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CONTROL OF JUMP PROCESSES AND APPLICATIONS

by

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RÉSUMÉ. — L'objet de cet article est d'étendre les méthodes développées par l'auteur pour le contrôle des diffusions à des processus de sauts, et particulièrement aux diffusions à sauts. On étudie également des jeux à somme nulle et des techniques d'approximation.

ABSTRACT. — The purpose of this paper is to extend the methods developped by the author for the control of diffusions to jump processes, and especially to jumping diffusions. Zero-sum games are solved, and approximation techniques are indicated.

0. Introduction

The purpose of this paper is to extend the methods which we have developed in [1] for the control of diffusions to more general processes, and especially for diffusion processes with jumps.

In the case of diffusions, we controlled in [1] a process governed by an equation of type

$$(0.1)\qquad \begin{cases} dx = b(t, x, u(t, x))\,dt + \sigma(t, x).d\beta, \\ x_s = x, \end{cases}$$

and we minimized

$$(0.2)\qquad e^{ps} E \int_s^{+\infty} e^{-pt} L(t, x_t, u(t, x_t))\,dt.$$

We then used certain fundamental features of the diffusion processes, in the general framework developed by STROOCK and VARADHAN in [20].

When u varies in the set of Borel functions on $R^+ \times R^d$:

— all the measures on the space of the continuous functions defined by (0.1) stay equivalent on each M_t^s, M_t^s being the σ-field $\mathscr{B}(x_u; s \leqslant u \leqslant t)$;

— the processes defined by (0.1) are strong Feller processes;

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– there is a common reference measure μ for these processes (i. e. for a Borel set to have a negligible potential, it is necessary and sufficient that it is μ-negligible);

– it is possible to represent the "pay-off" criteria process by means of square integrable additive functional martingales.

These methods are applied, in this paper, to cases which are simpler or more complicated.

In the first two sections, we treat the case of jump processes on Z^d, and their control.

We handle this relatively simple case with powerful probability theory methods, in order to apply them later to far more general cases, especially to diffusion with jumps.

In Section I, we start with a given „basic" jump process on Z^d, for instance a Poisson process, and we modify this process by a measure transformation, which will be equivalent to a modification of the speed rate of each possible jump.

If the initial process has a Levy system $N(x, dy)$ on Z^d, we build all all the processes which have as Levy system $(1+\tilde{b}(t, x, y)\, N(x, dy)$. We follow here the approach of JACOD in [11] for the definition of jump processes. The new measure will then have a density relative to the initial measure, with an explicit density given by Doléans-Nade formula in [9].

It is then possible to prove the weak continuity of the density as a function of $\tilde{b}$ relative to a weak topology on the set where $\tilde{b}$ is taken.

At this point, the reader should be aware that the new processes are not necessarily equivalent to the initial one. In particular, if $\tilde{b}$ does not depend on time, and if $\tilde{b}(x, y) = -1$ for $x \neq y$, the possible connection from x to y is "closed" in the new process.

In Section II, we solve various problems of the control of jump processes. In particular, if we assume that $\tilde{b}$ is of the form $\tilde{b}(t, x, y, u)$, and if the cost of connection between x and y is $N(x, dy)\, L(t, x, y, u)$, we minimize

$$(0.3)\qquad V_{(\tilde{b}, \tilde{L})}(s, x) = e^{ps} E_{(s, x)}^{\tilde{Q}^{\tilde{b}_u}} \int_s^{+\infty} e^{-pt}\, dt \times \int_{Z^d} N(x_t, dy)\, L(t, x_t, y, u(t, x_t, y))\, dt.$$

where u is the control variable, and depends on (t, x, y).

When $\tilde{b}$ and L depend only on (t, x, u), we minimize

$$(0.4)\qquad V'_{(\tilde{b}.\tilde{L})}(s, x) = e^{ps} E^{\tilde{\tilde{Q}}^{b_u}}_{(s, x)} \int_s^{+\infty} e^{-pt} L(t, x_t, u(t, x_t))\, dt,$$

where u depends on (t, x).

To prove existence of an optimal control, we use methods formally identical to the methods already developed by the author in [1]. In particular, convexity is an intermediary step, but is not necessary for getting an existence result.

A basic probabilistic difficulty comes from the fact that we must be able to represent the process $V_{(\tilde{b}, \tilde{L})}(t, x_t)$ for any of the measures $\tilde{\tilde{Q}}^{b'}$. The measures $\tilde{\tilde{Q}}^{b'}$ not necessarily being equivalent, this representation is not as "easy" as in the case of diffusions. However, the problem is solved very easily by going back to the resolvent equations.

Non purely probabilistic methods obviously work for this case, but the technique used here is generalized later to other processes where the resolvent equation is no longer manageable.

Similar techniques are used by BOEL and VARAIYA in [6] for the definition of jump processes in the general non anticipating case. The existence of an optimal control is obtained by different methods than here, essentially by representing the cost function of the problem as a semimartingale for each of the attainable processes, and deriving indirectly an optimal control. The proof proceeds here in the inverse order: an existence result is first derived very simply under a convexity assumption, and generalized to the non convex case. This method is directly adapted to the Markov case, and is generalizable to all the Markov optimization problem studied in [1], as to the control problem of Section V, where the techniques of Sections I and II and of [1] are combined. The strong Feller property and the existence of a common reference measure are the essential tools which allow us in all cases to derive the existence of an optimal Markov control.

In Section III, some applications are given. In particular, the optimal stopping time problem is treated very easily as an optimal control problem. An approximation scheme, very similar to the technique already found for diffusions in [3], is given. This approximation technique is then applied to the optimal stopping time problem.

We define an algorithm which gives a decreasing sequence of sets whose intersection is an optimal stopping set.

In Section IV, zero-sum games are introduced, and are solved by techniques very similar to the techniques of [2],

Finally, in Section V, we give the main outline of the argument for the control of jumping diffusions with a "continuous" set of possible jumps in the framework developed by STROOCK in [22].

I. — Jump processes and densities

The purpose of this chapter is to define some of the basic properties of jump processes. We start from a fixed process with integer jumps and with a given Levy system, and we consider processes whose Levy system is absolutely continuous with respect to the initial one. Following JACOD [11], we prove that the new processes can be defined in a unique way, and have a probability density relative to the initial process. Finally, we study the dependence of the densities on the coefficients of the Levy measure.

1. The martingale problem

Ω is the space $D(R^+; Z^d)$ of functions defined on R^+ with values in Z^d, which are right continuous and have left-hand limits. Ω is endowed with the Skorokhod metric.

M_t^s is the σ-algebra of Ω, defined by $M_t^s = \mathscr{B}(x_u; s \leqslant u \leqslant t)$.

$N(t, x, .)$ is a family of positive kernels defined on Z^d, such that:

(*a*) $N(t, x, Z^d)$ is uniformly bounded;

(*b*) $N(t, x, \{x\}) = 0$;

(*c*) for any $A \subset Z^d$, $N(t, x, A)$ is a Borel function on $R^+ \times Z^d$ with values in R^+.

We now define a simple martingale problem:

DÉFINITION I.1. — A measure $P_{(s, x)}$ on Ω is said to be a solution of the martingale problem relative to (s, x, N), if:

(*a*) $P_{(s,x)}(x_s = x) = 1$;

(*b*) for any function f, defined on $R^+ \times Z^d \times Z^d$, Borel measurable and bounded, if Sf is the process

$$Sf_t^s = \sum_{s \leqslant u \leqslant t,\, x_{u^-} \neq x_u} f(u, x_{u^-}, x_u) \tag{I.1}$$

then, the process $S^c f$, defined by

$$S^c f_t^s = Sf_t^s - \int_s^t (Nf)(u, x_u)\, du \tag{I.2}$$

is a martingale.

2. Existence and Uniqueness of the solution to the martingale problem

We assume from now on that $N(x, .)$ is a family of time homogeneous kernels which satisfy the previous assumptions. We then make the following fundamental assumption:

for any $x \in Z^d$, a solution $\tilde{P}_x$ to the martingale problem relative to (x, N) exists.

Example. — Let $\lambda_1, \ldots, \lambda_d$ be finite positive measures on Z such that

$$\lambda_1(0) = \ldots = \lambda_n(0) = 0.$$

For h defined on Z^d with values in R, uniformly bounded, we define N by

$$N h(x) = (\lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_d) \star h(x) \tag{I.3}$$

($\star$ is the convolution). Then, it is a well known result (*see*, for instance, [5], I, T-2.18) that the measure P_x, relative to the independent increment Poisson process which starts at x at time 0, is a solution of the problem.

N will now be fixed for the rest of the paper. M will be an upper bound for $N(x, Z^d)$.

$\tilde{b}$ is a Borel function, defined on $R^+ \times Z^d \times Z^d$, with values in R, which is supposed to be bounded and $\geqslant -1$.

We will consider the kernel $N(x, dy)(1 + \tilde{b}(t, x, y))$. Let A be the kernel

$$A(x, dy) = N(x, dy) - N(x, Z^d)\, \varepsilon_x(dy) \tag{I.4}$$

and $A^{\tilde{b}}$ the kernel

$$A^{\tilde{b}}(t, x, dy) = N(x, dy)(1 + \tilde{b}(t, x, y)) - \int_{Z^d} N(x, dz)(1 + \tilde{b}(t, x, z))\, \varepsilon_x(dy). \tag{I.5}$$

We will often write

$$N(x, dy) = n(x, y)\, dy. \tag{I.6}$$

We then have the following fundamental result.

THEOREM I.1. — *For any (s, x), the martingale problem relative to $(N(1 + \tilde{b}), s, x)$ has one and only one solution, $\tilde{Q}^{\tilde{b}}_{(s,x)}$. If $\tilde{P}_{(s,x)}$ is the unique solution of the problem relative to (N, s, x), $\tilde{Q}^{\tilde{b}}_{(s,x)}$ has a density relative to $\tilde{P}_{(s,x)}$ on M^s_t $(s \leqslant t < +\infty)$ given by*

$$Z^{\tilde{b}}_t = \exp\left(-\int_s^t (N\tilde{b})(u, x_u)\, du\right) \prod_{x_{u^-} \neq x_u,\, s \leqslant u \leqslant t} (1 + \tilde{b}(u, x_{u^-}, x_u)). \tag{I.7}$$

Moreover, if $\|\tilde{b}\|$ *is the norm of* $\tilde{b}$ *in* $L_\infty(R^+ \times Z^d \times Z^d)$, *then*

$$E^{\tilde{P}(s,x)}|Z_t^{\tilde{b}}|^2 \leqslant \exp M\|\tilde{b}\|^2(t-s). \tag{I.8}$$

The process defined by the measure $\tilde{Q}^{\tilde{b}}_{(s,x)}$ *is a strong Feller process. If* $\tilde{b}$ *and* $\tilde{b}'$ *are equal,* $dt \otimes dx \otimes dy$ a. e., *the* $Q^{\tilde{b}}$ *and* $Q^{\tilde{b}'}$ *are equal.*

Proof. — Uniqueness of the measure $\tilde{Q}^{\tilde{b}}_{(s,x)}$ follows from a result of JACOD [11] (Theorem 3.4). The measure $\tilde{P}_{(s,x)}$ exists, because it is associated to the process $\tilde{P}_x$ starting at time s.

