K. Ramachandra

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TWO REMARKS IN PRIME NUMBER THEORY

BY

K. RAMACHANDRA

[Bombay]

RESUME. — Nous considérons deux problèmes de théorie additive des nombres premiers. Le premier concerne une inégalité dont un cas particulier est le suivant : min |p−qα| (quand α est un nombre réel fixe plus grand que 1; p et q sont des nombres premiers, et le minimum est pris sur l’ensemble des entiers positifs inférieurs ou égaux à 4/ε, où ε est un nombre fixe avec 0 < ε < 1/2 est inférieur à p^ε pour une infinité de couples (p, q). Le second résultat montre que si θ > 7/72, l’intervalle X, X+X^θ contient au moins X^θ nombres distincts qui sont sommes de deux nombres premiers impairs.

SUMMARY. — Two questions in additive prime number theory are considered. First is an inequality of which a special case is this min |p−qα| (where α is a fixed real number exceeding 1; p, q are primes, and the minimum is over all positive integers a not exceeding 4/ε [0 < ε < 1/2, ε fixed]) is less than p^θ for an infinity of prime pairs (p, q). The second result is that if θ > 7/72 then the interval X, X+X^θ containing at least X^θ distinct numbers which are expressible as a sum of two odd primes.

1. Introduction

In this note, we consider two questions of an additive nature on prime numbers. Our results are as follows.

THEOREM 1. — Let ε be a positive constant less than 1, and let N be any natural number exceeding 2 ε^−1. Let α_1, . . . , α_N be any given positive real numbers no two of which are equal. Then there exist two of the numbers α_j say β and γ such that the inequality

|β p−γ q| < p^ε,

where p and q are required to be prime numbers has infinity of solutions in p, q.

By choosing α_1, . . . , α_N appropriately, we get the following corollary.

COROLLARY. — Let α be a real number exceeding 1 and ε as before. Then there exists a natural number a satisfying 1 ≤ a < 1+[2 ε^−1], such that the inequality

|p−qα^a| < p^ε,
where $p$ and $q$ are required to be prime numbers has infinity of solutions in $p$, $q$.

In other words, the inequality

$$\min_{a=1, 2, \ldots, [2^e - 1]} |\alpha - \left(\frac{p}{q}\right)^{1/a}| < p^{-1+\varepsilon}$$

has infinity of solutions in prime pairs $p$, $q$.

**THEOREM 2.** Let $\theta$ be a constant exceeding $7/72$, $X > X_0 (\theta)$ and $h = X^\theta$. Let $G$ denote the number of Goldbach numbers (a natural number $n$ is said to be Goldbach if there exist two odd prime numbers whose sum is $n$) in the interval $[X, X+h]$. Then $G$ exceeds $ch$ where $c$ is an absolute positive constant.

The weaker version $G \geq 1$ of theorem 2 is due to H. L. Montgomery and R. C. Vaughan and was communicated to me by Professor H. L. Montgomery in a letter to me a few years back, and is now published [2]. I am very much indebted to Professor H. L. Montgomery both for his letter and for his preprint.

### 2. Proof of theorem 1

Let $H = X^\varepsilon$ and for $X \leq x \leq 2X$ and any positive constant $\alpha$, put

$$f(\alpha, x) = \theta \left(\frac{x + [\alpha H]}{\alpha}\right) - \theta \left(\frac{x}{\alpha}\right),$$

where, for positive real $u$, we have written $\theta(u) = \sum_{p \leq u} \log p$. A simple application of the prime number theorem shows that

$$\sum_{x \leq n \leq 2x} f(\alpha, n) = HX (1+o(1))$$

and so

$$\sum_{x \leq n \leq 2x} \sum_{j=1}^{N} f(\alpha_j, n) = NXH (1+o(1)).$$

From this equality it follows that, for some integer $n$ satisfying $X \leq n \leq 2X$,

$$\sum_{j=1}^{N} f(\alpha_j, n) \geq NH (1+o(1))$$

(it may be remarked that here actually equality holds for some $n$). In view of the inequality (note that $N > 2e^{-1}$),

$$\pi(x) - \pi(x-y) \leq \frac{2y}{\log y} \left(1 + \frac{8}{\log y}\right)$$

TOME 105 — 1977 — N° 4
(this is a consequence of Selberg sieve; the first result in this direction is due to G. H. Hardy and J. E. Littlewood who obtained a bigger constant in place of 2 by the use of the sieve method of V. Brun) valid for all \( x, y \) satisfying \( 1 < y \leq x \) (see page 107 of [1]), it follows that there exist \( k_1, k_2 \) \((k_1 \neq k_2)\) for which \( f(\alpha_{k_1}, n) \neq 0 \) and \( f(\alpha_{k_2}, n) \neq 0 \). From these, it follows that there exist primes \( p, q \) satisfying

\[
\frac{n}{\alpha_{k_1}} < p \leq \frac{n + \lceil \alpha_{k_1} H \rceil}{\alpha_{k_1}}, \quad \frac{n}{\alpha_{k_2}} < q \leq \frac{n + \lceil \alpha_{k_2} H \rceil}{\alpha_{k_2}}
\]

(note that \( X \leq n \leq 2X \)) and so

\[
\left| p \alpha_{k_1} - q \alpha_{k_2} \right| \leq (\alpha_{k_1} + \alpha_{k_2}) H.
\]

This is true for all \( X \) and, in particular, for \( X = 2^M \) \((M = 1, 2, 3, \ldots)\). The pair \((k_1, k_2)\) depends on \( M \), but there are only finitely many \((\leq N^2)\) pairs and so there exists some pair say \((1, 2)\) for simplicity, which is the same for an infinite subsequence of integers \( M \). This proves Theorem 1.

