WOLDZIMIERZ M. TULCZYJEW

The Lagrange complex


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RÉSUMÉ. — Nous définissons le complexe de co-chaînes (A, δ), et nous prouvons le lemme de Poincaré pour l’opérateur δ. L’opérateur δ est utilisé dans le calcul des variations en vue de déduire les équations d’Euler-Lagrange. Le lemme de Poincaré fournit alors le critère suivant lequel un système d’équations est un système d’Euler-Lagrange.

ABSTRACT. — A cochain complex (A, δ) is defined, and the δ-Poincaré lemma is proved. The work is motivated by applications to the calculus of variations. The operator δ is used in the calculus of variations to construct the Euler-Lagrange equations, and the δ-Poincaré lemma provides criteria for partial differential equations to be Euler-Lagrange equations.

The present paper generalizes results contained in earlier publications ([6], [8]) which were applicable to ordinary differential equations of the Euler-Poisson type.

1. Jets and tangent vectors

Let \( M \) be a \( C^\infty \)-manifold. We denote by \( T^{(k)} M \) the manifold \( J^k_0 (\mathbb{R}^p, M) \) of jets of order \( k \) from \( \mathbb{R}^p \) to \( M \) with source 0 called by EHRESMANN [1] \( p^k \)-vitesse in \( M \). Elements of \( T^{(k)} M \) are equivalence classes of smooth mappings of \( \mathbb{R}^p \) into \( M \). Two mappings \( \gamma \) and \( \gamma' \) are equivalent if \( D^n (f \circ \gamma) (0) = D^n (f \circ \gamma') (0) \) for each \( C^\infty \)-function \( f \) on \( M \) and each \( n = (n_1, \ldots, n_p) \in \mathbb{N}^p \) such that \(|n| = n_1 + \ldots + n_p \leq k\). The symbol \( D^n g (0) \) is used to denote the partial derivative of a function \( g \):

\[
R^p \to \mathbb{R} : (t_1, \ldots, t_p) \mapsto g(t_1, \ldots, t_p)
\]

of orders \( n_1, \ldots, n_p \) with respect to the arguments \( t_1, \ldots, t_p \) respectively at \((t_1, \ldots, t_p) = (0, \ldots, 0)\). We denote by \( j^k_0 (\gamma) \) the jet of the mapping \( \gamma \).

For each \( k \in \mathbb{N} \), there is the projection

\[
\tau^{(k)} : T^{(k)} M \to M : j^k_0 (\gamma) \mapsto \gamma(0)
\]
and, if $k' \leq k$, then there is the projection

$$\rho_{(k')}^{(k)} : T^{(k)} M \to T^{(k')} M : j^{k'}_0 (\gamma) \mapsto j^k_0 (\gamma).$$

The manifold $T^{(0)} M$ is identified with $M$, and $T^{(1)} M$ is the tangent bundle $TM$ of $M$. For each $n \in \mathbb{N}^p$ such that $|n| \leq k$ and each $C^\infty$-function $f$ on $M$ there is a $C^\infty$-function $f_n$ defined on $T^{(k)} M$ by $f_n (j^k_0 (\gamma)) = D^n (f \circ \gamma) (0)$.

For each $k \in \mathbb{N}$, we introduce an equivalence relation in the set of smooth mappings of $\mathbb{R}^{p+1}$ into $M$. Two mappings $\chi$ and $\chi'$ will be considered equivalent if $D^{(r,n)} (f \circ \chi) (0) = D^{(r,n)} (f \circ \chi') (0)$ for each $C^\infty$-function $f$ on $M$, each $n \in \mathbb{N}^p$ such that $|n| \leq k$ and $r = 0, 1$. The symbol $D^{(r,n)} g (0)$ denotes the partial derivative of a function $g$:

$$\mathbb{R}^{p+1} \to \mathbb{R} : (s, t_1, \ldots, t_p) \mapsto g(s, t_1, \ldots, t_p)$$

defined on $T^{(k)} M$ in such a way that

$$\langle j^{(1,k)}_0 (\chi), df_n \rangle = D^{(1,n)} (f \circ \chi) (0)$$

for each function $f$ on $M$ and each $n \in \mathbb{N}^p$ such that $|n| \leq k$ and also

$$\chi_0 : \mathbb{R}^p \to M : (t_1, \ldots, t_p) \mapsto \chi(0, t_1, \ldots, t_p) \quad [7].$$