By [11] (Theorem 4.5), $Z_t^{\tilde{b}}$ is the density of $\tilde{Q}^{\tilde{b}}_{(s,x)}$ on M_t^s if $E^{\tilde{P}(s,x)} Z_t^{\tilde{b}} = 1$. By the result of DOLÉANS-DADE [9], $Z_t^{\tilde{b}}$ is the unique solution of

$$\begin{cases} dZ = Z - dS^{co}\tilde{b}, & t \geqslant s, \\ Z_s = 1, \end{cases} \tag{I.9}$$

where $S^{co}\tilde{b}$ is the sum $S^c\tilde{b}$, calculated for the process $\tilde{P}_{(s,x)}$.

We now prove that Z_t is a square integrable martingale. Let T_n be the stopping time.

$$T_n = \inf\{t \geqslant s;\ Z_{t-} \geqslant n\}. \tag{I.10}$$

Then, by (I.9),

$$E^{\tilde{P}(s,x)} Z^2_{t\wedge T_n} = 1 + E^{\tilde{P}(s,x)} \int_s^{t\wedge T_n} |Z_{u-}|^2 \, d\langle S^{co}\tilde{b}, S^{co}\tilde{b}\rangle, \tag{I.11}$$

where $\langle S^{co}\tilde{b}, S^{co}\tilde{b}\rangle$ is the quadratic variation of $S^{co}\tilde{b}$.

But, we have

$$d\langle S^{co}\tilde{b}, S^{co}\tilde{b}\rangle = N\tilde{b}^2(u, x_u)\,du. \tag{I.12}$$

Then,

$$E^{\tilde{P}(s,x)}|Z^2_{t\wedge T_n}| \leqslant 1 + E^{\tilde{P}(s,x)} \int_s^{t\wedge T_n} M\|b\|^2 Z_u^2\,du. \tag{I.13}$$

By Gronwall's lemma,

$$E^{\tilde{P}(s,x)}|Z^2_{t\wedge T_n}| \leqslant \exp M\|b\|^2(t-s). \tag{I.14}$$

This implies that Z_t is a square integrable martingale. For $T \geqslant s$, $Z_T\,d\tilde{P}_{(s,x)}$ is then a probability measure on M_T^s.

If $\tilde{b} = \tilde{b}'$, $dt \otimes dx \otimes dy$ a. e., we may write

$$E^{\tilde{P}(s,x)}|S^{co}\tilde{b}_t^s - S^{co}\tilde{b}_t'^s|^2 = E^{\tilde{P}(s,x)} \int_s^t N(\tilde{b}-\tilde{b}')^2\,du = 0. \tag{I.15}$$

This implies that $S^{co}\tilde{b} = S^{co}\tilde{b}'$, and then by (I.9) that $Z_t^{\tilde{b}} = Z_t^{\tilde{b}'}$. The measures $\tilde{\tilde{Q}}^{b}_{(s,x)}$ and $\tilde{\tilde{Q}}^{b}_{(s,x)}$ are necessarily equal. Moreover $\tilde{P}_{(s,x)}$ is a continuous function of $(s, x) \in R^+ \times Z^d$.

$\tilde{\tilde{Q}}^{b}_{(s,x)}$ is then a measurable function of (s, x). Let us now prove the strong Markov property of $\tilde{\tilde{Q}}^{b}$. If T is a stopping time for $\tilde{\tilde{Q}}^{b}_{(s,x)}$, let $\tilde{Q}_\omega$ be the regular conditional probability measure of $\tilde{\tilde{Q}}^{b}_{(s,x)}$ relative to M^s_T. It is then easily proved that the probability measure which is equal to $\tilde{\tilde{Q}}^{b}_{(s,x)}$ on M^s_T, and whose conditional distribution relative to M^s_T is $\tilde{\tilde{Q}}^{b}_{(T,x_T)}$, is also a solution to the martingale problem. By the uniqueness result, this implies that $\tilde{Q}_\omega = \tilde{Q}_{(T,x_T)}$.

Finally, let L be a bounded $\geqslant 0$ Borel function defined on $R^+ \times Z^d$ with values in R. We then consider the "differential equation":

$$\text{(I.16)}\qquad \begin{cases} \dfrac{dV}{dt} + A^{\tilde{b}}\, V = -L, & t \leqslant T, \\ V(T) = 0. \end{cases}$$

If we look at (I.16) as a differential equation in the Banach space $L_\infty(Z^d)$, we find one, and only one, solution for (I.16) which is uniformly bounded. V is then continuous on $R^+ \times Z^d$. We may write that, $\tilde{\tilde{Q}}^{b}_{(s,x)}$ a. s., we have;

$$\text{(I.17)}\qquad V(T, x_T) = V(0, x) + \int_0^t \frac{\partial V}{\partial t}(t, x_t)\, dt + \sum_{x_{s-} \neq x_s;\ 0 \leqslant s \leqslant T} (V(s, x_s) - V(s, x_{s-})).$$

If f is the function, defined by

$$\text{(I.18)}\qquad f(s, x, y) = V(s, y) - V(s, x),$$

the sum in (I.17) is nothing but Sf. By (I.2), we then know that $S^{c\tilde{b}} f$ (which is $S^c f$ for the measure $\tilde{\tilde{Q}}^{b}_{(s,x)}$) is a martingale for $\tilde{\tilde{Q}}^{b}_{(s,x)}$. This implies

$$\text{(I.19)}\qquad 0 = V(s, x) + E^{\tilde{b}} \int_s^t \left(\frac{\partial V}{\partial t} + A^{\tilde{b}}\, V\right)(t, x_t)\, dt,$$

or equivalently

$$\text{(I.20)}\qquad V(s, x) = E^{\tilde{\tilde{Q}}^{b}_{(s,x)}} \int_s^T L(t, x_t)\, dt.$$

The potential of $1_{t \leqslant T} L$ for $\tilde{\tilde{Q}}^{b}$ is a continuous function on $R^+ \times Z^d$. $\tilde{\tilde{Q}}^{b}$ is then strongly Feller.

Remark I.1. — In contrast to the case of diffusions, the processes $\tilde{Q}^{\tilde{b}}$ are not mutually absolutely continuous. In particular, if, for a couple $(x, y) \in Z^d$ with $x \neq y$, $\tilde{b}(x, y) = -1$, the "bridge" from x to y is "closed" for $\tilde{Q}^{\tilde{b}}$, whereas it may not be closed for $\tilde{P}$.

COROLLARY. — *The measure $dt \otimes dx$ on $R^+ \times R^d$ is a reference measure for any $\tilde{Q}^{\tilde{b}}$, i. e., a set A in $R^+ \times Z^d$ has a null potential for $\tilde{Q}^{\tilde{b}}$ if, and only if, A is $(dt \otimes dx)$-negligible.*

Proof. — A $(dt \otimes dx)$-negligible set is obviously of null potential. Moreover, for any $\tilde{Q}^{\tilde{b}}_{(s,x)}$, the process x_t remains a non-null time at x. A set which has a null potential is then, necessarily, $(dt \otimes dx)$-negligible.

3. Weak dependences

In this part, we prove results comparable to Proposition IV.4 and Theorem IV.3 of [1], which give the weak dependences of the density on the coefficient $\tilde{b}$.

First, we have the following result.

PROPOSITION I.1. — *For any (s, x) in $R^+ \times Z^d$, if $\{L_n\} \subset L_\infty(R^+ \times Z^d)$ converges weakly to $L \in L_\infty(R^+ \times Z^d)$, then $\int_s^T L_n(u, x_u)\, du$ converges to $\int_s^T L(u, x_u)\, du$ in $L_2(\Omega)$, when Ω is endowed with measure $\tilde{P}_{(s,x)}$.*

Proof. — If dy is the counting measure on Z^d, $P(s, x, t, y)$ can be defined as the probability of transition to y at time t.

If the function V_n is defined by

$$V_n(s, x) = E^{\tilde{P}(s,x)} \int_s^T L_n(u, x_u)\, du, \tag{I.21}$$

we have

$$V_n(s, x) = \int_{Z^d} \int_s^T L_n(u, y) P(s, x, u, y)\, du\, dy. \tag{I.22}$$

This implies that $V_n \to V$. Moreover,

$$\left| \int_s^T (L_n - L)(u, x_u)\, du \right|^2 = 2\left\{ \int_s^T (L_n - L)(u, x_u)\, du \int_u^T (L_n - L)(t, x_t)\, dt \right\}. \tag{I.23}$$

By the Markov property, we then necessarily have.

$$(\text{I.24}) \qquad E^{\tilde{P}(s,x)}\left|\int_s^T (L_n - L)(u, x_u)\,du\right|^2 = 2E^{\tilde{P}(s,x)}\int_s^T (L_n - L)(V_n - V)(u, x_u)\,du.$$

This will imply that the left hand side of (I.24) converges to 0. The result is proved.

We then have the fundamental result.

THEOREM I.2. — *Let $\tilde{b}_n$ be a sequence of elements of $L_\infty(R^+ \times Z^d \times Z^d)$ such that $\tilde{b}_n \geqslant -1$, converges weakly to $\tilde{b} \in L_\infty(R^+ \times Z^d \times Z^d)$.*

Then, if $\tilde{Z}_T^{\tilde{b}_n}$ is the density of $\tilde{\tilde{Q}}_{(s,x)}^{\tilde{b}_n}$ relative to $\tilde{P}_{(s,x)}$ on M_T^s, $\tilde{Z}_T^{\tilde{b}_n}$ converges weakly to $\tilde{Z}_T^{\tilde{b}}$ for the weak topology $\sigma(L_2(\Omega), L_2(\Omega))$ where Ω is endowed with measure $\tilde{P}_{(s,x)}$.

Proof. — It is sufficient to prove that for any sequence $\{n_k\}$ on N, it is possible to find a subsequence n_{k_l} such that $Z_T^{\tilde{b}_{n_{k_l}}}$ converges weakly to $Z_T^{\tilde{b}}$. By changing the indices, we come back to the sequence n. The sequence $\{\tilde{b}_n\}$ stays uniformly bounded.

By (I.8), the sequence $\{Z_T^{\tilde{b}_n}\}$ is weakly relatively compact in $L_2(\Omega)$, when Ω is endowed with measure $\tilde{P}(s, x)$. Let Z be a weak limit of a subsequence $\{Z_T^{b_{n_k}}\}$. Then $Z \geqslant 0$, and $Z\,d\tilde{P}_{(s,x)}$ defines a probability measure on M_T^s.

We know that, for f Borel measurable and bounded on $R^+ \times Z^d \times Z^d$,

$$(\text{I.25}) \qquad S_t^{c\tilde{b}_n} f_t^s = S f_t - \int_s^t (N(1+\tilde{b}_n)f)(u, x_u)\,du$$

is a martingale for the measure $Z_T^{\tilde{b}_n}\,d\tilde{P}_{(s,x)}$.

Let us prove that

$$(\text{I.26}) \qquad X_n(t, x) = \int_{Z^d} N(x, dy)(1+\tilde{b}_n(t, x, y))\,f(t, x, y)$$

converges weakly in $L_\infty(R^+ \times Z^d)$ to

$$(\text{I.27}) \qquad X(t, x) = \int_{Z^d} N(x, dy)(1+\tilde{b}(t, x, y))\,f(t, x, y).$$

We notice first that these functions are uniformly bounded in $R^+ \times Z^d$.

