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ON THE GEL'FAND-KIRILLOV CONJECTURE FOR INDUCED IDEALS IN THE SEMISIMPLE CASE

BY

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RÉSUMÉ. — Soient $\mathfrak{g}$ une algèbre de Lie semi-simple complexe, $\mathfrak{p}$ une sous-algèbre parabolique de $\mathfrak{g}$, E un $U(\mathfrak{p})$-module simple de dimension finie, M le $U(\mathfrak{g})$-module induit par E, et $I = \operatorname{Ann} M$. La représentation de $U(\mathfrak{g})$ dans M définit une injection ψ de $U(\mathfrak{g})/I$ dans la partie ad $\mathfrak{g}$ finie de $\operatorname{Hom}_{\mathbf{C}}(M, M)$. Supposons que M est simple et ψ surjective. Alors I est un idéal primitif, donc $U(\mathfrak{g})/I$ admet un anneau de fractions Fract $U(\mathfrak{g})/I$. Notons $m = \operatorname{codim} \mathfrak{p}$, et $\mathscr{A}_m$ l'algèbre de Weyl d'indice m sur $\mathbf{C}$. On démontre que Fract $U(\mathfrak{g})/I = \operatorname{End} E \otimes \operatorname{Fract} \mathscr{A}_m$, à isomorphisme près et que Fract $U(\mathfrak{g})/I$ contient un sous-corps maximal commutatif qui est ad $\mathfrak{g}$ stable. Pour $\mathfrak{g}$ simple $\neq sl(n+1) : n \in \mathbf{N}^+$, on montre que ce dernier résultat n'est pas toujours vrai pour un idéal primitif quelconque.

ABSTRACT. — Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$, E a finite dimensional $U(\mathfrak{p})$-module, M the $U(\mathfrak{g})$-module induced from E, and $I = \operatorname{Ann} M$. The representation of $U(\mathfrak{g})$ in M defines an injection ψ of $U(\mathfrak{g})/I$ into the ad $\mathfrak{g}$ finite part of $\operatorname{Hom}_{\mathbf{C}}(M, M)$. Suppose that M is simple and ψ surjective. Then I is a primitive ideal, and so $U(\mathfrak{g})/I$ admits a ring of fractions Fract $U(\mathfrak{g})/I$. Set $m = \operatorname{codim} \mathfrak{p}$ and let $\mathscr{A}_m$ denote the Weyl algebra of index m over $\mathbf{C}$. It is shown that Fract $U(\mathfrak{g})/I = \operatorname{End} E \otimes \operatorname{Fract} \mathscr{A}_m$, up to isomorphism and that Fract $U(\mathfrak{g})/I$ admits a maximal commutative subfield which is ad $\mathfrak{g}$ stable. For $\mathfrak{g}$ simple $\neq sl(n+1)$: $n \in \mathbf{N}^+$, it is shown that this last result is not a general property of an arbitrary primitive ideal.

1. Introduction

1.1: Let k be a commutative, algebraically closed field of characteristic zero, $\mathfrak{g}$ a k – Lie algebra which is finite dimensional and algebraic. Let $U(\mathfrak{g})$ (resp. $K(\mathfrak{g})$) denote the enveloping algebra (resp. field) of $\mathfrak{g}$, and Prim $U(\mathfrak{g})$

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(resp. Spec $U(\mathfrak{g})$) the set of primitive (resp. prime) ideals of $U(\mathfrak{g})$. Given $I \in \operatorname{Spec} U(\mathfrak{g})$, then by Goldie's theorem $U(\mathfrak{g})/I$ admits a ring of fractions which for some $n \in \mathbf{N}^+$ is isomorphic to the matrix ring M_n over a skew field K. We call n the Goldie rank, and K the Goldie field of I.

1.2: Given $r, s \in \mathbf{N}$, let $\mathscr{A}_{r,s}$ denote the generalized Weyl algebra of order r and index s over k. $\mathscr{A}_{r,s}$ is isomorphic to the associative k-algebra with identity and generators x_i, $\partial/\partial x_j : i = 1, 2, \ldots, r+s; j = 1, 2, \ldots, r$. GEL'FAND and KIRILLOV conjectured ([12]-[14]) that $K(\mathfrak{g})$ is isomorphic to the generalized Weyl field Fract $\mathscr{A}_{r,s}$, with $\dim \mathfrak{g} = 2r+s$, index $\mathfrak{g} = s$. This generalizes naturally to the conjecture that the Goldie field of any $I \in \operatorname{Spec} U(\mathfrak{g})$ is isomorphic to the Weyl field Fract $\mathscr{A}_{r,s}$ with $\operatorname{Dim} U(\mathfrak{g})/I = 2r+s$, Dim Cent (Fract $U(\mathfrak{g})/I) = s$, where Dim denotes GEL'FAND-KIRILLOV dimension. This has been established for $\mathfrak{g}$ solvable ([3], [25]) and for the minimal primitive ideals in $\mathfrak{g}$ semisimple [6]. Yet outside $\mathfrak{g}$ solvable the conjecture may be too strong. This is indicated by representation theory [14] and algebraic geometry. Indeed take $\mathfrak{r} = sl(2, k)$, $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is commutative, satisfies $3 < \dim \mathfrak{m} < \infty$ and is simple as an ad $\mathfrak{r}$ module. Let $S(\mathfrak{m})$ denote the symmetric algebra over $\mathfrak{m}$, and $C(\mathfrak{g})$ the centre of $K(\mathfrak{g})$. Since $C(\mathfrak{g}) = (\text{Fract } S(\mathfrak{m}))^{\mathfrak{r}}$, the most natural construction of the Weyl field for the zero ideal of $U(\mathfrak{g})$ requires Fract $S(\mathfrak{m})$ to be a pure transcendental extension of $(\text{Fract } S(\mathfrak{m}))^{\mathfrak{r}}$; but this is generally false (SERRE, unpublished). To avoid this difficulty one might demand that only some algebraic extension of the Goldie field be isomorphic to a Weyl field. This is established in [14] for the zero ideal in $\mathfrak{g}$ semisimple, with the Weyl group being the Galois group of the extension. Yet it is technically easier and perhaps more natural to just restrict the conjecture to primitive ideals as suggested in [17]. Furthermore the orbital method for constructing induced ideals suggests the further conjecture [17] that $I \in \operatorname{Prim} U(\mathfrak{g})$ is induced if, and only if, the Goldie field of I admits a maximal commutative subfield which is ad $\mathfrak{g}$ stable. The analysis of [19] shows that this holds for $\mathfrak{g}$ solvable.

1.3: Under certain minor technical restrictions, the main result of this paper (Theorem 4.3) establishes the first conjecture for induced primitive ideals in $\mathfrak{g}$ semisimple. It obtains by refining the Goldie rank computation of CONZE-BERLINE and DUFLO [7] (Sect. 8), through the preparation theorem of [20]. A new feature derives from having to consider primitive ideals of Goldie rank > 1 and for this we derive a general ring theoretic result in Sect. 2. The second conjecture is discussed in Sect. 5.

I should like to thank M. DUFLO for explaining to me what had been proved in [7].

Conventions. — Terms like noetherian mean left noetherian. The symbol # denotes the *smash product* defined in [25] (Remarks 2.9). It defines a *skew polynomial extension* in the sense of [3] (Sect. 4). An element of a ring is called *regular* if it is a non-zero divisor on both the left and the right. A left ideal of a ring is called *essential* if it intersects non-trivially with every non-zero left ideal of the ring. A ring R is said to be *torsion-free* if $mx = 0, m \in \mathbf{Z}, x \in R$ implies $m = 0$, or $x = 0$. For $\mathfrak{g}$ semisimple, a primitive ideal I is said to be *induced* if I Ann M; $M = \operatorname{ind}(N, \mathfrak{a} \uparrow \mathfrak{g}$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ (possibly $\mathfrak{g}$ itself) and $\dim N < \infty$.

2. A stepping-up theorem

2.1: Let A be a torsion-free ring, X a locally nilpotent derivation of A, and set $A^X = \{ a \in A;\ X a = 0 \}$. From A being prime Goldie, it does not follow that A^X is prime Goldie. For example, take $A = M_2$, $X = \operatorname{ad} e$ with e upper triangular. Yet we show that the converse does hold and we analyse the structure of A with respect to A^X.

Given V a subspace of A, set $V^X = V \cap A^X$. Call V, *X-stable* if $XV \subset V$. Clearly $V^X \neq 0$, for any non-zero X-stable subspace V. Let S (resp. T) denote the set of regular elements of A (resp. A^X).

LEMMA. — $T = S^X$.

$T \supset S^X$ trivially. Conversely given $t \in T$, $a \in A$ such that $ta = 0$ (or $at = 0$), choose $n \in \mathbf{N}$ such that $X^n a \in A^X - \{ 0 \}$. Then $t (X^n a) = 0$ (or $(X^n a) t = 0$) and so $X^n a = 0$, contradicting the choice of n.

2.2: From now on we assume that A^X is a prime Goldie ring.