The tangent mapping $T\rho_{(k')}^{(k)} : TT^{(k)} M \to TT^{(k')} M$ is given by

$$T\rho_{(k')}^{(k)} (j^{(1,k)}_0 (\chi)) = j^{(1,k')}_0 (\chi).$$

For each $k \in \mathbb{N}$ and each $m \in \mathbb{N}^p$ there is the mapping

$$F_m : TT^{(k)} M \to TT^{(k+m)} M : j^{(1,k)}_0 (\chi) \mapsto j^{(1,k+m)}_0 (\chi_m),$$

where $\chi_m$ is the mapping

$$\chi_m : \mathbb{R}^{p+1} \to M : (s, t_1, \ldots, t_p) \mapsto \chi(st^m, t_1, \ldots, t_p),$$

and $t^m = t_1^{m_1} \ldots t_p^{m_p}$. Diagrams

$$\begin{align*}
TT^{(k)} M &\xrightarrow{F_m} TT^{(k+m)} M \\
T^{(k)} M &\xrightarrow{T\rho_{(k')}^{(k)}} T^{(k')} M
\end{align*}$$

TOME 105 — 1977 — N° 4
and
\[ TT^{(k)} M \xrightarrow{F_m} TT^{(k)} M \]
\[ T p(k') (k) \downarrow \quad T p(k') (k) \]
\[ TT^{(k')} M \xrightarrow{F_m} TT^{(k')} M \]
are commutative.

For each \( \alpha = 1, \ldots, p \) and each \( k \in \mathbb{N} \), there is the mapping
\[ T^\alpha : \quad T^{(k+1)} M \rightarrow TT^{(k)} M : \quad j_0^{k+1} (\gamma) \mapsto j_0^{(1,k)} (\gamma^\alpha), \]
where \( \gamma^\alpha \) is the mapping
\[ \gamma^\alpha : \quad \mathbb{R}^{p+1} \rightarrow M : \quad (s, t_1, \ldots, t_p) \mapsto \gamma(t_1, \ldots, t_\alpha + s, \ldots, t_p) \quad (1). \]

Diagrams
\[ T^{(k+1)} M \xrightarrow{T^\alpha} TT^{(k)} M \]
\[ p(k) (k+1) \quad T^{(k)} M \]
\[ T^{(k+1)} M \xrightarrow{T^\alpha} TT^{(k)} M \]
are commutative.

2. Forms and derivations

Let \( \Omega_q^{(k)} \) denote the \( \mathbb{R} \)-linear space of \( q \)-forms on \( T^{(k)} M \), and let \( \Omega_{(k)} \) be the nonnegative graded linear space \{ \( \Omega_{(k)}^q \) \}. The exterior differential \( d \) is a collection \{ \( d^q \) \} of linear mappings
\[ d^q : \quad \Omega_{(k)}^q \rightarrow \Omega_{(k)}^{q+1} \]
and the exterior product \( \wedge \) is a collection \{ \( \wedge (q,q') \) \} of operations \( \wedge (q,q') : \Omega_{(k)}^q \times \Omega_{(k)}^{q'} \rightarrow \Omega_{(k)}^{q+q'} \). For each \( k' \leq k \) and each \( q \), there is the cotangent mapping \( \rho^{(k')} \) : \( \Omega_{(k')}^q \rightarrow \Omega_{(k)}^q \) corresponding to the mapping \( \rho^{(k')} : T^{(k)} M \rightarrow T^{(k')} M \), and, if \( k'' \leq k' \leq k \), then
\[ \rho^{(k')} (k') \circ \rho^{(k')} (k') = \rho^{(k')} (k'). \]

(1) The mappings \( T^\alpha \) are related to the holonomic lift \( \lambda \) defined by KUMPERA [3].

BULLETIN DE LA SOCIETE MATHEMATIQUE DE FRANCE
Hence $\left(\Omega^r, \rho^r_{(k)}(k)\right)$ is a directed system. Let $\Omega^q$ denote the direct limit of this system, and let $\Omega$ be the graded linear space $\{\Omega^q\}$. The underlying set of $\Omega^q$ is the quotient set of $\bigcup_k \Omega^q_{(k)}$ by the equivalence relation according to which two forms $\mu \in \Omega^q_{(k)}$ and $v \in \Omega^q_{(k')}$ are equivalent if $k' \leq k$ and $\mu = \rho^r_{(k)}(k') v$, or $k' \geq k$ and $v = \rho^r_{(k)}(k') \mu$. The exterior differential $d$ and the exterior product $\wedge$ extend in a natural way to the direct limits giving the graded linear space $\Omega$ the structure of both a cochain complex and a commutative graded algebra. We write $\mu \in \Omega^q_{(k)}$ for an element $\mu$ of $\Omega^q$ if $\mu$ has a representative in $\Omega^q_{(k)}$. This notation could be justified by identifying $\Omega^q_{(k)}$ with the image of the canonical injection $\Omega^q_{(k)} \to \Omega^q$. A collection $a = \{a^q\}$ of linear mappings $a^q : \Omega^q \to \Omega^{q+r} : \mu \to a^q \mu$ is called a graded linear mapping of degree $r$. We write $a$ instead of $a^q$ if this can be done without causing any confusion. The exterior differential $d$ is a graded linear mapping of degree 1.