Let $\varphi \in L_1 (R^+ \times Z^d)$, then

$$(I.28) \qquad \langle \varphi, X_n \rangle = \int_{R^+ \times Z^d \times Z^d} \varphi(t, x) n(x, y)(1+\tilde{b}_n(t, x, y)) \times f(t, x, y)\, dt\, dx\, dy.$$

But, now

$$(I.29) \qquad \varphi n f \in L_1 (\mathrm{R}^+ \times Z^d \times Z^d).$$

This yields

$$(I.30) \qquad \langle \varphi, X_n \rangle \to \langle \varphi, X \rangle.$$

Let A be a M_t^s-measurable set in Ω, with t such that $s \leqslant t \leqslant T$. Then, by the martingale property (I.2), we have

$$(I.31) \qquad E^{\tilde{P}(s,x)}\left(Z_T^{b_{n_k}}\left(S f_T^s - \int_s^T X_{n_k}(u, x_u)\, du\right) 1_A\right) = E^{\tilde{P}(s,x)}\left(Z_T^{b_{n_k}}\left(S f_t^s - \int_s^t X_{n_k}(u, x_u)\, du\right) 1_A\right).$$

We must then pass to the limit in (I.31). We know that:

$Z_T^{b_{n_k}} \to Z$ weakly in $L_2(\Omega)$;

$\int_s^T X_{n_k}(u, x_u)\, du \to \int_s^T X(u, x_u)\, du$ strongly in $L_2(\Omega)$;

$S f_t^s$ is in $L_2(\Omega)$ for any t, because we have

$$(I.32) \qquad E^{\tilde{P}(s,x)} | S^{co} f_t^s |^2 = E^{\tilde{P}(s,x)} \int_s^t (N f^2)(u, x_u)\, du,$$

$$(I.33) \qquad \| S f_t^s \|_{L_2(\Omega)} < \| S^{co} f_t^s \|_{L_2(\Omega)} + \left[E^{\tilde{P}(s,x)} \left(\int_s^t (N f)(u, x_n)\, du \right)^2 \right]^{1/2}.$$

We then see that

$$(I.34) \qquad E^{\tilde{P}(s,x)}\left(Z\left(S f_T^s - \int_s^T X(u, x_u)\, du\right) 1_A\right) = E^{\tilde{P}(s,x)}\left(Z\left(S f_t^s - \int_s^t X(u, x_u)\, du\right) 1_A\right).$$

This implies that

$$(I.35) \qquad S f_t^s - \int_s^t N((1+\tilde{b}) f)(u, x_u)\, du$$

is a martingale for $s \leqslant t \leqslant T$ relative to the measure $Z\, d\tilde{P}_{(s,x)}$.

By the uniqueness of the measure $\tilde{Q}^{\tilde{b}}_{(s,x)}$ on each M^s_T, we have necessarily

$$Z = Z^{\tilde{b}}_T, \tag{I.36}$$

and $Z^{\tilde{b}_n}_T$ converges then necessarily to $Z^{\tilde{b}}_T$.

II. — The problems of control

1. The problems

In this part, we define the problems of control. U is a metrizable compact space. p is a strictly positive constant.

We define *a first problem of control:*

$(\tilde{b}(t, x, y, u), L(t, x, y, u))$ is a function defined on $R^+ \times Z^d \times Z^d \times U$ with values in $]-1, +\infty[\times R$, uniformly bounded, measurable in (t, x, y), and continuous in u.

For u Borel on $R^+ \times Z^d \times Z^d$, $(\tilde{b}_u, L_u)$ is the function

$$(\tilde{b}(t, x, y, u(t, x, y)), L(t, x, y, u(t, x, y))).$$

DEFINITION II.1. — The problem of control $(\mathscr{P})$ consists of the search for u_0 Borel on $R^+ \times Z^d \times Z^d$ minimizing

$$e^{ps} E^{\tilde{Q}^{\tilde{b}_u}_{(s,x)}} \int_s^{+\infty} e^{-pt} (N L_u)(t, x_t)\, dt, \tag{II.1}$$

simultaneously for all (s, x) in $R^+ \times Z^d$.

Next, we define *a second problem of control:*

$(\tilde{b}(t, x, u), L(t, x, u))$ is a function defined on $R^+ \times Z^d \times U$ with values in $]-1, +\infty[\times R$, uniformly bounded, measurable in (t, x), and continuous in u.

DEFINITION II.2. — The problem of control $(\mathscr{P}')$ consists in the search for u Borel on $R^+ \times Z^d$ minimizing

$$e^{ps} E^{\tilde{Q}^{\tilde{b}_u}_{(s,x)}} \int_s^{+\infty} e^{-pt}\, L_u(t, x_t)\, dt, \tag{II.2}$$

simultaneously for all (s, x) in $R^+ \times Z^d$.

We then have the fundamental results.

THEOREM II.1. — *Problem $(\mathscr{P})$ has a solution.*

THEOREM II.2. — *Problem $(\mathscr{P}')$ has a solution.*

We devote the next sections to the proof of Theorems II.1 and II.2, concentrating mostly on the proof of Theorem II.1. At an essential step of the proof, we will use a maximum principle adapted to this problem.

2. A new definition for the problem

As in [1], IV.1, we change problem $(\mathscr{P})$ in a standard way. Let K be a Borel set valued mapping, defined on $R^+ \times Z^d \times Z^d$, with values in $]-1, +\infty[\times R$, nonempty, compact, and uniformly bounded.

DEFINITION II.3. — $\mathscr{L}$ is the set of equivalence classes of the Borel selections of K, defined on $R^+ \times Z^d \times Z^d$, with values in $]-1, +\infty[\times R$ for the measure $dt \otimes dx \otimes dy$.

For $c = (\tilde{b}, L) \in \mathscr{L}$, we define V_c by

$$V_c(s, x) = e^{ps} E^{\tilde{\tilde{Q}}^{\tilde{b}}_{(s, x)}} \int_s^{+\infty} e^{-pt} N L(t, x_t)\, dt. \tag{II.3}$$

DEFINITION II.4. — The problem of control $(\mathscr{P})$ consists of the search for $c_0 \in \mathscr{L}$ minimizing $c \to V_c(s, x)$ on $\mathscr{L}$ for all (s, x) simultaneously.

Another problem $(\mathscr{P}')$ is defined similarly, with K, defined on $R^+ \times Z^d$, with values in $]-1, +\infty[\times R$, and a functional

$$V'_c(s, x) = e^{ps} E^{\tilde{\tilde{Q}}^{\tilde{b}}_{(s, x)}} \int_s^{+\infty} e^{-pt} L(t, x_t)\, dt. \tag{II.4}$$

3. The convex case

We assume in this part only that K has convex values. $\mathscr{L}$ is then compact in the weak topology of $L_\infty(R^+ \times Z^d \times Z^d)$. Let μ be a probability measure on $R^+ \times Z^d$.

THEOREM II.3. — *The functional*

$$c \to I_\mu(c) = \int V_c(s, x)\, d\mu(s, x) \tag{II.5}$$

has a minimum in $\mathscr{L}$.

Proof. — By Proposition I.1 and Theorem I.2.

$$c \to E^{\tilde{\tilde{Q}}^{\tilde{b}}_{(s, x)}} \int_s^T e^{-pt} N L(t, x_t)\, dt \tag{II.6}$$

is continuous on $\mathscr{L}$. By a uniform convergence argument, $c \to V_c(s, x)$ will be continuous on $\mathscr{L}$.

I_μ is then continuous on $\mathscr{L}$. $\mathscr{L}$ being compact, the result follows.

We will now represent the process $V_c(t, x_t)$. Since the measures $\tilde{\tilde{Q}}^b$ may not necessarily be equivalent, the "simple" argument, given in [1], to represent this process for diffusions does not work so easily.

Let $\mathscr{V}$ be the function

$$\mathscr{V}(t, x) = e^{-pt} V_c(t, x). \tag{II.7}$$

We know by reasoning as in (I.16)-(I.20) that

$$\frac{d\mathscr{V}}{dt}+A^{\tilde{b}}\mathscr{V}=-e^{-pt}NL. \tag{II.8}$$

Then, $\tilde{P}$ a. s. (i. e. $\tilde{P}_\mu$ a. s. for all initial measure μ on $R^+\times Z^d$), we have

$$\mathscr{V}(t, x_t)-\mathscr{V}(s, x)=\int_s^t\left(\frac{d\mathscr{V}}{dt}+A\mathscr{V}\right)(u, x_u)\,du+S^{co}h_t^s \tag{II.9}$$

with

$$h(t, x, y)=\mathscr{V}(t, y)-\mathscr{V}(t, x). \tag{II.10}$$

The functional $S^{co}h$ is a square integrable additive martingale in the sense of [17] and is equally defined P_μ a. s. because, by the corollary of Theorem I.1, $dt\otimes dx$ is a reference measure for $\tilde{Q}^{\tilde{b}}$.

If f_c is defined by

$$f_c(t, x, y)=V_c(t, y)-V_c(t, x), \tag{II.11}$$

we will have $\tilde{P}$ a. s.

$$\begin{aligned} V_c(t, x_t)-V_c(s, x_s) \\ =\int_s^t\left(p\,V_c+e^{pt}\left(\frac{d\mathscr{V}}{dt}+A\mathscr{V}\right)\right)(u, x_u)+S^{co}f_c. \end{aligned} \tag{II.12}$$

We now have the following result.

PROPOSITION II.1. — *A sufficient condition for $c=(\tilde{b}, L)$ to be optimal for $I_{e^{-ps}\mu}$ is that $dt\otimes dx\otimes dy$* a. e.

$$\begin{aligned} n(x, y)(L(t,x, y)+(V_c(t, y)-V_c(t, x))\tilde{b}(t, x, y)) \\ =\min_{(\tilde{b}', L')\in K(t, x, y)}n(x, y)(L'+(V_c(t, y)-V_c(t, x))\tilde{b}'). \end{aligned} \tag{II.13}$$

Proof. — $\tilde{Q}^{\tilde{b}}_{(s, x)}$ is absolutely continuous to $\tilde{P}_{(s,x)}$ on each M_t^s. We then have $\tilde{Q}^{\tilde{b}'}$ a. s.

$$\begin{aligned} e^{-pt}V_c(t, x_t)-e^{-ps}V_c(s, x_s) \\ =-\int_s^t e^{-pu}NL(u, x_u)\,du+\int_s^t e^{-pu}(N(\tilde{b}'-\tilde{b})f_c)(u, x_u)\,du \\ +\{S^{c\tilde{b}'}(e^{-pu}f_c\}_t^s \end{aligned} \tag{II.14}$$

(we recall that $S^{c\tilde{b}'}h$ is the sum $S^c h$ calculated for $\tilde{Q}^{\tilde{b}'}$).
Then

$$E^{\tilde{Q}^{\tilde{b}'}_{(s, x)}}(S^{c\tilde{b}'}e^{-pu}f_c)_\infty^s=0. \tag{II.15}$$

If (II.13) holds then necessarily

$$\text{(II.16)}\qquad \begin{aligned}&\int e^{-ps}V_c(s,x)\,d\mu(s,x)\\ &= E^{\tilde{Q}^{\tilde{b}'}_{e^{-ps}\mu}}\int_s^{+\infty} e^{-pu}N(L+(\tilde{b}-\tilde{b}')f_c)(u,x_u)\,du\\ &\leqslant E^{\tilde{Q}^{\tilde{b}'}_{e^{-ps}\mu}}\int_s^{+\infty} e^{-pu}(NL')(u,x_u)\,du\\ &= \int e^{-ps}V_{c'}(s,x)\,d\mu(s,x).\end{aligned}$$

We then have the fundamental auxiliary result.