LEMMA. — *Given $A \neq A^X$, then $B := \{ a \in A; X a \in T \} \neq \emptyset$.*

Set $U = \{ u \in A;\ u = X v$, for some $v \in A \}$. Then U^X is a two-sided ideal of A^X. If $A \neq A^X$, then $U^X \neq 0$. Since A^X is prime Goldie, $U^X \cap T \neq \emptyset$ and so $B \neq \emptyset$.

2.3: From now on we assume that $A \neq A^X$.

LEMMA. — *For each pair $b \in B$, $t \in T$, there exists a pair $b' \in B$, $t' \in T$ such that $t' b = b' t$.*

Set $s = Xb \in T$. Since T is an Ore set for A^X, there exists a pair $s' \in T$, $a \in A^X$ such that $as = s' t$. Then $a \in T$ and

$$X(bs' t - sab) = ss' t - sas = 0.$$

Hence there exists $s'' \in T$, such that

$$s''(bs' t - sab) \in A^X t \subset A t.$$

This gives $b' \in A$ such that $s'' sab = b' t$. Then $(X b') t = s'' sas = s'' ss' t$. Now $t \in T$ so by 2.1, we obtain $X b' = s'' ss' \in T$. Hence $b' \in B$ and setting $t' = s'' sa$, we obtain $t' \in T$ and $t' b = b' t$, as required.

2.4: Given C a subring of A containing A^X and for which T is an Ore subset, we denote by $C_T = \{ t^{-1} c; t \in T, c \in C \}$ the localization of C at T. Let D be the subring of A generated by A^X and B. By 2.1 and 2.3, T is an Ore subset for D.

LEMMA:

(i) *For each $a \in A$ there exists $s \in T$ such that $sa \in D$;*

(ii) *T is an Ore subset for A and $A_T = D_T$.*

We recall our assumption that $A \neq A^X$. Then by 2.2, there exists $x \in D_T$ such that $Xx = 1$. Since A is torsion-free, one has

$$x_n := (n!)^{-1} x^n \in D_T \quad \text{for each } n \in \mathbf{N}^+,$$

and this element satisfies $X^n x_n = 1$. For each $m \in \mathbf{N}$, set

$$A_m = \{ a \in A; X^{m+1} a = 0 \}.$$

Consider $a \in A_n \setminus A_{n-1}$. Choose $t \in T$, such that $b := tx_n \in D$. Then $X^n (b(X^n a) - ta) = 0$ and so $b(X^n a) - ta \in A_{n-1}$. Yet $b(X^n a) \in D$ and so $ta \in D + A_{n-1}$. Since $A_0 = A^X \subset D$, we obtain (i) by induction on n, and (ii) follows from (i).

Remark. — The above rather indirect proof of (ii) is necessitated by the fact that we do not know that A is Goldie.

2.5: Set $R = \text{Fract}\, A^X$. Obviously $(A_T)^X = R$. We have seen that there exists $x \in A_T$ such that $Xx = 1$. From $X[x, R] = [1, R] = 0$, we obtain $[x, R] \subset R$. Define the skew polynomial extension ($R \# \mathbf{Q}[x]$)

of R by $\mathbf{Q}[x]$ to be the $\mathbf{Q}$-algebra with underlying vector space $R \otimes \mathbf{Q}[x]$ (where $\otimes = \otimes_{\mathbf{Q}}$) and multiplication

$$(a \# x^r)(b \# x^s) = ab \# x^{r+s} + ra[x, b] \# x^{r+s-1}.$$

Since A^X is a prime Goldie ring, there exists $n \in \mathbf{N}^+$ and a skew field K such that $R = M_n \otimes K$, up to isomorphism. By [20] (Lemma 6.3), we may adjust ad x by an inner derivation of $M_n \otimes K$ to obtain

$$(R \# \mathbf{Q}[x]) = M_n \otimes (K \# \mathbf{Q}[x]),$$

up to an isomorphism. Furthermore $K \# \mathbf{Q}[x]$ is an Ore domain and we set $K' = \text{Fract}\,(K \# \mathbf{Q}[x])$.

THEOREM:

(i) $A_T = (R \# \mathbf{Q}[x])$, *up to an isomorphism*;

(ii) A *is a prime Goldie ring*;

(iii) Fract $A = M_n \otimes K'$, *up to an isomorphism*;

(iv) A_T *is the largest X-stable subring of* Fract A *on which X is locally nilpotent.*

Recall that X is a locally nilpotent derivation of $A_T \ni x$, $Xx = 1$, and $(A_T)^X = R$. By [18] (2.2 (i), Taylor's lemma), this gives (i). We have seen that $\text{Fract}\,(R \# \mathbf{Q}[x]) = M_n \otimes K'$. Through the converse of Goldie's theorem, this gives (ii) and hence (iii). (iv) obtains from [18] (2.6 (iv)) applied to (iii).

2.6: Let k be a commutative field of characteristic zero, A an associative k-algebra, and $\mathfrak{m}$ a finite dimensional k-Lie algebra of locally nilpotent derivations of A. By [18] (2.2), $\mathfrak{m}$ is a nilpotent Lie algebra. Set $A^{\mathfrak{m}} = \bigcap \{ A^X ; X \in \mathfrak{m} \}$. The following immediate consequence of 2.5 is needed later on.

THEOREM. — *Suppose* $A^{\mathfrak{m}}$ *is a prime Goldie ring. Then A is a prime Goldie ring and* Fract $A^{\mathfrak{m}} = (\text{Fract}\,A)^{\mathfrak{m}}$.

Remark. — Set $R = \text{Fract}\,A^{\mathfrak{m}}$. Then $R = M_n \otimes K$, for some $n \in \mathbf{N}^+$ and some skew field K. By 2.5, Fract $A = M_n \otimes K'$, for some skew field K' which is the quotient field of some skew polynomial extension Q of K. Furthermore each $X \in \mathfrak{m}$ is a locally nilpotent derivation of Q and $Q^{\mathfrak{m}} = K$. Given $Q^{\mathfrak{m}} \subset \text{Cent}\,Q$ (or $K \subset \text{Cent}\,K'$) it follows by [18] (3.2), that Q is a generalized Weyl algebra over K. This generalizes [18], (3.2), to the prime situation.

3. A theorem of Conze-Berline and Duflo

3.1: Let $\mathfrak{g}$ denote a complex, semisimple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ ([8], 1.10.14). Let $u \to \check{u}$ denote the principle antiautomorphism of $U(\mathfrak{g})$. Let R (resp. R^+) denote the set of non-zero (resp. positive) roots for $\mathfrak{h}$ in $\mathfrak{g}$. Let $u \to {}^t u$ denote the antiautomorphism of order 2 of $U(\mathfrak{g})$ defined by ${}^t X_\alpha = X_{-\alpha}$ for all $\alpha \in R$, and ${}^t H = H$ for all $H \in \mathfrak{h}$. Set $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$, and given $\mathfrak{p} \supset \mathfrak{b}$ a subalgebra of $\mathfrak{g}$, we write

$$\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}, \qquad \mathfrak{s} = [\mathfrak{r}, \mathfrak{r}], \qquad h_1 = \mathfrak{h} \cap \mathfrak{s} \qquad \text{and} \qquad \mathfrak{m}^- = {}^t\mathfrak{m},$$

where $\mathfrak{m}$ (resp. $\mathfrak{r}$) is the nilradical (resp. reductive part) of $\mathfrak{p}$. Let $R_\mathfrak{s}^+$ (resp. $R_\mathfrak{s}^-$) denote the positive (resp. negative) roots for $\mathfrak{h}$ in $\mathfrak{r}$. Let W (resp. $W_\mathfrak{s}$) denote the Weyl group for $(\mathfrak{g}, \mathfrak{h})$ (resp. $(\mathfrak{s}, \mathfrak{h}_1)$), and $w_\mathfrak{s}$ the unique element of $W_\mathfrak{s}$ taking $R_\mathfrak{s}^+$ into $R_\mathfrak{s}^-$, Let ρ (resp. $\rho_\mathfrak{s}$) denote the half sum of roots of R^+ (resp. $R^+ \setminus R_\mathfrak{s}^+$). One has $\rho_\mathfrak{s} \in \mathfrak{h}_1^\perp$. Set $\sigma_\mathfrak{s} = \rho - \rho_\mathfrak{s}$. Let $\mathscr{P}_\mathfrak{s}$ (resp. $\mathscr{P}_\mathfrak{s}^+$) denote the set of integral (resp. dominant integral) weights defined with respect to $\mathfrak{s}$. Given $\lambda_1 \in \mathscr{P}_\mathfrak{s}^+$, let E_{λ_1} denote the simple, finite dimensional $U(\mathfrak{s})$-module with highest weight λ_1. Given $\lambda \in \mathfrak{h}^*$ such that $\lambda_1 := \lambda|_{\mathfrak{h}_1} \in \mathscr{P}_\mathfrak{s}^+$, let E_λ denote the simple $U(\mathfrak{p})$-module whose restriction to $U(\mathfrak{s})$ coincides with E_{λ_1} and whose restriction to $\mathfrak{h}$ is λ, and let e_λ denote its highest weight vector. Set

$$M_\mathfrak{p}(\lambda) = \operatorname{ind}^\sim (E_\lambda, \mathfrak{p} \uparrow \mathfrak{g}) \quad ([8],\ 5.1.1).$$

One has

$$M_\mathfrak{p}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E_{\lambda - \rho_\mathfrak{s}} = U(\mathfrak{m}^-) \otimes E_{\lambda - \rho_\mathfrak{s}} = U(\mathfrak{n}^-) e_{\lambda - \rho_\mathfrak{s}},$$

up to isomorphisms. In particular, $M_\mathfrak{p}(\lambda)$ is generated as a $U(\mathfrak{n}^-)$-module by a highest weight vector of weight $\lambda + \sigma_\mathfrak{s} - \rho$. Set $I_{\lambda + \sigma_\mathfrak{s}} = \operatorname{Ann} M_\mathfrak{p}(\lambda)$.