**DEFINITION 2.1.** A graded linear mapping $a = \{a^q\}$ of degree $r$ is called a derivation of $\Omega$ of degree $r$ if

$$a(\mu \wedge v) = a \mu \wedge v + (-1)^q \mu \wedge a v,$$

where $q = \text{degree} \mu$.

The exterior differential $d$ is a derivation of $\Omega$ of degree 1. If $a$ and $b$ are derivations of $\Omega$ of degrees $r$ and $s$ respectively, then

$$[a, b] = \{a^{q+s} b^q - (-1)^{rs} b^q a^q + r a^q\}$$

is a derivation of $\Omega$ of degree $r+s$ called the commutator of $a$ and $b$.

It follows from the general theory of derivations [2] that derivations of $\Omega$ are completely characterized by their action on $\Omega^0$ and $\Omega^1$. In fact, a derivation is completely determined by its action on equivalence classes of $f_n$ and $df_n$ for each function $f$ on $M$ and each $n \in \mathbb{N}$. Following FRÖLICHER and NIJENHUIS [2], we call a derivation $a$ a derivation of type $i_*$ if it acts trivially on $\Omega^0$. We call $a$ a derivation of type $d^*$ if $[a, d] = 0$.

For each $m \in \mathbb{N}$, each $k \in \mathbb{N}$ and each $q > 0$ there is a linear mapping

$$i_{F_m} : \Omega^q_{(k)} \to \Omega^q_{(k)} : \mu \mapsto i_{F_m} \mu,$$

defined by

$$\langle w_1 \wedge \ldots \wedge w_q, i_{F_m} \mu \rangle = \langle F_m(w_1) \wedge w_2 \wedge \ldots \wedge w_q, \mu \rangle + \ldots + \langle w_1 \wedge w_2 \wedge \ldots \wedge F_m(w_q), \mu \rangle,$$

TOME 105 — 1977 — N° 4
where $w_1, \ldots, w_q$ are vectors in $TT^{(k)} M$ such that $\tau_{T^{(k)} M} \circ (w_1) = \cdots = \tau_{T^{(k)} M} \circ (w_q)$ and $F_m : TT^{(k)} M \to TT^{(k)} M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$
\begin{array}{c}
\Omega^q_{\ell(k)} \xrightarrow{i_{F_m}} \Omega^q_{\ell(k')}
\end{array}
\downarrow
\begin{array}{c}
p_{\ell(k')}(k')
\end{array}
\begin{array}{c}
\rho_{\ell(k')}(k)
\end{array}
\Omega^q_{\ell(k)} \xrightarrow{i_{F_m}} \Omega^q_{\ell(k')}
$$

the mappings $i_{F_m}$ extend to a derivation $i_{F_m}$ of $\Omega$ of type $i_\ell$ and degree 0. If $\mu \in \Omega_{K}^q$, then $i_{F_m} \mu \in \Omega_{K+1}^q$ and $i_{F_m} \mu = 0$ if $\mu \in \Omega_{K}^q$ and $|m| > k$.

For each $\alpha = 1, \ldots, p$, each $k \in \mathbb{N}$, and each $q \in \mathbb{N}$, there is a linear mapping

$$i_{T_\alpha}: \Omega_{K}^{q+1} \to \Omega_{K+1}^{q}: \mu \mapsto i_{T_\alpha} \mu,$$

defined by

$$\langle w_1 \wedge \ldots \wedge w_q, i_{T_\alpha} \mu \rangle = \langle x \wedge u_1 \wedge \ldots \wedge u_q, \mu \rangle,$$

where

$$x = T^\alpha (v), \quad v = \tau_{T^{(k+1)} M} (w_1) = \cdots = \tau_{T^{(k+1)} M} (w_q),$$

