathbb{Z}^1(\pm \text{id}, U^k(N))$. Then $\bar{u} = 0$ if and only if u is a boundary. The argument is similar for $U_-^k(\text{torsion})$. This proves Corollary 4.15.

When $k = 1$, the results of Corollary 4.15 (i) and (ii) were previously obtained by BASS [B] and YAMAMOTO [Y] in the even and odd case respectively. In the case $k = 2$, the calculation of $H^0(\pm \text{id}, U^k)$ has an interpretation in the theory of modular forms (see [K 2]). We will calculate these cohomology groups in a following paper.

Appendix

The following broader notion has also proved useful in certain applications, e. g. Bernoulli polynomials on $\mathbb{Q}/\mathbb{Z}$, and also [Ma], [Mi]. Let A be an abelian group and let

$$g: \mathbb{Q}/\mathbb{Z}^k \rightarrow A,$$

be a map. We say that g is a distribution of weight w (a positive integer), if for each positive integer N we have

$$(A1) \quad N^w \sum_{Nb=a} g(b) = g(a).$$

For $k = 1$, the Bernoulli polynomial $B_w(X)$ yields a distribution of weight $w-1$. One may clearly speak of the universal distribution on $\mathbb{Q}^k/\mathbb{Z}^k$ of weight w , and we denote it by $U^{k,w}$. We may also speak of the level

groups $U^{k,w}(N)$. There is a rational isomorphism of $U^{k,w}(N)$ into the group ring $\mathbb{Q}[C^k(N)]$ given by

$$(A2) \quad r^{k,w}(N)(a) = f(a)^{-w} s(X(a)) \prod_{p|f(a)} \left(1 - p^w \frac{s(X_p(N))}{|X_p(N)|} \right),$$

where $a \in (1/N) \mathbb{Z}^k / \mathbb{Z}^k$. We have the following analogue of Theorem 4.17.

THEOREM :

- (i) $U^{k,w}(N)$ is a free $\mathbb{Z}$ -module of rank $|C^k(N)|$.
- (ii) $U^{k,w}(N) \otimes \mathbb{Q}$ is isomorphic to the free $\mathbb{Q}$ -vector space on $Z_k^*(N)$ as a $GL_k(\mathbb{Z}/N\mathbb{Z})$ -module.

The proof is as follows. By (A2) we see as before that $U^{k,w}(N)$ has free rank at least $|C^k(N)|$. The Theorem will therefore follow if we can show that $U^{k,w}(N)$ is generated as an abelian group by at most $|C^k(N)|$ elements. It is clear from (A1) that the elements we chose in Section 1 in the case $w = 0$ will no longer generate $U^{k,w}(N)$ for $w > 0$. We now proceed as follows. Define

$$(A3) \quad \langle a \rangle = \sum_{(N/f(a))b=a} (b),$$

so

$$(A4) \quad (N/f(a))^w \langle a \rangle = (a).$$

We prove the following, distribution law'' for $\langle a \rangle$.

LEMMA. — Let p be a prime such that p divides $N/f(a)$.

(i) If $p | f(a)$, then

$$\sum_{pb=a} \langle b \rangle = \langle a \rangle.$$

(ii) If $p \nmid f(a)$, then

$$\sum_{pb=a, f(b)=pf(a)} \langle b \rangle = \langle a \rangle - p^w \langle p^{-1}a \rangle.$$

Proof. — For (i), we have

$$\begin{aligned} \sum_{pb=a} \langle b \rangle &= \sum_b \sum_{(N/p)f(a)c=b} (c) \\ &= \sum_{(N/f(a))c=a} (c) = \langle a' \rangle. \end{aligned}$$

For (ii), we have

$$\begin{aligned} \sum_{pb=a, f(b)=pf(a)} \langle b \rangle &= \sum_b \sum_{(N/p)f(a)c=b} (c) \\ &= \sum_{(N/f(a))c=a} (c) - \sum_{(N/p)f(a)c=p^{-1}a} (c) \\ &= \langle a \rangle - p^w \sum_{(N/f(a))d=p^{-1}a} (d) \quad \text{by (A1)} \\ &= \langle a \rangle - p^w \langle p^{-1}a \rangle. \end{aligned}$$

We may now proceed as in Section 1 to show that the family of elements $\{\langle a \rangle\}$ with $a \in T(N)$ generates $U^{k,w}(N)$ by using the distribution relations for $\langle a \rangle$. This completes the proof of the Theorem.

One special feature when $w > 0$ is that $U^{k,w}(N) \otimes \mathbb{Q}$ has as a $\mathbb{Q}$ -basis the elements (x) , where $x \in Z_k^*(N)$. This may be proved easily by induction. We must only show that the elements of exact denominator N/p belong to the $\mathbb{Q}$ -vector space generated by $Z_k^*(N)$. This is obvious from the distribution relations if $(N/p, p) \neq 1$. Suppose $p \nmid (N/p)$. Let $x \in Z_k^*(N/p)$. Then modulo the distribution relations and the image of $Z_k^*(N)$ in $U^{k,w}(N)$, the element (x) is congruent to $p^w(p^{-1}x)$, where

$$(p^{-1}x) \in Z_k^*(N/p).$$

Let v be the order of p in $(\mathbb{Z}/(N/p)\mathbb{Z})^*$. Then by induction we see that (x) is congruent to $p^{wv}(x)$ modulo the image of $Z_k^*(N)$ in $U^{k,w}(N)$. So $(p^{wv}-1)(x)$ belongs to the group generated by $Z_k^*(N)$ in $U^{k,w}(N)$. Since $w > 0$, we have $p^{wv}-1 \neq 0$, and the result follows by tensoring with $\mathbb{Q}$. The corresponding statement when $w = 0$ is not true.

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