Identify $U := U(\mathfrak{g}) \otimes U(\mathfrak{g})$ canonically with $U(\mathfrak{g} \oplus \mathfrak{g})$, and consider $\operatorname{Hom}_\mathbf{C}(M_\mathfrak{p}(\lambda), M_\mathfrak{p}(\lambda'))$ as a U-module through

$$((a \otimes b).T)\, m = ({}^t\check{a}\, T \check{b})\, m,$$

for all

$$a, b \in U(\mathfrak{g}), \quad m \in M_\mathfrak{p}(\lambda), \quad T \in \operatorname{Hom}_\mathbf{C}(M_\mathfrak{p}(\lambda), M_\mathfrak{p}(\lambda')).$$

Define the embedding $j : \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, through $j(X) = (X, -{}^t X)$, for all $X \in \mathfrak{g}$. Set $\mathfrak{k} = j(\mathfrak{g})$ and let $L(M_\mathfrak{p}(\lambda), M_\mathfrak{p}(\lambda'))$ denote the U-submodule of $\operatorname{Hom}_\mathbf{C}(M_\mathfrak{p}(\lambda), M_\mathfrak{p}(\lambda'))$ of $\mathfrak{k}$ finite elements.

Consider $U(\mathfrak{g})$ as a U-module through

$$(a \otimes b).u = {}^t\check{a}\, u\, b, \quad \text{for all } a, b, u \in U(\mathfrak{g}).$$

The representation of $U(\mathfrak{g})$ in $M_{\mathfrak{p}}(\lambda)$ induces an injective homomorphism Ψ of $U(\mathfrak{g})/I_{\lambda+\sigma_{\mathfrak{s}}}$ into $L(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda))$. By [7] (2.12, 4.7, 5.5 and 6.3), we have the following proposition.

PROPOSITION. — *Suppose that* $2((\lambda+\sigma_{\mathfrak{s}}), \alpha)/(\alpha, \alpha) \notin \mathbf{N}^+$, *for all* $\alpha \in R^+ \setminus R_{\mathfrak{s}}^+$, *then:*

(i) $M_{\mathfrak{p}}(\lambda)$ *is irreducible*;

(ii) Ψ *is surjective.*

Remarks. — Clearly Ψ is surjective if, and only if, Ψ (1) is a cyclic vector for $L(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda))$. Given $\mathfrak{p} \not\supseteq \mathfrak{b}$, it can happen that $M_{\mathfrak{p}}(\lambda)$ is irreducible, yet Ψ is not surjective [7] (6.5). On the other hand, it can also happen that $I_{\lambda+\sigma_{\mathfrak{s}}} = I_{\lambda'+\sigma_{\mathfrak{s}}}$, given $\lambda+\sigma_{\mathfrak{s}} \in W(\lambda'+\sigma_{\mathfrak{s}})$, even though $\lambda \neq \lambda'$. Furthermore, if $-w_{\mathfrak{s}}(\lambda+\sigma_{\mathfrak{s}})$ lies in the closure of the positive Weyl chamber, then the hypothesis of the lemma is satisfied. For integral λ these induced ideals are just the primitive ideals minimal for given "τ-invariant" [4] (2.17 *d*). An implicit conjecture in [7] is that these are no further induced ideals; but this is false [24] (3.7). (i) is a special case of [12] (Satz 3).

3.2: We assume from now on that $M_{\mathfrak{p}}(\lambda)$ is irreducible (in which case $I_{\lambda+\sigma_{\mathfrak{s}}}$ is primitive) and that Ψ is surjective, so we can identify $U(\mathfrak{g})/I_{\lambda+\sigma_{\mathfrak{s}}}$ with $L(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda))$. Set $A = U(\mathfrak{g})/I_{\lambda+\sigma_{\mathfrak{s}}}$, and let S denote the set of regular elements of A. The following theorem is due to CONZE-BERLINE and DUFLO [7].

THEOREM. — *For each* $b \in (\mathrm{Hom}(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda)))^{j(\mathfrak{n})}$ *there exists* $s \in S$ *such that* $a := sb \in A$. *Furthermore* $a \neq 0$ *if* $b \neq 0$.

Remarks. — The theorem is not explicitly stated in [7]; but DUFLO pointed out to me that it follows from their analysis. In detail, given $\mu \in \mathscr{P}_{\mathfrak{g}}^+$ such that $\mu|_{\mathfrak{h}_1} = 0$, then the identity map on $U(n^-)$ induces by passage to the quotient, a linear isomorphism θ_μ of $M_{\mathfrak{p}}(\lambda)$ into $M_{\mathfrak{p}}(\lambda-\mu)$ Given b as above, then by [7] (5.9), one can choose μ such that $\theta_\mu b \in L(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda-\mu))$ and by [7] (8.4 and 4.8), such that $M_{\mathfrak{p}}(\lambda-\mu)$ is irreducible. Then by [7] (5.8), $L(M_{\mathfrak{p}}(\lambda-\mu), M_{\mathfrak{p}}(\lambda))$ is non-zero and by [7] (8.5), there exists $\varphi_\mu \in L(M_{\mathfrak{p}}(\lambda-\mu), M_{\mathfrak{p}}(\lambda))$ such that $\varphi_\mu \theta_\mu$ is a regular element of $L(M_{\mathfrak{p}}(\lambda), M_{\mathfrak{p}}(\lambda))$. The first part obtains on setting $a = \varphi_\mu \theta_\mu b, s = \varphi_\mu \theta_\mu$. The last part is explicitly proved in [23] (5.9).

4. Main theorem

4.1: Retain the notation of 3.1, and set $M = M_{\mathfrak{p}}(\lambda)$, $E = E_{\lambda-\rho_{\mathfrak{s}}}$. Let $\tau : U(\mathfrak{p}) \to \operatorname{End} E$ define the irreducible representation of $U(\mathfrak{r})$ acting in E. Given $a \in U(\mathfrak{m}^-)$, $b \in U(\mathfrak{s})$, define

$$r_a \otimes \tau(b) \in (\operatorname{Hom}(M, M))^{j(\mathfrak{m})}$$

through

$$(r_a \otimes \tau(b))(u \otimes e) = u\check{a} \otimes \tau(b)e, \quad \text{for all } u \in U(\mathfrak{m}^-),\ e \in E.$$

A straightforward computation gives the following lemma.

LEMMA:

(i) *the map* $a \otimes \tau(b) \mapsto r_a \otimes \tau(b)$ *extends linearly to an isomorphism of* $U(\mathfrak{m}^-) \otimes \operatorname{End} E$ *onto* $(\operatorname{Hom}(M, M))^{j(\mathfrak{m})}$;

(ii) *for all* $X \in \mathfrak{r}$, *one has*;

$$-(j({}^tX).(r_a \otimes \tau(b)) = (r_{[X,a]} \otimes \tau(b)) + (r_a \otimes \tau[X, b]);$$

(iii) $(\operatorname{Hom}(M, M))^{j(\mathfrak{m})}$ *is* $j(\mathfrak{s})$ *finite and so generated by the action of* $j(\mathfrak{s})$ *on* $(\operatorname{Hom}(M, M))^{j(\mathfrak{m})}$.

4.2: Set $B = (\operatorname{Hom}(M, M))^{j(\mathfrak{m})}$. By 4.1 (1), B is a prime, noetherian ring and $\operatorname{Fract} B = K(\mathfrak{m}^-) \otimes \operatorname{End} E$, up to an isomorphism. Assume the hypotheses and notation of 3.2, and in particular identify $A := U(\mathfrak{g})/I_{\lambda+\sigma_{\mathfrak{s}}}$ with $\mathrm{L}(M, M)$.

PROPOSITION. — $A^{\mathfrak{m}^-}$ *admits a ring of fractions, and* $\operatorname{Fract} A^{\mathfrak{m}^-} = \operatorname{Fract} B$.