THEOREM II.4. – *If* μ *is a probability measure equivalent to* $dt\otimes dx$ *on* $R^+\times Z^d$, *then a necessary and sufficient condition for* $c\in\mathscr{L}$ *to be minimal for* $I_{e^{-ps}\mu}$ *is that* (II.13) *holds* $dt\otimes dx\otimes dy$ a. e.

Proof. – We prove that condition (II.13) is also necessary. If this condition is not verified, it is possible to find $c'=(b',L')\in\mathscr{L}$ such that:

$$\text{(II.17)}\qquad \begin{aligned}&n(x,y)(L'(t,x,y)+f_c(t,x,y)\tilde{b}'(t,x,y))\\ &\leqslant n(x,y)(L(t,x,y)+f_c(t,x,y)\tilde{b}(t,x,y))\end{aligned}$$

with strict inequality on a $dt\otimes dx\otimes dy$ non-negligible set.

By the Corollary of Theorem I.1, a $dt\otimes dx$ non-negligible set has a non-null potential for any $\tilde{Q}^{\tilde{b}}$. For any $\tilde{Q}^{\tilde{b}}$, the measure $e^{-p}\mu\,V^{\tilde{b}}$ defined by

$$\text{(II.18)}\qquad e^{-p}\mu\,V^{\tilde{b}}(\varphi)=\int e^{-ps}V_{(\tilde{b},\varphi)}(s,x)\,d\mu(s,x)$$

is a reference measure. From (II.14) and (II.17), this will imply:

$$\text{(II.19)}\qquad \begin{aligned}&\int e^{-ps}V_c(s,x)\,d\mu(s,x)\\ &= E^{\tilde{Q}^{\tilde{b}'}_{e^{-ps}\mu}}\int_s^{+\infty} e^{-pu}N(L+(\tilde{b}-\tilde{b}')f_c)(u,x_u)\,du\\ &> E^{\tilde{Q}^{\tilde{b}'}_{e^{-ps}\mu}}\int_s^{+\infty} e^{-pu}NL'(u,x_u)\,du=\int e^{-ps}V_{c'}(s,x)\,d\mu(s,x)\end{aligned}$$

(II.19) is a contradiction to the optimality of c.

We now prove the existence of a solution for problem $(\mathscr{P})$.

Proof. — By Theorem II.3, $I_{e^{-p}\mu}$ has a minimum. When μ is equivalent to $dt \otimes dx$, by Theorem II.4, (II.13) holds $dt \otimes dx \otimes dy$ a. e. By Proposition II.1, applied to $\mu = \delta_{(s, x)}$, it follows

$$V_c(s, x) \leqslant V_{c'}(s, x). \tag{II.20}$$

The result is proved.

Let us now define q by

$$q = \inf_{c' \in \mathscr{L}} V_{c'}. \tag{II.21}$$

Then

$$q = V_c. \tag{II.22}$$

Remark II.1. — For problem $(\mathscr{P}')$, the condition corresponding to (II.13) would have been

$$\begin{aligned} L(t, x)+(A V_c)(t, x)\tilde{b}(t, x) \\ = \min_{(\tilde{b}', L') \in K(t, x)} L'+A V_c(t, x)\tilde{b}'\, dt \otimes dx \quad \text{a.e.} \end{aligned} \tag{II.23}$$

4. The general case

We now assume that $K(t, x, y)$ does not necessarily have convex values. We consider its closed convex hull $\hat{K}(t, x, y)$. $\hat{K}$ is then Borel measurable by Corollary 3.3 of [19].

The problem $\hat{\mathscr{P}}$ associated to $\hat{K}$ has an optimal solution, and for c to be an optimal solution in $\hat{\mathscr{L}}$, it is necessary and sufficient that (II.13) holds.

$K(t, x, y)$ and $\hat{K}(t, x, y)$ having the same extremal points, it is possible to choose the optimal solution in $\mathscr{L}$.

This completes the proof of Theorems II.1 and II.2.

Remark II.2. — When the problem is time-homogeneous, the control can be taken time-homogeneous by using the methods of [1], V.2.

III. — APPLICATIONS

1. Processes killed on a Borel set

Let A be a Borel subset of $R^+ \times Z^d$, and be T_A is the stopping time

$$T_A = \inf\{t > s;\ (t, x_t) \in A\}. \tag{III.1}$$

We will consider the functional

$$e^{ps} E^{\tilde{\tilde{Q}}(s, x)} \int_s^{T_A} e^{-pt} (NL)(t, x_t)\, dt. \tag{III.2}$$

The problem of minimizing (III.2) can be solved by the methods of [1], V

In the homogeneous case where N, $\tilde{b}$, $\tilde{L}$, A do not depend on time, an optimal solution can be taken time-homogeneous by the method given in [1], V.2. Moreover, in this case, the problem can be looked at as the minimization of

$$(III.3) \qquad e^{ps} E^{\tilde{Q}^{b'(1+\tilde{b})-1}} \int_0^{+\infty} e^{-pt} b'(N L)(x_t)\, dt,$$

where b' is 1 on CA and 0 on A.

We come back to a problem of the type which we have already solved in Section II.

2. Processes with controlled death

The criterion is

$$(III.4) \qquad e^{ps} E^{\tilde{Q}^{\xi_{s,x}}} \int_s^{T_A} \exp\left(-\int_s^t N m(u, x_u)\, du\right)(N L)(t, x_t)\, dt,$$

where m is also a control $\geqslant p > 0$. The method is identical to that in [1], Chapter VI.

3. The optimal stopping time problem

The optimal stopping time problem can be changed into a standard control problem for time homogeneous processes. Let g be a function defined on Z^d with values in R, which is uniformly bounded. We want to find $\bar{A}$ in Z^d minimizing for all x:

$$(III.5) \qquad E^{\tilde{P}_x} e^{-pT_{\bar{A}}} g(x_{T_{\bar{A}}}).$$

For a general discussion of this problem, we refer to [10], [14] and [4]. (III.5) may be written as

$$(III.6) \qquad g(x) + E^{\tilde{P}_x} \int_0^{T_{\bar{A}}} e^{-pt} (A g - pg)(x_t)\, dt.$$

We then want to minimize

$$(III.7) \qquad E^{\tilde{P}_x} \int_0^{T_{\bar{A}}} e^{-pt} (A g - pg)(x_t)\, dt.$$

Let $K(x)$ be defined by

$$(III.8) \qquad K(x) = \{0, 1\}.$$

The problem is equivalent to finding a measurable selection b' of K minimizing:

$$(III.9) \qquad E^{\tilde{Q}^{b'-1}_{x}} \int_0^{+\infty} e^{-pu} b'(A g - pg)(x_u)\, du.$$

This problem has a homogeneous solution by Theorem II.2.

Let q be the function

$$q(x) = \inf_{\bar{A}} E^{\tilde{P}_x} e^{-pT_{\bar{A}}} g(x_{T_{\bar{A}}}). \tag{III.10}$$

By [14], $-q$ is the least p-excessive function $\geqslant -g$.
Then, on the optimal A, we obviously have

$$q(x) = g(x). \tag{III.11}$$

A characterization of the optimal b' is

$$[b'(Ag-pg)+b'A(q-g)](x) \leqslant [b(Ag-pg)+bA(q-g)](x) \tag{III.12}$$

when $b \in \{0, 1\}$, or

$$b'(Aq-pg)(x) \leqslant b(Aq-pg)(x). \tag{III.13}$$

This implies

$$\begin{cases} b'(x) = 0 & \text{if} \quad Aq-pg > 0, \\ b'(x) = 1 & \text{if} \quad Aq-pg < 0. \end{cases} \tag{III.14}$$

The region $A\,q-pg = 0$ is indifferent.

By noticing that

$$b'Aq = pq+pg(b'-1), \tag{III.15}$$

we find that by defining

$$\begin{cases} A^0 = \{x;\ Aq-pg > 0\}, \\ B^0 = \{x;\ Aq-pg < 0\}, \\ C^0 = \{x;\ Aq-pg = 0\}. \end{cases} \tag{III.16}$$

We have

$$\begin{cases} A^0 \cup C^0 = \{x;\ q(x) = g(x)\}, \\ \quad B^0 = \{x;\ q(x) < g(x)\}. \end{cases} \tag{III.17}$$

Then, any set $\bar{A}$, containing A^0, and included in $A^0 \cup C^0$, is a solution for the optimal stopping time problem. This is a special case of a general result of Bismut-Skalli [4].

4. An approximation

By reasoning as in [3], it is possible to define an approximation method for solving problems ($\mathscr{P}$) and ($\mathscr{P}'$). We shall give one word of the proof for problem ($\mathscr{P}$). We start from $c_1 = (b_1, L_1) \in \mathscr{L}$.

Let c_2 an element of $\mathscr{L}$, such that, $dt \otimes dx \otimes dy$ a. e.:

$$\text{(III.18)} \quad \begin{aligned} & n(x, y)(L_2(t, x, y)+\tilde{b}_2(t, x, y)(V_{c_1}(t, y)-V_{c_1}(t, x))) \\ & \quad = \min_{(\tilde{b}', L') \in K(t, x, y)} n(x, y)(L'+\tilde{b}'(V_{c_1}(t, y)-V_{c_1}(t, x))); \end{aligned}$$

$c_3 \ldots c_n$ are determined in the same way.

THEOREM III.1. — *The sequence V_{c_n} decreases to q. Any weak limit c of c_n is such that:*

$$\text{(III.19)} \qquad V_c = q.$$

Proof. — We will assume that K has convex values. μ is taken as in Theorem II.4. Either

$$\text{(III.20)} \qquad I_\mu(c_{n+1}) < I_\mu(c_n)$$

or

$$\text{(III.21)} \qquad I_\mu(c_n) = \min_{c \in \mathscr{L}} I_\mu(c) \quad \text{and} \quad I_\mu(c_{n+1}) = I_\mu(c_n).$$

Let n_k be a subsequence of N such that

$$\text{(III.22)} \qquad \begin{cases} c_{n_k} \to c \in \mathscr{L}, \\ c_{n_k+1} \to c' \in \mathscr{L}. \end{cases}$$

Then, by (III.20) and (III.21),

$$\text{(III.23)} \qquad I_\mu(c) = I_\mu(c').$$

Moreover, we have, $dt \otimes dx \otimes dy$ a. e.:

$$\text{(III.24)} \quad \begin{aligned} & n(x, y)(L_{n_k+1}(t, x, y)+\tilde{b}_{n_k+1}(t, x, y)(V_{c_{n_k}}(t, y)-V_{c_{n_k}}(t, x))) \\ & \quad \leqslant n(x, y)(\bar{L}+\bar{b}(V_{c_{n_k}}(t, y)-V_{c_{n_k}}(t, x))), \end{aligned}$$

when $(\bar{b}, \bar{L}) \in K(t, x, y)$.