3. Proof of Theorem 2

For integral \( h, x, \) and \( Y \) with \( h \leq x^2 \) and \( x^2 \leq Y \leq x/3 \), consider the sum

\[
S = \sum_{y = 1}^{2Y} (\theta(x + h - y) - \theta(x - y)) (\theta(y+h) - \theta(y)) = \sum_{y = 1}^{2Y} (\theta(x + h - y) - \theta(x - y)) (\theta(y+h) - \theta(y) - h + h),
\]

\[
= h \sum_{y = 1}^{2Y} (\theta(x + h - y) - \theta(x - y)) + O(\max_{y \leq y \leq 2Y} (\theta(x + h - y) - \theta(x - y)) Y^{1/2} \times (\sum_{Y \leq y \leq 2Y} (\theta(y+h) - \theta(y) - h)^2)^{1/2}).
\]

The O-term is easily proved to be \( O(h^2 Y \exp(- (\log x) 1/6)) \) provided \( h \geq Y^{(7/12) + \varepsilon} \) (these results are due to A. Selberg and M. N. Huxley, see [3]). The main term is easily seen to be

\[
h \sum_{x - 2Y < n < x - 2Y + h - 1} (\theta(n + Y) - \theta(n)),
\]

which is asymptotic to \( h^2 Y \) provided that \( Y \geq x (7/12) + \varepsilon \) (these results are due to A. E. Ingham and M. N. Huxley, see [3]). Thus we have following result.
LEMMA 1. — If $h, x, Y$ are integers with $Y \geq h \geq Y^{1/6+\varepsilon} Y \geq x^{7/12+\varepsilon}$ and $Y \leq x/3$, then
\[ \sum_{p}^{2Y} (9(x+h-y)-9(x-y))(9(y+h)-9(y)) = h^2 Y(1+o(1)). \]

Next we record the following lemma.

LEMMA 2. — There exists, under the conditions of lemma 1, an integer $y_0$ satisfying $Y \leq y_0 \leq 2Y$ such that
\[ (9(x+h-y_0)-9(x-y_0))(9(y_0+h)-9(y_0)) \geq h^2 (1+o(1)) \]
(actually equality may be secured for a suitable $y_0$).

Proof. — Follows from lemma 1.

LEMMA 3. — Let $r(n)$ denote the number of solutions of the equation $n=p_1+p_2$ where $p_1$ and $p_2$ are prime numbers satisfying $x-y_0 \leq p_1 \leq x+h-y_0$ and $y_0 \leq p_2 \leq y_0+h$. Then
\[ (\log x)^2 \sum_{n=x}^{x+2h} r(n) \geq h^2 (1+o(1)). \]

Proof. — Follows from lemma 2.

LEMMA 4. — We have
\[ r(n) \leq 16 \prod_{p \geq 2} \left( 1 - \frac{1}{(p-1)^2} \right) \times \prod_{2 < p \mid n} \left( 1 + \frac{1}{p-2} \right) \frac{h}{(\log h)^2} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right) \]
where the constant implied by the $O$-symbol is absolute.

Proof. — This is corollary 5.8.3 on page 179 of [1].

LEMMA 5. — We have
\[ \sum_{n=x}^{x+2h} \prod_{2 < p \mid n} \left( 1 + \frac{1}{p-2} \right)^2 \leq 2h (1+o(1)) \prod_{p \geq 2} \left( 1 + \frac{2p-3}{p(p-2)^2} \right). \]

Proof. — In the product on the left, the contributions from the primes $p > \log x$ are negligible, and so we may restrict to those $p \leq \log x$.

Accordingly the left side is
\[ < 2h + 1 + \sum_{x \leq p_1 < \ldots < p_2 \leq x+2h} \prod_{j=1}^{l} (2(p_j-2)^{-1} + (p_j-2)^{-2}), \]
with $2 < p_1 < p_2 < \ldots < p_r \leq \log x$,

$$< 2h + \sum_{r=1}^{r} \frac{2h}{p_1 \cdots p_r} \prod_{j=1}^{r} (2(p_j - 2)^{-1} + (p_j - 2)^2)$$

with $2 < p_1 < p_2 < \ldots < p_r \leq \log x$, and this proves lemma 5.

We now fix $h = \lceil Y^{(1/6)+\varepsilon} \rceil$, $Y = \lceil x^{(7/12)+\varepsilon} \rceil$ and apply Hölder's inequality to the inequality of lemma 3 and use lemmas 4 and 5. We see that theorem 2 is proved with any positive constant $c$ satisfying

$$1/2 > 16 \left( \frac{72}{5} \right)^2 c^{1/2} \prod_{p > 2} \left( 1 - \frac{2p - 3}{p(p-2)^2} \right).$$

REFERENCES


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K. RAMACHANDRA,
School of Mathematics,
Tata Institute of Fundamental Research,
Homi Bhabha Road,
Bombay 400 005 (India).