Let us recall the basic structure theorem for $U(\mathfrak{g})$ given in [20] (3.4). Define the subalgebra $\mathfrak{c}(\mathfrak{m})$ (or, simply, $\mathfrak{c}$) of $\mathfrak{b}$ as in [20] (2.6), and set $\mathfrak{c}^- = {}^t\mathfrak{c}$. Let $\mathfrak{m}_0^-$ denote the nilradical of $\mathfrak{c}^-$. Then $\mathfrak{c}^- = \mathfrak{m}_0^- \oplus \mathfrak{l}$ with $\mathfrak{l}$ a subalgebra of $\mathfrak{h}$ and $\mathfrak{n}^- \supset \mathfrak{m}_0^- \supset \mathfrak{m}^-$. Furthermore by [20] (2.6 (iii)), we have $Z(\mathfrak{m}_0^-) \subset U(\mathfrak{c}^-)^{\mathfrak{m}^-} = Z(\mathfrak{m}^-)$. By [20] (2.4), there exists $z \in Z(\mathfrak{m}_0^-)$ such the localized algebra $U(\mathfrak{g})_z$ is defined and takes the form

$$U(\mathfrak{g})_z = (U(\mathfrak{g})^{\mathfrak{m}_0} \#_{Z(\mathfrak{m}_0^-)} U(\mathfrak{c}^-))_z = ((U(\mathfrak{g})^{\mathfrak{m}_0^-} \otimes_{Z(\mathfrak{m}_0^-)} U(\mathfrak{m}_0^-)) \# U(\mathfrak{l}))_z.$$

Here the smash product $\#$ is defined through the adjoint action of $\mathfrak{c}^-$ in $U(\mathfrak{g})$. Suppose $I \in \operatorname{Spec} U(\mathfrak{g})$ satifies $I \cap Z(\mathfrak{m}_0^-) = 0$. Then through the methods of [3] (Sect. 4) applied to the above formula (as show in [20], Sect. 6) it follows that the localized algebra $(U(\mathfrak{g})/I)_z^{\mathfrak{m}_0^-}$ is defined and is prime noetherian. Take $I = I_{\lambda+\sigma_s}$. Since the restriction of $\operatorname{ind}^\sim(\tau; \mathfrak{p}\uparrow\mathfrak{g})$ to $U(\mathfrak{m}^-)$ identifies with its left regular representation, it follows that $I \cap U(\mathfrak{m}^-) = 0$. Hence the localized algebra $A_z^{\mathfrak{m}_0^-}$ is defined and is prime noetherian. By 2.6 applied to $\mathfrak{m}_0^-/\mathfrak{m}^-$, it follows that $A_z^{\mathfrak{m}^-}$ is defined and is prime, Goldie. By 2.6 applied to $\mathfrak{m}^-$, we have

$$(\operatorname{Fract} A)^{\mathfrak{m}^-} = (\operatorname{Fract} A_z)^{\mathfrak{m}^-} = \operatorname{Fract} A_z^{\mathfrak{m}^-} = \operatorname{Fract} A^{\mathfrak{m}^-}.$$

On the other hand by 3.2 and 4.1 (iii), we have $\operatorname{Fract} A \supset B$ and so

$$(\operatorname{Fract} A)^{\mathfrak{m}^-} \supset \operatorname{Fract} B \supset \operatorname{Fract} A^{\mathfrak{m}^-},$$

which with the previous equality gives the required assertion.

Remark. — From $I \in \operatorname{Spec} U(\mathfrak{g})$, it does not follow that $(U(\mathfrak{g})/I)^{\mathfrak{n}}$ is prime. For example take I of finite codimension > 1.

4.3 (Notations 3.1, 3.2, 4.1): Set

$$I = I_{\lambda+\sigma_s}, \qquad n = \dim E, \qquad 2m = 2 \operatorname{codim} \mathfrak{p} = \operatorname{Dim} U(\mathfrak{g})/I$$

([1], 2.3 (*b*)). Recall that $I = \operatorname{Ann} M$.

THEOREM. — *Suppose that* $M = \operatorname{ind}^\sim(E, \mathfrak{p}\uparrow\mathfrak{g})$ *is simple, and that the representation of* $U(\mathfrak{g})$ *in* M *sets up a bijection of* $U(\mathfrak{g})/I$ *onto the* $\mathfrak{k}$*-finite part* $L(M, M)$ *of* $\operatorname{Hom}_{\mathbf{C}}(M, M)$ (*cf.* (3.1)). *Then*

(i) $\operatorname{Fract} U(\mathfrak{g})/I = M_n \otimes \operatorname{Fract} \mathscr{A}_m$, *up to an isomorphism*;

(ii) *one may choose*

$$\mathscr{A}_m = \mathbf{C}[x_1, x_2, \ldots, x_m, \partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_m]$$

so that $\mathbf{C}[x_1, x_2, \ldots, x_m]$ *is* ad $\mathfrak{g}$-*stable*.

Define $\mathfrak{c}$, $\mathfrak{m}^-$, $\mathfrak{l}$ and z as in 4.2. As in 4.2, we obtain from [20] (3.4), that A_z is defined and takes the form

$$A_z = (A^{\mathfrak{m}^-} \#_{Z(\mathfrak{m}^-)} U(\mathfrak{c}^-))_z = (((A^{\mathfrak{m}^-} \otimes_{Z(\mathfrak{m}^-)} U(\mathfrak{m}_0^-)) \# U(\mathfrak{l}))_z.$$

We remark that $(A^{\mathfrak{m}^-} \otimes_{Z(\mathfrak{m}^-)} U(\mathfrak{m}_0^-)) = (A^{\mathfrak{m}_0^-} \otimes_{Z(\mathfrak{m}_0^-)} U(\mathfrak{m}_0^-))$, and that the smash product is defined through the adjoint action of $\mathfrak{l}$ in $U(\mathfrak{g})$. This coincides with the action of $j(\mathfrak{l})$ in $U(\mathfrak{g})$ considered as a U-module. Let S denote the set of regular elements of $A^{\mathfrak{m}}$. Then $z \in S$ and

$$\begin{aligned} S^{-1}A &= (\operatorname{Fract} A^{\mathfrak{m}^-} \otimes_{Z(\mathfrak{m}^-)} U(\mathfrak{m}_0^-)) \# U(\mathfrak{l}), \\ &= ((K(\mathfrak{m}^-) \otimes \operatorname{End} E) \otimes_{Z(\mathfrak{m}^-)} U(\bar{\mathfrak{m}}_0)) \# U(\mathfrak{l}), \quad \text{by } 4.2, \\ &= \operatorname{End} E \otimes ((K(\mathfrak{m}^-) \otimes_{Z(\mathfrak{m}^-)} U(\mathfrak{m}_0^-)) \# U(\mathfrak{l})), \end{aligned}$$

up to an isomorphism. The last step was obtained by adjusting each ad X : $X \in \mathfrak{l}$ by an inner derivation of End $E \otimes ((K(\mathfrak{m}^-) \otimes_{Z(\mathfrak{m}^-)} U(\mathfrak{m}^-))$. This is achieved by dropping the second term in the right hand side of 4.1 (ii).

It follows that up to an isomorphism

$$\operatorname{Fract}((K(\mathfrak{m}^-) \otimes_{Z(\mathfrak{m}^-)} U(\mathfrak{m}_0^-)) \# U(\mathfrak{l})) = \operatorname{Fract} U(\mathfrak{m}^- \otimes \mathfrak{c}^-)/J,$$

where J is the two-sided ideal generated by the semi-invariants

$$y \otimes 1 - 1 \otimes y : y \in Z(\mathfrak{m}^-).$$

Obviously J is a prime ideal and the Lie algebra $\mathfrak{m}^- \oplus \mathfrak{c}^-$ is solvable and algebraic. Noting that

$$\operatorname{Dim} U(\mathfrak{g})/I = \operatorname{Dim} U(\mathfrak{m}^- \oplus \mathfrak{c}^-)/J = \dim \mathfrak{m} + \dim \mathfrak{c} - \operatorname{Dim} Z(\mathfrak{m}^-) = 2 \dim \mathfrak{m}^-,$$

by [20] (2.6 (ii)); we obtain (i) from [25] (Cor. 6.4 (1)) or from [3] (6.8).

(ii) obtains on adapting [19] to prime ideals or as follows. Observe that in the identification of $U(\mathfrak{g})/I$ with $L(M, M)$, the restriction to $U(\mathfrak{m}^-)$ defines the left regular representation of $U(\mathfrak{m}^-)$ and furthermore from Fract $A^{\mathfrak{m}^-}$ we get a second copy of $U(\mathfrak{m}^-)$ which (*cf.* 4.1) identifies with the right regular representation of $U(\mathfrak{m}^-)$. Then since

$$\operatorname{Dim}((U(\mathfrak{m}^-) \otimes_{Z(\mathfrak{m}^-)} U(\mathfrak{m}_0^-)) \# U(\mathfrak{l})) = 2 \dim \mathfrak{m}^- = 2m = \operatorname{Dim} \mathscr{A}_m,$$

it follows from [20] (5.2) that $\mathscr{A}_m$ may be chosen so that $\mathfrak{g}$ is represented by first order differential operators on $\mathbf{C}[x_1, x_2, \ldots, x_m]$ (with a zero order part depending also on τ). Hence (ii).

Remarks. — The proof of the theorem is entirely algebraic and consequently the base field can be any algebraically closed commutative field of characteristic zero. Yet excepting the case $\mathfrak{p} = \mathfrak{b}$ ([6], Sect. 6) the proof of 3.2 (ii) is partly analytical. Part (ii) asserts that the Goldie field of I admits a maximal commutative subfield which is $\mathfrak{g}$-stable. Yet excepting $sl(2)$ and $sl(3)$ not all primitive ideals are induced ([4], [6]).