Then, knowing that

$$\text{(III.25)} \qquad V_{c_n} \to V_c,$$

if $c' = (\tilde{b}', L')$, $dt \otimes dx \otimes dy$ a. e., for $(\bar{b}, \bar{L}) \in K(t, x, y)$, we have

$$\text{(III.26)} \quad \begin{aligned} & n(x, y)(L'(t, x, y)+\tilde{b}'(t, x, y)(V_c(t, y)-V_c(t, x))) \\ & \quad \leqslant n(x, y)(\bar{L}+\bar{b}(V_c(t, y)-V_c(t, x))). \end{aligned}$$

In particular, $dt \otimes dx \otimes dy$ a. e.:

$$\text{(III.27)} \quad \begin{aligned} & n(x, y)(L'(t, x, y)+\tilde{b}'(t, x, y)(V_c(t, y)-V_c(t, x))) \\ & \quad \leqslant n(x, y)(L(t, x, y)+\tilde{b}(t, x, y)(V_c(t, y)-V_c(t, x))). \end{aligned}$$

If this inequality held strictly on a $dt \otimes dx \otimes dy$ nonnegligible set, we would have

$$I_\mu(c') < I_\mu(c). \tag{III.28}$$

By comparison with (III.23), we find that there is equality in (III.27). By (III.26), $dt \otimes dx \otimes dy$ a. e., if $(\bar{b}, \bar{L}) \in K(t, x, y)$:

$$\begin{aligned} n(x, y)(L(t, x, y)+\tilde{b}(t, x, y)(V_c(t, y)-V_c(t, x))) \\ \leqslant n(x, y)(\bar{L}+\bar{b}(V_c(t, y)-V_c(t, x))), \end{aligned} \tag{III.29}$$

and we then find that

$$V_c = q. \tag{III.30}$$

The result follows.

5. Approximation of the optimal stopping time problem

We will now apply the previous results to the optimal stopping time problem. We start with a set A_0, for instance $A_0 = Z^d$ (i. e., $b_0 = 0$ everywhere). A_1 is defined as

$$A_1 = \{x; (Ag-pg)(x) > 0\}. \tag{III.31}$$

If $A_1 = A_0$, A_0 is optimal. If not, we define g_1 by

$$g_1(x) = E^{\tilde{P}_x} e^{-pT_{A_1}} g(x_{T_{A_1}}). \tag{III.32}$$

Then $g_1 \leqslant g$ and g_1 is strictly less than g at some points.

If b_1 is 0 on A_1, and 1 on CA_1, we have

$$b_1 A g_1 = pg_1 + pg(b_1 - 1). \tag{III.33}$$

Then

$$\{x; g_1(x) = g(x)\} = A_1 \cup \{x; Ag_1 - pg = 0\}, \tag{III.34}$$

$$\{x; g_1(x) < g(x)\} \subset \{Ag_1 - pg_1 = 0\}. \tag{III.35}$$

A_2 is defined as

$$A_2 = \{x; Ag_1 - pg > 0\}. \tag{III.36}$$

A_2 is then obviously included in A_1, because $g \geqslant g_1$.

g_2 is defined as

$$g_2(x) = E^{\tilde{P}_x} e^{-pT_{A_2}} g(x_{T_{A_2}}), \tag{III.37}$$

$A_3, A_4, \ldots, A_n \ldots$ are defined similarly. The sequence of sets A_n decreases.

Moreover, by Theorem III.1, we know that any weak limit of b_n is optimum for the "convexified" problem. $\bigcap_{n=1}^{+\infty} A_n$ is then an optimal region for the optimal stopping time problem.

IV. – Games

In this chapter, we shall define the zero-sum games which are the natural extensions of problems $(\mathscr{P})$ and $(\mathscr{P}')$.

K and K' are two set value mappings defined on $R^+ \times Z^d \times Z^d$ with valued subsets of $[-1, +\infty[\times R$ which are non empty, compact, and uniformly bounded.

$\mathscr{L}$ and $\mathscr{L}'$ are defined as in Definition II.3.

DEFINITION IV.1. – Problem (G) is defined as the search for $c_0 = (\tilde{b}_0, L_0) \in \mathscr{L}$ and $c'_0 = (\tilde{b}'_0, L'_0) \in \mathscr{L}'$ such that, if for all $(c, c') \in \mathscr{L} \times \mathscr{L}'$,

$$V_{c_0+c'} \leqslant V_{c_0+c'_0} \leqslant V_{c+c'_0}. \tag{IV.1}$$

A problem (G′) corresponding to problem $(\mathscr{P}')$ may be defined as well.

We then have the following theorem.

THEOREM IV.1. – *Problem* (G) *and* (G′) *have solutions.*

We shall give only a brief outline of the proof since it is very similar to the proof given in [2] for diffusions.

1. The convex case

Let μ be the same measure as in Theorem II.4. We will assume that K and K' have convex values.

For $c \in \mathscr{L}$ (resp. $c' \in \mathscr{L}'$), we define Γ'_c (resp. $\Gamma_{c'}$):

$$\Gamma'_c = \{ c' \in \mathscr{L}' ; \forall \tilde{c}' \in \mathscr{L}', \ V_{c+c'} \geqslant V_{c+\tilde{c}'} \}, \tag{IV.2}$$

$$(\text{resp.}\ \Gamma_{c'} = \{ c \in \mathscr{L} ; \forall \tilde{c} \in \mathscr{L}, \ V_{c+c'} \leqslant V_{\tilde{c}+c'} \}). \tag{IV.2′}$$

PROPOSITION IV.1. – Γ'_c *(resp.* $\Gamma_{c'}$*) has non-empty compact convex values in* $\mathscr{L}'$ *(resp.* $\mathscr{L}$*).*

Proof. – The non-emptyness of Γ'_c (resp. $\Gamma_{c'}$) follows from Theorem II.1. Moreover, Theorem II.4 proves that Γ'_c (resp. $\Gamma_{c'}$) is convex, because K and K' have convex values.

PROPOSITION IV.2. – *The set valued mapping defined on* $\mathscr{L} \times \mathscr{L}'$ *with values in* $\mathscr{L} \times \mathscr{L}'$ *by*

$$(c, c') \to \Gamma_{c'} \times \Gamma'_c \tag{IV.3}$$

is upper semicontinuous.

Proof. — This follows from the continuity of V_c on $\mathscr{L}$.

THEOREM IV.2. — *Problem* (G) *has a solution.*

Proof. — $(c, c') \to \Gamma_{c'} \times \Gamma'_c$ has non-empty compact convex values and is upper semicontinuous. It has a fixed point by Kakutani's theorem. Such a fixed point is a solution of the game.

We now proceeed as in [2]. The game has, in this case, a value q, and if (c_0, c'_0) is a solution of the game, then

$$V_{c_0+c'_0} = q. \tag{IV.4}$$

$V_{c_0+c'_0}$ is then a fixed function.

As in [2], Theorem 3.2, it is possible to prove that for $c_0 = (\tilde{b}_0, L_0)$ and $c'_0 = (\tilde{b}'_0, L'_0)$ to be solutions of the game, it is necessary and sufficient that $dt \otimes dx \otimes dy$ a. e.:

$$\begin{aligned} &n(x, y)(L_0(t, x, y)+\tilde{b}_0(t, x, y)(q(t, y)-q(t, x))) \\ &\quad = \min_{(\tilde{b}, \tilde{L}) \in K(t, x, y)} n(x, y)(\tilde{L}+\tilde{b}(q(t, y)-q(t, x))), \end{aligned} \tag{IV.5}$$

$$\begin{aligned} &(\text{resp.}\, n(x, y)(L'_0(t, x, y)+\tilde{b}'_0(t, x, y)(q(t, y)-q(t, x))) \\ &\quad = \max_{(\tilde{b}', \tilde{L}') \in K'(t, x, y)} n(x, y)(L'+\tilde{b}'(q(t, y)-q(t, x))). \end{aligned} \tag{IV.5'}$$

We now extend the above result to the general case.

2. The general case

We will give only a brief outline of the proof. We convexity and close K and K'. The convexified problem has a solution. By the necessary and sufficient conditions given in (IV.5) and (IV.5′) a solution can be found in the $\mathscr{L} \times \mathscr{L}'$. The argument is developped more fully in [2].

Remark IV.1. — The proof of the existence of a solution for problem (G′) proceeds in exactly the same way.

Remark IV.2. — All the previous results may be extended to control problems analogous to those treated in Section III.

V. — Control of jumping diffusions

In this chapter, we shall give the basic steps of the proof for the control of the jumping diffusions considered by STROOCK in [22].

1. The martingale problem

Let Ω be the space $D(R^+; R^d)$, and M_t^s be the σ-field $\mathscr{B}(x_u \mid s \leqslant u \leqslant t)$. a is a function defined on $R^+ \times R^d$, with positive definite values in $R^d \otimes R^d$, which is bounded, continuous and uniformly elliptic.

Let $M(t, x, .)$ be $a \geqslant 0$ σ-finite measure on $R^d/\{0\}$ such that for any $\Gamma \in \mathscr{B}(R^d/\{0\})$:

$$\text{(V.1)} \qquad \int_\Gamma \frac{|y|^2}{1+|y|^2} M(t, x, dy)$$

is bounded and continuous on $R^+ \times R^d$.

DEFINITION V.1. — Let L_t be the operator

$$\text{(V.2)} \qquad L_t f(x) = \frac{1}{2}\sum a_{ij}(t, x) f_{x_i x_j}(x) + \int_{R^d}\left(f(x+y) - f(x) - \frac{\langle y, f_x(x)\rangle}{1+|y|^2}\right) M(t, x, dy).$$

A measure P on Ω is said to be a solution of the martingale problem for $(s, x) \in R^+ \times R^d$ if:

(*a*) $P(x_s = x) = 1$;

(*b*) for any $f \in C_0^\infty(R^d)$,

$$\text{(V.3)} \qquad f(x_t) - \int_s^t L_u f(x_u)\, du \qquad \text{is a martingale.}$$

We then have the result of STROOCK in [22].

THEOREM V.1. — *The martingale problem has a unique solution* $\tilde{P}_{(s, x)}$ *which defines a strong Markov process* $\tilde{P}$. $\tilde{P}$ *is strong Feller.*

Proof. — This is Theorem 4.3 and Remark 4.1 of [22].

COROLLARY. — $\tilde{P}$ *has a reference measure* λ ([5], V, (1.1)) *on* $R^+ \times R^d$.

Proof. — From theorem V.1, the *p*-excessive functions will be l. s. c. The result follows from [5], V, (1.3).

This last result is fundamental for control theory.

2. Densities

Let b be a bounded Borel function on $R^+ \times R^d$ with values in R^d, and let $\tilde{b}$ be a bounded Borel function defined on $(R^+ \times R^d \times R^d/\{0\})$ with values in $]-1, +\infty[$ such that $(\tilde{b}(t, x, y))/|y|$ is a bounded function.

DEFINITION V.2. — Let L'_t be the operator

$$\text{(V.4)} \qquad L'_t f(x) = \frac{1}{2}\sum_{i,j} a_{ij}(t, x) f_{x_i x_j}(x) + \sum_k b_k(t, x) f_{x_k}(x) + \int_{R^d}\left(f(x+y) - f(x) - \frac{\langle y, f_x(x)\rangle}{1+|y|^2}\right) \times (1+\tilde{b}(t, x, y)) M(t, x, dy).$$

A measure Q on Ω is a solution of the martingale problem relative to $(s, x) \in R^+ \times R^d$ if:

(a) $Q(x_s = x) = 1$;

(b) for $f \in C_0^\infty(R^d)$,

$$f(x_t) - \int_s^t L'_u f(x_u)\, du \quad \text{is a martingale.} \tag{V.5}$$

We now prove, by using results of STROOCK [22], DOLÉANS-DADE [9] and JACOD-MEMIN [13], that the previous martingale problem has on unique solution, which may be defined by its density relative to the measure $\tilde{P}_{(s,x)}$ on each σ-field M_t^s. Finally, we prove that the processes $Q^{(b,\tilde{b})}$ have a common reference measure, and define the same fine topology.