Thus it still remains to show that for such non-induced ideals the Goldie field does not admit a maximal commutative subfield which is $\mathfrak{g}$-stable. This is discussed in Sect. 5.

4.4: We round off the discussion by showing that an induced ideal is primitive only if it is induced from a parabolic subalgebra. (We remark that this condition is not also a sufficient one, even if the module from which one induces is simple and finite dimensional ([4], [21]).)

Let $Y(\mathfrak{g})$ denote the invariants of the symmetric algebra $S(\mathfrak{g})$ and let Y_+ be the subspace of $Y(\mathfrak{g})$ spanned by homogeneous invariants of positive degree. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form and recall [8] (8.1.3 (ii)), that the zero variety of $S(\mathfrak{g})\,Y_+$ is the cone $\mathcal{N}$ of nilpotent elements of $\mathfrak{g}$.

LEMMA. — *Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$. If $Y_+ \subset S(\mathfrak{g})\,\mathfrak{a}$, then $\mathfrak{a}$ is a parabolic subalgebra of $\mathfrak{g}$.*

We have $\mathfrak{a}^\perp \subset \mathcal{N}$ and $[\mathfrak{a}, \mathfrak{a}^\perp] \subset \mathfrak{a}^\perp$. Thus if we can show that $\mathfrak{a}^\perp \subset \mathfrak{a}$, it will follow that $\mathfrak{a}^\perp$ is a nilpotent subalgebra and hence that $\mathfrak{a}$ is parabolic. Let $\mathfrak{a}_1$ denote the algebraic hull of $\mathfrak{a}$ and $\mathfrak{m}$ the nilradical of $\mathfrak{a}_1$, and $\mathfrak{h}_1$ a Cartan subalgebra for the reductive part $\mathfrak{r}$ of $\mathfrak{a}_1$. By [8] (1.10.16), there exists a triangular decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- \quad \text{with} \quad \mathfrak{n} \supset \mathfrak{m} \quad \text{and} \quad \mathfrak{h} \supset \mathfrak{h}_1.$$

If $X \in \mathfrak{r}$ is an $\mathfrak{h}_1$ weight vector of non-zero weight, then $X \in \mathfrak{n} \oplus \mathfrak{n}^-$ and so

$$\mathfrak{h}_1^\perp \cap \mathfrak{h} \subset \mathfrak{a}_1^\perp \cap \mathfrak{h} \subset \mathfrak{a}^\perp \cap \mathfrak{h} \subset \mathcal{N} \cap \mathfrak{h} = 0.$$

Thus $\mathfrak{h}_1 = \mathfrak{h}$ and since $[\mathfrak{a}_1, \mathfrak{a}^\perp] \subset \mathfrak{a}^\perp$, it follows that $\mathfrak{a}^\perp$ is spanned by root vectors. Thus $\mathfrak{a} = \mathfrak{a}^{\perp\perp}$ is also spanned by root vectors (and in particular $\mathfrak{a}_1 = \mathfrak{a}$). Suppose $\mathfrak{a}^\perp \not\subseteq \mathfrak{a}$. Then there exists $\alpha \in R$, such that $X_\alpha \in \mathfrak{a}^\perp$, $X_\alpha \notin \mathfrak{a}$. This gives $X_{-\alpha} \in \mathfrak{a}^\perp$, which contradicts the fact that $X_\alpha + X_{-\alpha} \notin \mathcal{N}$.

4.5: Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$ and W a $U(\mathfrak{a})$-module. Set $M = \operatorname{ind}(W, \mathfrak{a} \uparrow \mathfrak{g})$, $I = \operatorname{Ann} M$.

PROPOSITION. — *If $I \in \operatorname{Prim} U(\mathfrak{g})$, then $\mathfrak{a}$ is a parabolic subalgebra.*

Let gr be the gradation functor for the canonical filtration of $U(\mathfrak{g})$. If $I \in \operatorname{Prim} U(\mathfrak{g})$, then $(I \cap Z(\mathfrak{g})) \in \operatorname{Max} Z(\mathfrak{g})$ and so $\operatorname{gr}(I) \cap Y(\mathfrak{g}) = Y_+$. Yet $I \subset U(\mathfrak{g}) \operatorname{Ann} W$ and so $\operatorname{gr}(I) \subset S(\mathfrak{g})\,\mathfrak{a}$, in virtue of the Poincaré-Birkhoff-Witt theorem. This gives $Y_+ \subset S(\mathfrak{g})\,\mathfrak{a}$ and so $\mathfrak{a}$ is parabolic by 4.4.

5. The subfield criterion for induced ideals

Here we show that the Goldie field of a certain non-induced primitive ideal constructed in [16] does not admit a maximal commutative subfield which is ad $\mathfrak{g}$ stable. These ideas are inspired by the work of NGHIÊM ([26], [27]); but since we require results valid outside solvable Lie algebras our analysis has to be somewhat different. It is motivated by [9].

5.1: Let $\mathfrak{g}$ be a finite dimensional k-Lie algebra. Extend the gradation functor gr for the canonical filtration of $U(\mathfrak{g})$ to $K(\mathfrak{g})$ through

$$\operatorname{gr}(a^{-1}b) = \operatorname{gr}(a)^{-1}\operatorname{gr}(b); \qquad a, b \in U(\mathfrak{g}), \quad a \neq 0$$

(*cf.* [22], Chap. II or [13], Sect. 3). Through the identification gr $(U(\mathfrak{g})) = S(\mathfrak{g})$; gr defines a Poisson bracket $\{\, ,\, \}$ on $R(\mathfrak{g}) := \operatorname{Fract} S(\mathfrak{g})$ ([22], 2.1, 2.2). Given K a subfield of $K(\mathfrak{g})$, we set $K' = \operatorname{Fract} \operatorname{gr}(K)$. A subfield K' of $R(\mathfrak{g})$ is called *strongly commutative* if $\{a, b\} = 0$, for all $a, b \in K'$.

Now assume that K is a commutative ad $\mathfrak{g}$ stable subfield of $K(\mathfrak{g})$. Then K' is a strongly commutative ad $\mathfrak{g}$ stable subfield of $R(\mathfrak{g})$. Let $\mathfrak{g}(K)$ (resp. $\mathfrak{g}(K')$) denote $\mathfrak{g}K+K$ (resp. $\mathfrak{g}K'+K'$) considered as a subspace of $K(\mathfrak{g})$ (resp. $R(\mathfrak{g})$) over K (resp. K'). The set $\{X\}_{i=1}^{m} : X_i \in \mathfrak{g}$, is said to be a *cobase* for $\mathfrak{g}(K)$ over K (resp. $\mathfrak{g}(K')$ over K') if with the identity adjoined it becomes a basis for $\mathfrak{g}(K)$ (resp. for $\mathfrak{g}(K')$). Obviously $\mathfrak{g}(K)$ and $\mathfrak{g}(K')$ admit cobases though these may not coincide. Observe further that $\mathfrak{g}(K)$ (resp. $\mathfrak{g}(K')$) is closed under commutation (resp. Poisson bracket) and so are Lie algebras (The Lie algebra $\mathfrak{g}(K)$ and the notion of a cobase for $\mathfrak{g}(K)$ are due to NGHIÊM [26], 1.2.3.) Recall that for a commutative field, Dim coincides with transcendence degree (over k). Let dim denote $\dim_k$.

LEMMA:

(i) $\operatorname{Dim} K' \leqslant \operatorname{Dim} K$;

(ii) $1+\dim \mathfrak{g} \leqslant \dim_{K'} \mathfrak{g}(K')+\operatorname{Dim} K'$;

(iii) $\dim_{K'} \mathfrak{g}(K') \leqslant \dim_K \mathfrak{g}(K)$;

(i) is elementary. Let $\{X_i\}_{i=1}^{l}$ be a cobasis for $\mathfrak{g}(K')$ over K'. One has $\dim_{K'} \mathfrak{g}(K') = l+1$. Since $\{X_i\}_{i=1}^{l}$ generates $R(\mathfrak{g})$ over K' and $\operatorname{Dim} R(\mathfrak{g}) = \dim \mathfrak{g}$, this gives (ii).

Let $\{X_i\}_{i=1}^{n}$ be a basis for $\mathfrak{g}$. Let $\mathfrak{B}$ denote the set of all cobases for $\mathfrak{g}(K)$ over K formed from subsets of $\{X_i\}_{i=1}^{n}$. The set $\mathfrak{B}$ is trivially non-empty. Given $b \in \mathfrak{B}$, let $b(K')$ denote the linear span of $\{b, 1\}$ over K', and set $I(b) = \{i \in \{1, 2, \ldots, n\}; X_i \in b(K')\}$. Choose $b \in \mathfrak{B}$

maximizing card $I(b)$. For (iii), it suffices to show that card $I(b) = n$. We can assume that $b = \{ X_i \}_{i=1}^m$ and set $X_0 = 1$. Given card $I(b) < n$, choose $r \in \{ m+1, \ldots, n \}$, $r \notin I(b)$. Since b is a cobase for $\mathfrak{g}(K)$ over K, there exist $x_i \in K$ such that

$$(5.1) \qquad X_r = \textstyle\sum_{i=0}^m x_i X_i.$$

Yet $X_r \notin b(K')$ by hypothesis, so leading terms on the right-hand side of (5.1) must cancel. This defines a non-empty set $J \subset \{ 0, 1, 2, \ldots, m \}$ such that

$$(5.2) \qquad 0 = \textstyle\sum_{j \in J} (\mathrm{gr}\, x_j) X_j\,; \quad \mathrm{gr}\, x_j \in K' - \{ 0 \}.$$

Since K' is a field, card $J > 1$. Choose $j \in J \setminus \{ 0 \}$ and set $b' = \{ b \setminus X_j \} \cup \{ X_r \}$. Since $x_j \neq 0$, we obtain $b' \in \mathfrak{B}$ from (5.1). Since $X_r \in b'$ and $X_j \in b'(K')$ by (5.2), it follows that card $I(b') \geqslant 1 +$card $I(b)$, contradicting the choice of b. Hence (iii).

5.2: In addition to the hypothesis of the above lemma assume that K contains its commutant in $\mathfrak{g}(K)$. Set $\dim_K \mathfrak{g}(K) = m+1$, and let δ_{ij} denote the Kronecker delta.

LEMMA. – *There exist $x_i \in K$, $y_j \in \mathfrak{g}(K)$, $i, j = 1, 2, \ldots, m$, such that*

(i) $[x_i, y_j] = \delta_{ij}$;

(ii) $[y_i, y_j] \in K$;

(iii) $\{ y_i \}_{i=0}^m$, $y_0 = 1$, *is a basis for* $\mathfrak{g}(K)$.

Obviously $m = 0$, if and only if, $K = K(\mathfrak{g})$. Suppose $m > 0$, and let l be the largest non-negative integer such that $x_i \in K$, $y_j \in \mathfrak{g}(K)$, $i, j = 1, 2, \ldots, l$, can be chosen to satisfy (i). Then $\{1, y_1, y_2, \ldots, y_l\}$ are linearly independent over K and so $l \leqslant m$. Given $y \in \mathfrak{g}(K)$, then $a_i := [x_i, y] \in K$, and we set

$$y' = y - \textstyle\sum_{i=1}^t a_i y_i.$$

Then $[x_i, y'] = 0$. Suppose $a := [x, y'] \neq 0$, for some $x \in K$. Set $b_i = [x, y_i]$, $x = x_{l+1}$, $y'_{l+1} = a^{-1} y'$, $y'_i = y_i - b_i y'_{l+1}$. Then $\{ x_i, y'_i;\, i = 1, 2, \ldots, l+1 \}$ satisfy (i) contradicting the maximality of l. Hence $[K, y'] = 0$, and so $y' \in K$ since K contains its commutant in $\mathfrak{g}(K)$. It follows that $\{ 1, y_1, y_2, \ldots, y_l \}$ spans $\mathfrak{g}(K)$ over K. Hence $l \geqslant m$, which with the opposite inequality established above proves (i) and (iii). Through the Jacobi identity $[x_i, [y_j, y_k]] = 0$. Yet $[y_j, y_k] \in \mathfrak{g}(K)$, and so (ii) follows from (i) and (iii).

5.3: Retain the hypotheses and notation of 5.2. By 5.2 (iii), the y_i generate over K a subalgebra S of $K(\mathfrak{g})$ containing $U(\mathfrak{g})$. Consider ad x_i, $i = 1, 2, \ldots, m$, as spanning a commutative Lie algebra $\mathfrak{n}$ of locally nilpotent derivations of S. By 5.2 (i), $S^{\mathfrak{n}} = K$. By [18] (2.6 (iv)), S admits a left quotient field Fract S which must obviously coincide with $K(\mathfrak{g})$. By [18] (2.6 (vi)), K contains its commutant in $K(\mathfrak{g})$ and in particular $C(\mathfrak{g}) \subset K$. Set $K' =$ Fract gr (K).

Given L a skew field over k define $\mathrm{T}\deg_k L$ (or simply $T\deg L$) as follows. Let V denote a finite dimensional k subspace of L containing the identity and given $b \in L$, $m \in \mathbf{N}$, let $(b\,V)^m$ denote the k subspace of L spanned by $\{a_1 a_2 \ldots a_m;\, a_i \in b\,V\}$. Define

$$T\deg L = \sup_V \inf_{b \neq 0} \overline{\lim}_{m \to \infty} \frac{\log \dim (bV)^m}{\log m}.$$

PROPOSITION:

(i) $m + T\deg K \leqslant T\deg \operatorname{Fract} S = \dim \mathfrak{g}$;

(ii) $\dim_K \mathfrak{g}(K) + \operatorname{Dim} K = \dim \mathfrak{g} + 1$;

(iii) $\operatorname{Dim} K = \operatorname{Dim} K'$;

(iv) $\operatorname{Dim} K \leqslant 1/2\,(\dim \mathfrak{g} + \operatorname{index} \mathfrak{g})$, *with equality if* $\operatorname{Dim} C(\mathfrak{g}) = \operatorname{index} \mathfrak{g}$, *so in particular if* $\mathfrak{g}$ *is algebraic.*

The inequality in (i) follows from [18] (2.6 (iv), 4.2 (ii)) applied to 5.2. (Note that in [18], Dim is denoted by Dim^1 and $T\deg$ by Dim). The equality in (i) follows from the identity Fract $S = K(\mathfrak{g})$ and [18] (4.3). One has $\operatorname{Dim} K = T\deg K$, since K is commutative and then (ii) and (iii) follow from (i) and 5.1. Since K' is strongly commutative, we obtain $\operatorname{Dim} K' \leqslant 1/2\,(\dim \mathfrak{g} + \operatorname{index} \mathfrak{g})$ by [22] (2.7). By 5.2 (i), the x_i, $i = 1, 2, \ldots, m$, are algebraically independent over $C(\mathfrak{g})$ and so $m + \operatorname{Dim} C(\mathfrak{g}) \leqslant \operatorname{Dim} K = \dim \mathfrak{g} - m$, by (ii). Hence

$$m \leqslant 1/2\,(\dim \mathfrak{g} - \operatorname{Dim} C(\mathfrak{g})).$$

Then by (ii), we have

$$\operatorname{Dim} K = \dim \mathfrak{g} - m \geqslant 1/2(\dim \mathfrak{g} + \operatorname{Dim} C(\mathfrak{g})),$$

which gives (iii).

Remarks. — Unlike NGHIÊM we do not assume that K is generated by its intersection with $U(\mathfrak{g})$. This case is simpler; for example (iii) would be a consequence of [22] (2.5), which does not required K to be ad $\mathfrak{g}$ stable. However this restriction is inappropriate outside $\mathfrak{g}$ solvable.

Again for $\mathfrak{g}$ solvable one can show that any commutative ad $\mathfrak{g}$ stable subfield of $K(\mathfrak{g})$ can be embedded in a commutative ad $\mathfrak{g}$ stable subfield K which contains its commutant in $\mathfrak{g}(K)$. This generalizes [26] (III.4); but the argument given in [26] needs a little modification. Finally it is noted above that a commutative ad $\mathfrak{g}$ subfield K of $K(\mathfrak{g})$ which contains its commutant in $\mathfrak{g}(K)$ necessarily contains its commutant in $K(\mathfrak{g})$; i. e. it is maximal commutative. This generalizes [26] (III.7).

5.4: Now let $\mathfrak{g}$ be a simple Lie algebra. It is known that $\mathfrak{g}^*$ (identified with $\mathfrak{g}$ through the Killing form) admits a unique nilpotent orbit $\mathcal{O}_0$ of minimal non-zero dimension. In [16], we constructed a completely prime, primitive ideal J_0 for which $\mathcal{O}_0 \cup \{0\}$ is the zero variety $\mathcal{V}(\operatorname{gr} J_0)$ of $\operatorname{gr} J_0$. The ideal J_0 can be described as follows. First there exists a solvable algebraic Lie subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ satisfying $\dim \mathfrak{r} = \dim \mathcal{O}_0$, index $\mathfrak{r} = 0$, and containing the highest root eigenvector E (*see* [16], Sect. 4). Secondly there exists an algebra homomorphism $\overline{\Phi}: U(\mathfrak{g}) \rightarrow U(\mathfrak{r})_E$ satisfying $\overline{\Phi}|_{\mathfrak{r}} = Id$ (There is only one such homomorphism if $\mathfrak{g} \neq sl(n+1)$: $n \in \mathbf{N}^+$, [16], Theorem 4.3.) Set $J_0 = \ker \overline{\Phi}$. Then $\mathcal{V}(\operatorname{gr} J_0) = \mathcal{O}_0 \cup \{0\}$ ([16], Prop. 10.2). Furthermore $\overline{\Phi}$ defines, by passage to the quotient and localization at E, a bijection Φ:

$$(U(\mathfrak{g})/J_0)_E \rightarrow U(\mathfrak{r})_E$$

whose restriction to $U(\mathfrak{r})_E$ is the identity. In [16] (Sect. 8), it was shown that for $\mathfrak{g} \neq sl(n+1)$, $n \in \mathbf{N}^+$, the ideal J_0 is not induced from any proper subalgebra of $\mathfrak{g}$. This was obtained through a dimensionality estimate which gave little insight into why J_0 is not induced. This motivates the following theorem.