THEOREM V.2. — *The martingale problem has a unique solution* $\tilde{Q}^{(b,\tilde{b})}_{(s,x)}$, *which defines a strong Markov process* $\tilde{Q}^{(b,\tilde{b})}$. $\tilde{Q}^{(b,\tilde{b})}$ *is strongly Feller. Moreover* $\tilde{Q}^{(b,\tilde{b})}$ *and* $\tilde{P}$ *have a common reference measure* λ *on* $R^+ \times R^d$ *and define the same fine topology.*

On each σ-field M_t^s, $Q^{(b,\tilde{b})}_{(s,x)}$ has a density $Z_t^{(b,\tilde{b})}$ relative to $\tilde{P}_{(s,x)}$ given by

$$\begin{aligned} Z_t^{(b,\tilde{b})} = \exp\left\{ \int_s^t \langle b'(u, x_u), a^{-1}(u, x_u)\, d\gamma_u \rangle \right. & \\ \left. - \frac{1}{2}\int_s^t \langle b', a^{-1} b' \rangle (u, x_u)\, du \right\} & \\ \times \exp S^{c(0,0)} \tilde{b}_t^s & \\ \times \textstyle\prod_{s \leqslant u \leqslant t;\, x_{u-} \neq x_u} \{ (1 + \tilde{b}(u, x_{u-}, x_u - x_{u-})) & \\ \times \exp - \tilde{b}(u, x_{u-}, x_u - x_{u-}) \}, & \end{aligned} \tag{V.6}$$

where:

- $b' = b - M\tilde{b}y/(1 + |y|^2)$;
- γ is the purely continuous part of x defined in Corollary (1.3.2) of [22];
- $S^{c(0,0)}\tilde{b}_t^s$ is the martingale associated to the function $\tilde{b}(t, x, y - x)$ for the measure $\tilde{P}_{(s,x)}$ by [17] (p. 154) (2).

Moreover, $E^{\tilde{P}(s,x)} |Z_t^{(b,\tilde{b})}|^2$ stays bounded when $(b, \tilde{b}, \tilde{b}/|y|))$ stay in a bounded set.

---

(2) There is a notational discrepancy between [17] and [22] for Levy systems. In the first chapter, we have adopted the notation of [17]. Here we take the notation of [22], but we keep the notation of [17] for the definition of the sums S and S^c.

Proof. — Existence and uniqueness of $\tilde{Q}^{(b, \tilde{b})}$ follow from [22], Theorem 4.3 and Remark 4.3.

We need only to prove that (V.6) defines a measure solution of the problem.

By Corollary (1.3.4) in [22], there is a brownian motion β^0 for $\tilde{P}_{(s,x)}$ such that

$$d\gamma = \sigma(t, x_t)\, d\beta_t^0, \tag{V.7}$$

where σ is the positive square root of a. Moreover, $S^{c(0,0)}\, \tilde{b}_t^s$ is a square integrable martingale, because

$$E_{(s,x)}^{\tilde{P}} \int_s^t \int_{R^d} M(u, x_u, dy)\, |\tilde{b}(u, x_u, y)|^2 < +\infty \tag{V.8}$$

by (V.1) and the assumptions of $\tilde{b}$. Finally b' is uniformly bounded. We then consider the equation

$$\begin{cases} dZ_u = Z_{u^-}(\langle \sigma^{-1}(u, x_u)\, b'(u, x_u), d\beta_u^0 \rangle + dS^{c(0,0)}\, \tilde{b}_u^s), & u \geqslant s, \\ Z_s = 1. \end{cases} \tag{V.9}$$

By [9], (V.9) has a unique solution given by (V.6).

As in Part I, it is easily checked that Z defines a square integrable martingale, and that $E^{\tilde{P}(s,x)}\, |Z_t|^2$ stays bounded under the stated conditions.

We now check that the measure defined by (V.6) is a solution of the problem. For $f \in C_0^\infty(R^d)$, we have $\tilde{P}$ a. e.

$$f(x_t) = f(x_s) + \int_s^t L_u f(x_u)\, du + \int_s^t \langle f_x(x_u), \sigma(u, x_u)\, d\beta_u^0 \rangle + S^{c(0,0)} f_t^s, \tag{V.10}$$

where $S^{c(0,0)} f$ is the square integrable additive martingale $S^c g$ defined in [17] (p. 154) with

$$g(x, y) = f(y) - f(x). \tag{V.11}$$

This follows from Corollary (1.3.2) in [22], from (V.1), from the fact that g is bounded, and from the inequality

$$|g(x, x+y)| \leqslant k\, |y|. \tag{V.12}$$

Moreover,

$$d\langle \beta_i^0, Z \rangle_u = Z_u(\sigma^{-1}(u, x_u)\, b'(u, x_u))_i\, du, \tag{V.13}$$

(V.14) $$d\langle S^{c(0,0)} f, Z\rangle_u = Z_u\,du \int_{R^d} M(u, x_u, dy)\,\tilde{b}(u, x_u, y)(f(x_u+y)-f(x_u))$$

(V.10), (V.13) and (V.14) will imply that

(V.15) $$Z_t f(x_t) - \int_s^t Z_u L'_u f(x_u)\,du \quad \text{is a martingale for } \tilde{P}_{(s,x)}.$$

The density Z_t is then a solution of the problem. We refer to JACOD and MEMIN for an extension of this method to a more general class of problems [13].

Uniqueness and the strong Feller property follow from Remarks 4.1 and 4.3 of [22], p. 232-233 (it should be noted that once existence has been established, uniqueness follows from the argument of Section 4 in [22], and the strong Feller property from Theorem A1 in [22] and the techniques of Section 7 of [20]).

Let $\varepsilon > 0$ be such that $|\tilde{b}(t, x, y)| < 1/2$ when $|y| < \varepsilon$. We define then

(V.16) $$\begin{cases} 1+\tilde{b}_0 = 1_{|y|<\varepsilon}, & 1+\tilde{b}_1 = 1_{|y|<\varepsilon}(1+\tilde{b}), \\ b_0 = -\displaystyle\int \frac{M\,y\,1_{|y|\geq\varepsilon}}{1+|y|^2}, & b_1 = b - \displaystyle\int \frac{M\,1_{|y|\geq\varepsilon}\,y(1+\tilde{b})}{1+|y|^2}. \end{cases}$$

By (V.6), the measures $\tilde{Q}^{(b_0,\tilde{b}_0)}_{(s,x)}$ and $\tilde{Q}^{(b_1,\tilde{b}_1)}_{(s,x)}$ are equivalent on M^s_t.

Let L be a bounded positive Borel function on $R^+ \times R^d$, and let p be a strictly positive constant. Let V and V' be the functions

(V.17) $$\begin{cases} V(s, x) = e^{ps} E^{\tilde{P}_{(s,x)}} \displaystyle\int_s^{+\infty} e^{-pt} L(t, x_t)\,dt, \\ V'(s, x) = e^{ps} E^{Q^{(b,\tilde{b})}_{(s,x)}} \displaystyle\int_s^{+\infty} e^{-pt} L(t, x_t)\,dt. \end{cases}$$

If T is the stopping time

(V.18) $$T = \inf\{t \geq s;\ |x_t - x_{t^-}| \geq \varepsilon\}.$$

We have

(V.19) $$V(s, x) = e^{ps} E^{\tilde{P}_{(s,x)}} \left(\int_s^T e^{-pt} L(t, x_t)\,dt + e^{-pT}\,V(T, x_T) \right).$$

By Corollary (3.1.1) of [22], we may write

(V.20) $$V(s, x) = e^{ps} E^{\tilde{Q}^{(b_0,\tilde{b}_0)}_{(s,x)}} \int_s^{+\infty} \exp\left\{ -\left(pt + \int_s^t M\,1_{|y|\geq\varepsilon}\,du \right) \right\} \times (L + M\,1_{|y|\geq\varepsilon}\,V)(t, x_t)\,dt.$$

Similary, we have

$$(V.21)\qquad V'_{(s,x)} = e^{ps} E^{\tilde{Q}^{(b_1,\tilde{b}_1)}_{(s,x)}} \int_s^{+\infty} \exp\left\{ -pt + \int_s^t M(1+\tilde{b})\, 1_{|y|\geq \varepsilon}\, du \right\} \times (L + M(1+\tilde{b})\, 1_{|y|\geq \varepsilon}\, V'(.,.+y))(t, x_t)\, dt.$$

Since $\tilde{Q}^{(b_0,\tilde{b}_0)}$ and $\tilde{Q}^{(b_1,\tilde{b}_1)}$ have the same negligible sets, it will obviously suffice to prove that $\tilde{Q}^{(b,\tilde{b})}$ and $\tilde{Q}^{(b_1,\tilde{b}_1)}$ have the same negligible sets. If L has a zero potential for $\tilde{Q}^{(b,\tilde{b})}$, its potential for $\tilde{Q}^{(b_1,\tilde{b}_1)}$ is also null. Let us prove the converse.

By (V.21), if L has a zero potential for $\tilde{Q}^{(b_1,\tilde{b}_1)}$, then

$$(V.22)\qquad V'(s,x) = e^{ps} E^{\tilde{Q}^{(b_1,\tilde{b}_1)}_{(s,x)}} \int_s^{+\infty} \exp\left\{ -\left(pt + \int_s^t M(1+\tilde{b})\, 1_{|y|\geq \varepsilon}\, du \right) \right\} \times (M(1+\tilde{b})\, 1_{|y|\geq \varepsilon}\, V')(t, x_t)\, dt.$$

But, by (V.1), $M(1+\tilde{b})\, 1_{|y|\geq\varepsilon}$ is now uniformly bounded by a positive constant k. If $k' = \sup_{(s,x)\in R^+\times R^d} V'(s,x)$, this implies that by (V.22),

$$(V.23)\qquad k' \leqslant k' \frac{k}{p+k}$$

and necessarily $k' = 0$. L has a zero potential for $\tilde{Q}^{(b,\tilde{b})}$.

Finally $\tilde{Q}^{(b_0,\tilde{b}_0)}$ and $\tilde{Q}^{(b_1,\tilde{b}_1)}$ define the same fine topology, because they are equivalent on each M_t^s. But by Corollary (3.1.1) of [22], $\tilde{Q}^{(b,\tilde{b})}_{(s,x)}$ and $\tilde{Q}^{(b_1,\tilde{b}_1)}_{(s,x)}$ are identical on $M^s_{T^-}$. Since T is > 0, the fine topologies of $\tilde{Q}^{(b,\tilde{b})}$ and $\tilde{Q}^{(b_1,\tilde{b}_1)}$ are the same. Similarly, $\tilde{P}$ and $\tilde{Q}^{(b_0,\tilde{b}_0)}$ define the same fine topology. The theorem is now completely proved.

3. Weak dependences

We now give one word of the proofs of weak continuous dependences which are needed to develop a good control theory. The results we give are not the most general that it is possible to obtain, but they are sufficient for handling intricate control problems.