THEOREM. – *Suppose* $\mathfrak{g} \neq sl(n+1)$, $n \in \mathbf{N}^+$. Then Fract $U(\mathfrak{g})/J_0$ *does not admit a maximal commutative subfield which is* ad $\mathfrak{g}$ *stable.*

Let K be such a subfield. Through Φ, the subfield K may be considered as a subfield of $K(\mathfrak{r})$. Set $K' = \operatorname{Fract} \operatorname{gr}(K)$. From K', we construct for certain $f \in \mathcal{O}_0$, a polarization of $\mathfrak{g}$ in f. Yet for $\mathfrak{g} \neq sl(n+1)$, $n \in \mathbf{N}^+$, ([16], Prop. 3.5), no $f \in \mathcal{O}_0$ is polarizable and this contradiction will prove the theorem.

5.5: Define a map $\Phi' : \mathfrak{g} \rightarrow R(\mathfrak{r})$, through $\Phi'(X) = \operatorname{gr} \Phi(X)$, for all $X \in \mathfrak{g}$. Recall that deg extends to $R(\mathfrak{r})$ through $\deg a^{-1} b = \deg b - \deg a$.

LEMMA:

(i) $\Phi'(X) = X$, *for all* $X \in \mathfrak{r}$;

(ii) $\deg \Phi'(X) = 1$, *for all* $X \in \mathfrak{g}$;

(iii) $\{\Phi'(X), \Phi'(Y)\} = \Phi'[X, Y]$, *for all* $X, Y \in \mathfrak{g}$;

(iv) $(f, \Phi'(X)) = (f, X)$, *for all* $X \in \mathfrak{g}$ *and all* $f \in \mathcal{O}_0$ *satisfying* $f(E) \neq 0$.

(i) is immediate. (ii) obtains from the explicit formula [16] (Theorem 5.3), for Φ, or as an easy consequence of (i) and the fact that $R(\mathfrak{r})^{\mathfrak{r}}$ reduces to scalars. (iii) follows from (ii). For each $X \in \mathfrak{g}$, there exists $s \in \mathbf{N}$, such that $E^s(\bar{\Phi}(X) - X) \in J_0$. By (ii), $E^s(\Phi'(X) - X) \in \operatorname{gr} J_0$. Hence (iv).

5.6: Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form, and set $f = {}^tE$ (notation 3.1). Then $f \in \mathcal{O}_0$, $f(E) \neq 0$ and $\mathfrak{r}$ identifies with the tangent space to $\mathcal{O}_0$ at f ([16], Sect. 4). Consider $\Omega = \{g \in \mathcal{O}_0; g(E) \neq 0\}$ in $\mathfrak{r}^*$ by restriction. Then (always in the Zariski topology), we have the folloning lemma.

LEMMA. — Ω *is a non-empty open subset of* $\mathfrak{r}^*$.

5.7: Define a linear map $\varphi : \mathfrak{g} \otimes R(\mathfrak{r}) \to R(\mathfrak{r})$ through

$$\varphi(X \otimes x) = \Phi'(X)\, x.$$

Let $\{X_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$ such that $\{X_i\}_{i=1}^m$ is a basis for $\mathfrak{r}$ and for each $f \in R(\mathfrak{g})$, let $\partial_i f$ denote its partial derivative with respect to X_i. Define a linear map $d : R(\mathfrak{r}) \to \mathfrak{g} \otimes R(\mathfrak{r})$, through

$$da = \textstyle\sum_{i=1}^n (X_i \otimes (\partial_i a)) = \sum_{i=1}^m (X_i \otimes (\partial_i a)).$$

Define an antisymmetric bilinear two-form B on $\mathfrak{g} \otimes R(\mathfrak{r})$ through $B(x, y) = \varphi[x, y]$. Let V denote the linear span $\{da : a \in K'\}$ over $R(\mathfrak{r})$, and let $V^\perp$ denote the orthogonal complement of V with respect to B. Set $N = \ker B$.

LEMMA :

(i) $V \subset V^\perp$;

(ii) $[V^\perp, V^\perp] \subset V^\perp$;

(iii) $N \cap V = 0$;

(iv) $\dim_{R(\mathfrak{r})} V = 1/2 \dim \mathfrak{r}$;

(v) $\operatorname{rank} B = \dim \mathfrak{r}$;

(vi) $\dim_{R(\mathfrak{r})} V^\perp = \dim \mathfrak{g} - 1/2 \dim \mathfrak{r}$;

(vii) $B(V^\perp, V^\perp) = 0$.

Given a, $b \in K'$, then

$$\begin{aligned} B(da, db) &= \textstyle\sum_{i,j=1}^{m} \varphi([X_i, X_j] \otimes (\partial_i a)(\partial_j b)) \\ &= \textstyle\sum_{i,j=1}^{m} [X_i, X_j](\partial_i a)(\partial_j b), \quad \text{by } 5.6 \text{ (i)} \\ &= \{a, b\} = 0, \quad \text{since } K' \text{ is strongly commutative.} \end{aligned}$$

Hence (i).

Given $x, y \in \mathfrak{g} \otimes R(\mathfrak{r})$, $a \in R(\mathfrak{r})$, set $a_i = \{\Phi(X_i), a\}$ and

$$x = \textstyle\sum_{i=1}^{n} (X_i \otimes x_i), \qquad v = \textstyle\sum_{i=1}^{n} (X_i \otimes y_i),$$

$$b = \textstyle\sum_{i=1}^{n} y_i\, da_i, \qquad c = \textstyle\sum_{i=1}^{n} x_i\, da_i.$$

Then using 5.6 (i), (ii), a straightforward computation gives

$$B([x, y], da) = B(x, b) - B(y, c). \tag{5.3}$$

Now K is ad $\mathfrak{g}$ stable, that is $[\Phi(X), K] \subset K$ and so $\{\Phi'(X), K'\} \subset K'$, for all $X \in \mathfrak{g}$. Then $a \in K'$, implies that b, $c \in V$ and so (ii) follows from (5.3).

Given $x \in V \cap N$, write

$$x = \textstyle\sum_{t=1}^{s} x_t\, da_t : \quad x_t \in R(\mathfrak{r}), \qquad a_t \in K'.$$

By definition of N, we have, for all $i = 1, 2, \ldots, m$,

$$0 = B(X_i, x) = \textstyle\sum_{t=1}^{s} \sum_{j=1}^{m} [X_i, X_j]\, x_t(\partial_j a_t). \tag{5.4}$$

Now index $\mathfrak{r} = 0$ and so $\det[X_i, X_j] \neq 0$. Substitution in (5.4) gives $x = 0$, Hence (iii).

One has $\dim_{R(\mathfrak{r})} V = \operatorname{Dim} K' = 1/2 \dim \mathfrak{r}$, by 5.3 (iii), (iv) (the first equality is elementary). Hence (iv).

For all X, $Y \in \mathfrak{g}$, we have $B(X \otimes 1, Y \otimes 1) = \Phi'[X, Y] \subset S(\mathfrak{r})_E$. Given $f \in \mathfrak{r}^*$, such that $f(E) \neq 0$, define a two-form B_f on $\mathfrak{g}$ through $B_f(X, Y) = (f, \Phi'[X, Y])$. Then there exists a non-empty open set $\Omega' \subset \mathfrak{r}^*$ such that rank B = rank B_f for all $f \in \Omega'$. With Ω as defined in 5.6, choose $f \in \Omega \cap \Omega'$. Then

$$(f, \Phi'[X, Y]) = (f, [X, Y]), \quad \text{for all } X, Y \in \mathfrak{g}, \quad \text{by } 5.5 \text{ (iv)}.$$

Hence rank $B = \dim \mathcal{O}_0 = \dim \mathfrak{r}$, which gives (v). By (iii) and (iv),

$$\dim_{R(\mathfrak{r})} V = \dim_{R(\mathfrak{r})}(\mathfrak{g} \otimes R(\mathfrak{r})) - \dim_{R(\mathfrak{r})} V = \dim \mathfrak{g} - 1/2 \dim \mathfrak{r},$$

which gives (vi). Finally we have $V+N \subset V^{\perp}$, by (i) and by (iii)-(v), that $\operatorname{Dim}_{R(\mathfrak{r})}(V+N) = \dim_{R(\mathfrak{r})} V^{\perp}$. Hence (vii).

5.8: Set $V_1 = V^{\perp} \cap (\mathfrak{g} \otimes S(\mathfrak{r}))$. Given $f \in \mathfrak{r}^*$,

$$x = \textstyle\sum_{i=1}^{n} (X_i \otimes x_i) \in \mathfrak{g} \otimes S(\mathfrak{r}),$$

set

$$\langle x, f \rangle = \textstyle\sum_{i=1}^{n} X_i (x_i, f) \qquad \text{and} \qquad \mathfrak{p}_f = \{ \langle x, f \rangle ; x \in V_1 \}.$$

By 5.7 (ii), $\mathfrak{p}_f$ is a subalgebra of $\mathfrak{g}$. Since V_1 generates V over $R(\mathfrak{r})$, there exists a non-empty open set $\Omega'' \subset \mathfrak{r}^*$ such that $\dim \mathfrak{p}_f = \dim_{R(\mathfrak{r})} V^{\perp}$, for all $f \in \Omega''$. Choose $f \in \Omega \cap \Omega''$. Then by 5.5 (iv), we have for all $x, y \in V_1$, that

$$(f, [\langle x, f \rangle, \langle y, f \rangle]) = (f, B(x, y)).$$

Then by 5.7 (vii), $\mathfrak{p}_f$ is subordinate to f. By 5.7 (vi),

$$\dim \mathfrak{p}_f = \dim \mathfrak{g} - 1/2 \dim \mathfrak{r} = \dim \mathfrak{g} - 1/2 \dim \mathcal{O}_0.$$

Hence $\mathfrak{p}_f$ is a polarization of $\mathfrak{g}$ in f. This proves the theorem.

Remark. — Is a maximal commutative subfield of $K(\mathfrak{r})$ which is ad $\mathfrak{r}$ stable, necessarily a pure transcendental extension of k? (*cf.* [19], Theorem 2.3).

5.9: The above construction of a polarization differs from that given by NGHIÊM in [27] and which does not use the d map. NGHIÊM construction requires that K be generated by its intersection with $U(\mathfrak{g})$ (or some image of $U(\mathfrak{g})$). This is fine for $\mathfrak{g}$ solvable, but too restrictive for $\mathfrak{g}$ semisimple as the following lemma shows.

LEMMA. — *Take* $\mathfrak{g}$ *semisimple and* $I \in \operatorname{Prim} U(\mathfrak{g})$. *Let* K *be a commutative* ad $\mathfrak{g}$ *stable subfield of* Fract $U(\mathfrak{g})/I$. Then $(U(\mathfrak{g})/I) \cap K$ *reduces to scalars.*

After DUFLO [11] (Theorem 1), there exists $\lambda \in \mathfrak{h}^*$ such that $I = \operatorname{Ann} L(\lambda+\rho)$, where $L(\lambda+\rho)$ is the unique simple $U(\mathfrak{g})$-module with highest weight vector e of weight λ. Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I$, be the natural projection. It is clear that $A := (U(\mathfrak{g})/I) \cap K$ is an ad $\mathfrak{g}$ module. Then by [8] (4.2.5), A reduces to scalars or there exists $\mu \in \mathfrak{h}^* - \{0\}$, and a weight vector $a_{-\mu} \in U(\mathfrak{g})^{\mathfrak{n}^-}$ for which $\pi(a_{-\mu}) \in A - \{0\}$. We have $f_{\lambda-\mu} := a_{-\mu} e_{\lambda} \neq 0$, for otherwise $a_{-\mu} \in I$. Since $L(\lambda+\rho)$ has a unique highest weight, application of the simple root vectors to $f_{\lambda-\mu}$ provides a weight vector $a_{-\alpha} \in U(\mathfrak{g})$ of weight α, α simple, for which $\pi(a_{-\alpha}) \in A - \{0\}$, and $a_{-\alpha} e_\lambda \neq 0$. Let $\mathfrak{s}_\alpha$ denote the $sl(2)$ subalgebra generated by X_α and $X_{-\alpha}$. We can assume that $a_{-\alpha}$ is contained in a simple $\mathfrak{s}_\alpha$ submodule V_α of $U(\mathfrak{g})$ with $\pi(V_\alpha) \subset A$ and having lowest weight vector $a_{-t\alpha}$, $t \in \mathbf{N}^+$. Then $a_{-\alpha} = (\operatorname{ad} X_\alpha)^{t-1} a_{-t\alpha}$, up to a scalar, and so $a_{-t\alpha} e_\lambda \neq 0$. Let $\mathfrak{m}$ be the nilradical of $\mathfrak{b} \oplus k X_{-\alpha}$ and $P^\alpha : U(\mathfrak{g}) \to U(\mathfrak{s}_\alpha + \mathfrak{h})$ the projection defined by the decomposition $U(\mathfrak{g}) = U(\mathfrak{s}_\alpha + \mathfrak{h}) \oplus ({}^t\mathfrak{m}\, U(\mathfrak{g}) + U(\mathfrak{g})\,\mathfrak{m})$, and $\theta : U(\mathfrak{g})^{\mathfrak{h}} \to U(\mathfrak{h})$, the Harish-Chandra homomorphism [8] (7.4.3). Then $P^\alpha(a_{-t\alpha}) = X_{-\alpha}^t a$, for some $a \in U(\mathfrak{s}_\alpha + \mathfrak{h})^{\mathfrak{h}}$ satisfying $[\mathfrak{s}_\alpha, a] = 0$ and $P^\alpha(a_{t\alpha}) = X_\alpha^t a$, where $a_{t\alpha} := (\operatorname{ad}^{2t} X_\alpha)\, a_{-t\alpha} \in V_\alpha$. Now

$$a_{-t\alpha} e_\lambda = P^\alpha(a_{-t\alpha})\, e_\lambda = X_{-\alpha}\, \theta(a)\, e_\lambda.$$

Hence $\theta(a)(\lambda) \neq 0$, $X_\alpha^t X_{-\alpha}^t e_\lambda \neq 0$, through the characterization of Ann e_λ noted in [11] (1, *Modules de Verma*). Yet

$$[a_{t\alpha}, a_{-t\alpha}]\, e_\lambda = a_{t\alpha} a_{-t\alpha} e_\lambda = P^\alpha(a_{t\alpha})\, P^\alpha(\mu_{-t\alpha})\, e_\lambda = (\theta(a)(\lambda))^2 X_\alpha^t X_{-\alpha}^t e_\lambda \neq 0,$$

which contradicts the commutativity of A.

5.10: If $\mathfrak{g} = sl(n+1)$, $n \in \mathbf{N}^+$; then $\mathcal{O}_0$ admits polarization and the inducing procedure gives a family J_λ ($\lambda \in k$) of completely prime, primitive ideals [16] (3.5). These all take the form $J_\lambda = \ker \overline{\Phi}_\lambda$, for some homomorphism $\overline{\Phi}_\lambda : U(\mathfrak{g}) \to U(\mathfrak{r})_E$ (notation 5.4). Furthermore for each $\lambda \in k$, Fract $U(\mathfrak{g})/J_\lambda$ identifies with $K(\mathfrak{r})$ and admits a maximal commutative subfield K_λ which is ad $\mathfrak{g}$ stable.

K_λ can be chosen to be generated by its intersection with $U(\mathfrak{r})_E$. Indeed taking $m = 1/2 \dim r$, one knows that $U(\mathfrak{r})$ is isomorphic to the subalgebra of the Weyl algebra $k[x_1 x_2, \ldots, x_m, y_1 y_2, \ldots, y_m]$, $y_i = \partial/\partial x_i$ with generators

$$y_i \quad (i = 1, 2, \ldots, m),$$

$$x_{j-m} \quad (j = 1, 2, \ldots, m-1), \qquad (x_m y_m - 1/2 \textstyle\sum_{j=1}^{m-1} x_j y_j).$$

Then $\overline{\Phi}(\mathfrak{g} \oplus k)$ is spanned by y_i, $x_i y_j$, $x_i(\sum_{j=1}^m x_j y_j + \lambda + 2)$, $i, j = 1, 2, \ldots, m$, and we may choose $K = k(x_1, x_2, \ldots, x_m)$.

Let us examine the polarization construction in detail for $sl(2)$. Set $x = x_1$, $y = y_1$. The canonical basis $\{E, H, F\}$ for $sl(2)$ is represented above through $E = y$, $H = -yx - \lambda$, $F = yx^2 + 2x\lambda$ and satisfies the relations $[E, F] = -2H$, $[H, E] = E$, $[H, F] = -F$. Then

$$J_\lambda = U(\mathfrak{g})(EF - H(H+1) + \lambda(\lambda - 1)),$$

and we set $I = \operatorname{gr}(J_\lambda) = S(\mathfrak{g})(EF - H^2)$. We have

$$\mathfrak{r} = kH \oplus kE, \qquad K_\lambda = k(E^{-1}(H+\lambda)), \qquad K' = k(E^{-1}H).$$

Set

$$u = H \otimes E - E \otimes H, \qquad v = E \otimes H^2 - 2(H \otimes EH) + \mathrm{F} \otimes E^2,$$

considered as elements of $\mathfrak{g} \otimes S(\mathfrak{r})$. Then (notation 5.7), V is spanned by u, N by v, and $V^\perp$ by $\{u, v\}$. Furthermore $[u, v] = (1 \otimes E)\, v$. Taking $f = {}^tE$, $\mathfrak{p}_f$ has basis $\{H, F\}$, that is $\mathfrak{p}_f = {}^t\mathfrak{b}$ (notation 3.1). It is well-known that the J_λ are induced from the one-dimensional representation of ${}^t\mathfrak{b}$ defined in the obvious fashion by the character $\chi_\lambda : H \mapsto \lambda$.

5.11: The analysis of 5.5-5.8 applies in principle to any $I \in \operatorname{Prim} U(\mathfrak{g})$, though in practice certain technical difficulties arise. Yet just as the results of [16] (as summarized in 5.4) are generalized in [20] (6.10, 6.14, 6.15), so the results of 5.5-5.8 admit a corresponding generalization. The details are left to the reader.

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