Let λ be a reference probability measure on $R^+\times R^d$ for $\tilde{P} = \tilde{Q}^{(0,0)}$. $L^\lambda_\infty(R^+\times R^d)$ will be the space $L_\infty(R^+\times R^d)$ where $R^+\times R^d$ has the measure $d\lambda$.

We then have the following result.

PROPOSITION V.1. — *Proposition* I.1 *holds for* $\tilde{P}_{(s,x)}$.

Proof. — Same as Proposition I.1.

We now study the weak dependence of $Z_t^{(b,\tilde{b})}$ on $(b,\tilde{b})$.

First, we note that by the continuity of (V.1), all the measures $M(s, x, dy)$ are absolutely continuous relative to a fixed probability measure μ on R^d. We can then define $L_\infty^{\lambda\otimes\mu}(R^+ \times R^d \times R^d)$ to be the space $L_\infty(R^+ \times R^d \times R^d)$, for the measure $d\lambda \otimes d\mu$. For

$$(b, \tilde{b}) \in L_\infty^\lambda (R^+ \times R^d) \times L_\infty^{\lambda\otimes\mu}(R^+ \times R^d \times R^d),$$

it is then possible to define $Q^{(b, \tilde{b})}$ unambiguously, as any of the $Q^{(b', \tilde{b}')}$, with $(b', \tilde{b}')$ Borel and $(b', \tilde{b}') \in (b, \tilde{b})$, because under the assumptions of Theorem V.2, $S^{c(0,0)} \tilde{b}'$ depends only on $\tilde{b}$; by (V.9) $Q^{(b, \tilde{b})}$ is then well defined.

To avoid unnecessary complications, we make the following assumption on $\tilde{b}$: there is a function $\varphi \geqslant 0$ on R^d with:

$$\text{on } |y| \leqslant 1, \qquad |\varphi(y)| \leqslant |y|,$$
$$\text{on } |y| \geqslant 1, \qquad |\varphi(y)| = 1,$$

such that

$$\tilde{b}(t, x, y) = \tilde{b}'(t, x, y)\varphi(y), \tag{V.24}$$

where $\tilde{b}'$ is a bounded Borel function on $R^+ \times R^d \times R^d$, with values in $(-1, +\infty($. We then have the following theorem.

THEOREM V.3. — *If $(b_n, \tilde{b}'_n)$ is a sequence of elements of*

$$L_\infty^\lambda (R^+ \times R^d) \times L_\infty^{\lambda\otimes\mu}(R^+ \times R^d \times R^d)$$

such that $b'_n \geqslant -1$, if $(b_n, \tilde{b}'_n) \to (b, \tilde{b}')$ weakly and, if $\tilde{b}_n = \tilde{b}'_n \varphi$, then the density Z_t^n of $\tilde{Q}_{(s,x)}^{(b_n, \tilde{b}_n)}$ relative to $\tilde{P}_{(s,x)}$ on M_t^s converges to the density Z_t of $\tilde{Q}_{(s,x)}^{(b, \tilde{b})}$ in the topology $\sigma(L_1(\Omega), L_\infty(\Omega))$ when Ω is endowed with measure $\tilde{P}_{(s,x)}$.

Proof. — Theorem V.2 proves that the $E^{\tilde{P}(s,x)} |Z_t^n|^2$ stay bounded in L_2. The proof proceeds as in Theorem I.2, using definition V.2.

Remark V.1. — By using Theorem A1 in [22] and proceeding as in [1], Theorems V.1, V.2 and V.3 may be proved even if a is elliptic (i. e. non uniformly).

4. Martingales

In [1], an important step of the proof of existence of for an optimal control is the representation of square integrable additive martingales.

We will assume that $\tilde{b}/|y|$ is a bounded function. $Q^{(b, \tilde{b})}$ is then Hunt.

Let $\beta^{(b, \tilde{b})}$ be the Brownian motion associated to $Q^{(b, \tilde{b})}$ as in Corollary (1.3.4) of [22].

We then have the following result.

THEOREM V.4. — *Any square integrable additive functional martingale for $Q^{(b, \tilde{b})}$ may be written as*

$$\text{(V.25)} \qquad \int_s^t H(u, x_u)\, d\beta_u^{(b, \tilde{b})} + S^{c\,(b, \tilde{b})} g_t^s$$

where:

- *H is a Borel function such that*

$$\text{(V.26)} \qquad E_{(s, x)}^{(b, \tilde{b})} \int_s^t |H(u, x_u)|^2\, du < +\infty;$$

- *g is a Borel function on $R^+ \times R^d \times R^d$, such that*

$$\text{(V.27)} \qquad E_{(s, x)}^{(b, \tilde{b})} \int_s^t du \int_{R^d} M(u, x_u, dy) \times (1 + \tilde{b}(u, x_u, y))\, |g(x_u, x_u + y)|^2 < +\infty;$$

- *$S^{c(b, \tilde{b})} g$ is the square integrable additive functional martingale associated to g by* [17] (p. 154).

Proof. — This result has been proved in a more general case by JACOD [12]. We give here a short proof based on the Markov property of the considered process under consideration inspired by an argument of DELLACHERIE in [8].

Let us assume that M is a continuous martingale orthogonal to any martingale (V.25). By stopping M conveniently, we may assume $|M| \leqslant a < +\infty$. Let Q' be the measure

$$\text{(V.28)} \qquad dQ' = \left(1 + \frac{M}{2a}\right) dQ_{(s, x)}^{(b, \tilde{b})}.$$

Then, we know by (V.10) applied to $Q^{(b, \tilde{b})}$ that, if $f \in C_0^\infty(R^d)$,

$$\text{(V.29)} \qquad f(x_t) - f(x_s) - \int_s^t L'_u f(x_u)$$

can be written as (V.25). It will then be a martingale for Q'. Because of the uniqueness of the solution of the martingale problem, $Q' = Q_{(s, x)}^{(b, \tilde{b})}$ and $M = 0$.

By Theorem 5 of [17] (p. 156), any square integrable additive functional martingale which is a compensated sum of jumps may be written as $S^{c(b, \tilde{b})} g$, with g satisfying (V.27).

By decomposing a square integrable additive functional martingale into the sum of its continuous part and of its jump part, Theorem V.4 follows.

Remark V.2. — The interested reader can compare this proof with Annexes 1 and 3 of [1].

5. Representation of processes and measure transformations

An important step in [1] for the proof of existence results for control problems is to represent a square integrable additive functional martingale for a given $Q^{(b, \tilde{b})}$ as a local martingale on $Q_\mu^{(b', \tilde{b}')}$, where μ is a probability measure on $R^+ \times R^d$.

THEOREM V.5. — *Let N be a square integrable additive functional martingale for* $Q^{(b, \tilde{b})}$ *written as* (V.25). *Then if* $1+\tilde{b}' \leqslant k\,(1+\tilde{b})$, N_t *may be represented* $Q^{(b', \tilde{b}')}$ a. e. *as*

$$
\text{(V.30)} \qquad \begin{aligned} N_t - N_s = & \int_s^t H(u, x_u) . d\beta_u^{(b', \tilde{b}')} \\ & + \int_s^t \langle H(u, x_u), \sigma^{-1}(b'-b)(u, x_u) \rangle \, du \\ & - \int_s^t \left\langle H(u, x_u), \sigma^{-1} M\left(\frac{\tilde{b}' - \tilde{b})}{1+|y|^2} y\right)(u, x_u)\, du \right\rangle \\ & + S^{c\,(b', \tilde{b}')} g_t^s + \int_s^t M(\tilde{b}' - \tilde{b})\, g(x_u, x_u + .)\, du, \end{aligned}
$$

where $\int_s^t H(u, x_u) . d\,\beta_u^{(b', \tilde{b}')}$ *and* $S^{c(b', \tilde{b}')} g$ *are in* M_{loc}^2 (3) *for* $Q_\mu^{(b', \tilde{b}')}$.

Proof. — For $Q^{(b, \tilde{b})}$, we have

$$
\text{(V.31)} \qquad \begin{aligned} x_t - x_s = & \int_s^t b(u, x_u)\, du + \int_s^t \sigma(u, x_u) . d\beta_u^{(b, \tilde{b})} \\ & + \{ S^{c\,(b, \tilde{b})} 1_{|y-x|<\delta} (y-x) + S\, 1_{|y-x|\geqslant\delta} (y-x) \}_t^s \\ & + \int_s^t \frac{M(1+\tilde{b})}{1+|y|^2} (1_{|y|<\delta}\, y\, |y|^2 - 1_{|y|\geqslant\delta}\, y)(u, x_u)\, du \end{aligned}
$$

---

(3) M_{loc}^2 is the space of local martingales such that there is an increasing sequence of stopping times $T_n \to +\infty$ such that $M_{t \wedge T_n}$ are square-integrable martingales.

and a similar relation holds for $Q^{(b',\tilde{b})}$. Then, we have

$$(V.32)\quad S^{c\,(b,\tilde{b})}(1_{|y-x|<\delta}(y-x))_t^s = S^{c\,(b',\tilde{b}')}(1_{|y-x|<\delta}(y-x))_t^s + \int_s^t M((\tilde{b}'-\tilde{b})\,1_{|y|<\delta}\,y)(u, x_u)\,du.$$

To prove (V.32), it is sufficient to apply Corollary 1.3.1 of [22] by approximating $1_{|y|<\delta}\,y$ by $1_{\varepsilon \leqslant |y| < \delta}\,y$ when $\varepsilon \to 0$, and using the fact that $|\tilde{b}'-\tilde{b}| \leqslant k\,|y|$. Then, necessarily,

$$(V.33)\quad \int_s^t \left(b - \frac{M\,\tilde{b}y}{1+|y|^2}\right)(u, x_u)\,du + \int_s^t \sigma(u, x_u).\,d\beta_u^{(b,b)}$$

equals the corresponding expression for $(b', \tilde{b}')$. The relation between $\beta^{(b,\tilde{b})}$ and $\beta^{(b',\tilde{b}')}$ is then found and the first part of (V.30) is justified.

We now justify the second part. Let T_n be the stopping time

$$(V.34)\quad T_n = \inf\left\{ t > s;\ \int_s^t M\,|g|^2(1+\tilde{b})(u, x_u)\,du \geqslant n \right\}.$$

Then for $Q_\mu^{(b',\tilde{b}')}$, $S^{c(b',\tilde{b}')}\,(1_{s \leqslant u \leqslant T_n}\,g)_t^s$ is a square integrable martingale and (V.30) will hold until time T_n. In particular, the last term of (V.30) is finite because:

$$(V.35)\quad \int_s^t \{M(\tilde{b}'-\tilde{b})\,g\}(u, x_u)\,du \leqslant \left(\int_s^t M\frac{(\tilde{b}'-\tilde{b})^2}{(1+\tilde{b})}\,du\right)^{1/2} \left(\int_s M\,|g|^2\,(1+\tilde{b})\,du\right)^{1/2}$$

Because $Q_{(s,x)}^{(b',\tilde{b}')}$ is absolutely continuous relative to $Q_{(s,x)}^{(b,\tilde{b})}$, when $n \to +\infty$, $T_n \to +\infty$ $Q_{(s,x)}^{(b',\tilde{b}')}$ a. s. The theorem is proved.

6. Representation of potentials

The measures $\tilde{Q}_{(s,x)}^{(b,\tilde{b})}$ not being necessarily equivalent on M_t^s, we will have the same difficulty representing the potentials of one of the processes relative to another process as in Section **II**.

We give however a simple and straightforward result which will allow us to do the same manipulations as in Section II.

PROPOSITION V.2. — *If $\tilde{b}/|y|^2$ is bounded, if L is a bounded Borel function, and if p is a strictly positive constant, if function V is defined by*

$$(V.36)\quad V(s, x) = e^{ps}\,E^{\tilde{Q}_{(s,x)}^{(b,\tilde{b})}} \int_s^{+\infty} e^{-pt}\,L(t, x_t)\,dt$$

we then have

(V.37) $$V(s, x) = e^{ps} E^{\tilde{Q}^{(b',0)}} \int_s^{+\infty} e^{-pt} L'(t, x_t)\, dt$$

with

(V.38) $$b' = b - \frac{M\tilde{b}y}{1+|y|^2}.$$

(V.39) $$L'(t,x) = L(t, x) + \int_{R^d} M(t, x, dy)\tilde{b}(t, x, y)(V(t, x+y) - V(t, x)).$$

Proof. — Let ε be chosen as in the proof of Theorem V.2. We define then

(V.40) $$\begin{cases} b_0 = b - \dfrac{M 1_{|y|\geq\varepsilon} y(1+\tilde{b})}{1+|y|^2}, \\ \tilde{b}_0 = 1_{|y|<\varepsilon}(1+\tilde{b}) - 1, \\ b_1 = b - M\dfrac{\tilde{b}y}{1+|y|^2} - \dfrac{M 1_{|y|\geq\varepsilon} y}{1+|y|^2}, \\ \tilde{b}_1 = -1_{|y|\geq\varepsilon}. \end{cases}$$

Then, by (V.21), we have

(V.41) $$V_{(s,x)} = e^{ps} E^{\tilde{Q}^{(b_0,\tilde{b}_0)}_{(s,x)}} \int_s^{+\infty} \exp - \left\{ pt + \int_s^t (M(1+\tilde{b}) 1_{|y|\geq\varepsilon}) \right\} \times (L + M(1+\tilde{b}) 1_{|y|\geq\varepsilon} V(.,. + y))(t, x_t)\, dt.$$

This implies that $\tilde{Q}^{(b_0,\tilde{b}_0)}$ a. s.

(V.42) $$\begin{aligned} V(t, x_t) - V(s, x_s) = &- \int_s^t (L + M(1+\tilde{b}) 1_{|y|\geq\varepsilon} V(.,. + y) \\ &- VM(1+\tilde{b}) 1_{|y|\geq\varepsilon} - pV)(u, x_u)\, du \\ &+ \int_s^t H(u, x_u) . d\beta_u^{(b_0,\tilde{b}_0)} + S^{c\,(b_0,\tilde{b}_0)}\, V. \end{aligned}$$

But now $(1+\tilde{b}_1) \leq 2\,(1+\tilde{b}_0)$. Therefore we can apply Theorem V.5. We see then that $\tilde{Q}^{(b_1,\tilde{b}_1)}$ a. s.

(V.43) $$\begin{aligned} V(t, x_t) &- V(s, x_s) \\ = &- \int_s^t (L + M(1+\tilde{b}) 1_{|y|\geq\varepsilon} V(.,. + y) \\ &- VM(1+\tilde{b})1_{|y|\geq\varepsilon} - pV + M\tilde{b}1_{|y|<\varepsilon} V(.,. + y) \\ &- VM\tilde{b}1_{|y|<\varepsilon})(u, x_u) du \\ &+ \int_s^t H(u, x_u)\, d\beta^{(b_1,\tilde{b}_1)} + S^{c\,(b_1,\tilde{b}_1)}\, V \end{aligned}$$

or equivalently

$$\text{(V.44)}\qquad \begin{aligned} &V(t, x_t) - V(s, x_s) \\ &= -\int_s^t (L + M\tilde{b}\, V(.,. + y) - VM\tilde{b} - p\, V \\ &\qquad + M\, 1_{|y| \geqslant \varepsilon}\, V(.,. + y) - VM\, 1_{|y| \geqslant \varepsilon})(u, x_u) \\ &\qquad + \int_s^t H(u, x_u)\, d\beta^{(b_1, \tilde{b}_1)} + S^{c\,(b_1, b_1)}\, V. \end{aligned}$$

Let T_n be the stopping time

$$\text{(V.45)}\qquad \begin{aligned} T_n = \inf\left\{ t \geqslant s;\ \int_s^t |H|^2(u, x_u)\, du \geqslant n \right\} \\ \wedge \inf\left\{ t \geqslant s;\ \int_s^t M(1 + \tilde{b}_1)(V(u, x_u + .) - V(u, x_u))^2\, du \geqslant n \right\}. \end{aligned}$$

We see then that

$$\text{(V.46)}\qquad \begin{aligned} &e^{-ps}\, V(s, x_s) \\ &= E^{\tilde{Q}^{(b_1, \tilde{b}_1)}_{(s,x)}} \exp\left\{ -\left(p\, T_n + \int_s^{T_n} M\, 1_{|y| \geqslant \varepsilon}\, du \right) \right\} \\ &\times V(T_n, x_{T_n}) + E^{\tilde{Q}^{(b_1, \tilde{b}_1)}_{(s,x)}} \\ &\times \int_s^{T_n} \exp\left\{ -\left(pu + \int_s^u M\, 1_{|y| \geqslant \varepsilon}\, d\sigma \right) \right\} \\ &\times (L' + M\, 1_{|y| \geqslant \varepsilon}\, V(.,. + y))(u, x_u)\, du \end{aligned}$$

$\tilde{b}/|y|^2$ being bounded, L' is a bounded function. We may pass to the limit in (V.46), and write

$$\text{(V.47)}\qquad \begin{aligned} V(s, x) = e^{ps}\, E^{Q^{(b_1, \tilde{b}_1)}_{(s,x)}} \int_s^{+\infty} \exp\left\{ -\left(pu + \int_s^u M\, 1_{|y| \geqslant \varepsilon}\, d\sigma \right) \right\} \\ \times (L' + M\, 1_{|y| \geqslant \varepsilon}\, V(.,. + y))(u, x_u)\, du. \end{aligned}$$

But by (V.21), the function defined in (V.37) verifies the same identify. The equality of (V.37) and (V.47) follows then from the uniqueness of the fixed point in the transformation associated to the right hand member of (V.47): this uniqueness is obtained by the same argument as in (V.23).

7. Control problems

We now define a control problem for jumping diffusions, corresponding to problem ($\mathscr{P}$) of Section II. Another problem ($\mathscr{P}'$) can be defined similarly, and the existence of an optimal control can also be derived.

K is a Borel set valued mapping defined on $R^+ \times R^d$ with values in $R^d \times R \times (-1, +\infty($, with non empty, compact, uniformly bounded values.

Let $\mathscr{L}$ be the set of classes for the reference measure λ of Borel selections of K.

Let φ now be a function of R^d, such that $|\varphi(y)| \leqslant |y|^2$ on $|y| \leqslant 1$ and $\varphi(y) = 1$ on $|y| > 1$.

For $c = (b, L, \tilde{b}) \in \mathscr{L}$, we define V_c by

$$V_c(s, x) = e^{ps} E^{\tilde{Q}^{(b, \varphi\tilde{b})}_{(s, x)}} \int_s^{+\infty} e^{-pt} L(t, x_t)\, dt. \tag{V.48}$$

DEFINITION V.3. — Problem (P) is the search for $c \in \mathscr{L}$ such that for any $c' \in \mathscr{L}$:

$$V_c \leqslant V'_{c'}. \tag{V.49}$$

THEOREM V.6. — *Problem* (P) *has a solution.*

Proof. — We will not give full details of the proof, which is closely related to the proof of [1] for the control of diffusions and to Part II for the control of jumps. However, we do have the same basic elements as in [1] to prove the existence result:

- the processes are all strongly Feller;
- there is a common reference probability measure λ, by Theorem V.2;
- the fine topology is the same for all the processes $Q^{(b, \varphi\tilde{b})}$;
- V_c depends continuously on c, by Proposition V.1 and Theorem V.3;
- by (V.37), we have

$$V_c(s, x) = e^{ps} E^{\tilde{Q}^{(b', 0)}_{(s, x)}} \int_s^{+\infty} e^{-pt} L'(u, x_u)\, du. \tag{V.50}$$

It is then possible to write on $(\Omega, Q^{(b', 0)}_{(s, x)})$.

$$\begin{aligned} V_c(t, x_t) - V_c(s, x_s) \\ = -\int_s^t (L' - p V_c)(u, x_u)\, du + \int_s^t H_c(u, x_u)\, d\beta_u^{(b', 0)} + S^{c\,(b', 0)} g_c, \end{aligned} \tag{V.51}$$

where H_c and g_c are such that

$$E^{\tilde{Q}^{(b', 0)}_{(s, x)}} \int_s^t |H(u, x_u)|^2\, du < +\infty, \tag{V.52}$$

$$E^{\tilde{Q}^{(b', 0)}_{(s, x)}} \int_s^t M |g_c|^2 (u, x_u)\, du < +\infty. \tag{V.53}$$

In fact, we have

$$(V.54)\qquad g_c(t, x, y) = V_c(t, y) - V_c(t, x),$$

and g_c is a bounded function.

But now all the measures $\tilde{Q}^{(b^*, \varphi b^*)}$ are absolutely continued relative to $\tilde{Q}^{(b', 0)}_{(s, x)}$ on each M^s_t, and we are in the situation described in Theorem V.2, which allows us to represent (V.48) on any $\tilde{Q}^{(b^*, \varphi \tilde{b}^*)}_{(s, x)}$;

- when K has convex values, necessary and sufficient conditions are derived in the following way: for c to minimize

$$c' \to I_\lambda(c') = \int V_{c'}(s, x)\, e^{-ps}\, d\lambda(s, x)$$

it is necessary and sufficient that if $V_c(t, x_t)$ is represented by (V.51) then λ a. e.:

$$(V.55)\quad \left(L + \left\langle H_c, \sigma^{-1}\left(b - \tilde{b}\frac{M\varphi y}{1+|y|^2}\right)\right\rangle + \tilde{b} M\varphi g_c\right)(t, x)$$
$$= \inf_{(b', L', \tilde{b}') \in K(t, x)} \left(L' + \left\langle H_c, \sigma^{-1}\left(b' - \tilde{b}'\frac{M\varphi y}{1+|y|^2}\right)\right\rangle + \tilde{b}' M\varphi g_c\right)(t, x);$$

- (V.55) will imply that when c minimizes $I_{e^{-ps}\lambda}$, for any $c' \in \mathscr{L}$, $V_c \leqslant V_{c'}$;
- in the non-convex case, (V.55) is used to prove the existence result.

8. Extensions

All extensions to games, approximations, etc., are possible. The methods are the same as in [1] and in Parts III and IV.

In particular, if, instead of considering a criterion from time s to infinity, we stop the process at the hitting time T_A of a Borel set A of $R^+ \times R^d$, equality (5.29) in [1] is no longer true because there are inaccessible stopping times. In this case, it may easily be proved that in [1] (5.29), $\int_s^{t \wedge T_A} dA$ is the dual predictable projection of $1_{t \geqslant T_A} V'_c(T_A, x_{T_A})$ and is equal to

$$\int_s^{t \wedge T_A} (M 1_{. + y \in A}\, V'_c(., . + y))(u, x_u)\, du$$

The proof continues then as in [1], V.

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