$$u_1 = T_{p,K+1, K} (w_1), \quad \ldots, \quad u_q = T_{p,K+1, K} (w_q),$$

and $T^\alpha : T^{(k+1)} M \to TT^{(k)} M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$
\begin{array}{c}
\Omega_{K}^{q+1} \xrightarrow{i_{T_\alpha}} \Omega_{K+1}^{q+1}
\end{array}
\downarrow
\begin{array}{c}
\rho_{K+1}^{\ell(K+1)}(k+1)
\end{array}
\begin{array}{c}
\rho_{K+1}^{\ell(K+1)}(K+1)
\end{array}
\Omega_{K}^{q+1} \xrightarrow{i_{T_\alpha}} \Omega_{K+1}^{q+1}
$$

the mappings $i_{T_\alpha}$ extend to a derivation $i_{T_\alpha}$ of $\Omega$ of type $i_\ell$ and degree $-1$.

A derivation $d_{T_\alpha}$ of $\Omega$ of type $d_\ell$ and degree 0 is defined by $d_{T_\alpha} = [i_{T_\alpha}, d]$. If $\mu \in \Omega_{K}^{q+1}$, then $i_{T_\alpha} \mu \in \Omega_{K+1}^{q+1}$, and $d_{T_\alpha} \mu \in \Omega_{K+1}^{q+1}$.

For each $\alpha = 1, \ldots, p$ let $e^\alpha$ denote the element $(e_1^\alpha, \ldots, e_p^\alpha)$ of $\mathbb{N}^p$ defined by $e_0^\alpha = 1$ if $\alpha = \beta$, and $e_0^\alpha = 0$ if $\alpha \neq \beta$. Let $\geq$ denote the partial ordering relation in $\mathbb{N}^p$ defined by $(n_1, \ldots, n_p) \geq (n'_1, \ldots, n'_p)$ if

$$n_1 \geq n'_1, \ldots, n_{p-1} \geq n'_{p-1} \quad \text{and} \quad n_p \geq n'_p.$$

For each $m \in \mathbb{N}^p$, let $m!$ denote $m_1! \ldots m_p!$. 

**BULLETIN DE LA SOCIETE MATHematique DE FRANCE**
PROPOSITION 2.1. — If $m \geq e^a$ then
\[ [i_{f_n}, d_{T^a}] = \frac{m!}{(m-e^a)!} i_{f_{n-m}} \quad \text{and} \quad [i_{f_n}, d_{T^e}] = 0 \]
in all cases other than $m \geq e^a$.

Proof. — The commutator $[i_{f_n}, d_{T^a}]$ is a derivation and it is of type $i_{\hat{g}}$ since it acts trivially on $\Omega$. It can be easily shown for each $n \in \mathbb{N}^p$ and each function $f$ on $M$ that $i_{f_n} df_n = (n!/(n-m)!) df_{n-m}$ if $n \geq m$, and $i_{f_n} df_n = 0$ in all other cases. Also $d_{T^a} f_n = f_{n+e^a}$. It follows that
\[ [i_{f_n}, d_{T^a}] df_n = \frac{m!}{(m-e^a)!} i_{f_{n-m}} df_n \quad \text{if} \quad m \geq e^a, \]
and $[i_{f_n}, d_{T^e}] df_n = 0$ in all cases other than $m \geq e^a$. This completes the proof since a derivation of type $i_{\hat{g}}$ is completely determined by its action on equivalence classes of $df_n$ for each $f$ and each $n \in \mathbb{N}^p$.

PROPOSITION 2.2. — For each $\alpha$, $\beta = 1, \ldots, p$, $[d_{T^\alpha}, d_{T^\beta}] = 0$.

Proof. — Obvious.

3 The Lagrange complex $(\Lambda, \delta)$ (2)

Let $\tau = \{ \tau^q \}$ be the graded linear mapping of $\Omega$ into $\Omega$ of degree 0 defined by $\tau^0 = 1$ and
\[ \tau^q \mu = \frac{1}{q} \sum_{|m| \leq k} (-1)^{|m|} (m!)^{-1} d^m_T i_{f_m} \mu, \]
where $q > 0$, $\mu \in \Omega^q_{(k)}$ and $d^m_T = (d_{T^1})^m \ldots (d_{T^p})^{m^p}$. The sum in the above definition contains all nonzero terms $(-1)^{|m|} (m!)^{-1} d^m_T i_{f_m} \mu$ since $i_{f_m} \mu = 0$ unless $|m| \leq k$. We write
\[ \tau^q = \frac{1}{q} \sum_{|m| \leq k} (-1)^{|m|} (m!)^{-1} d^m_T i_{f_m} \]
without explicitly restricting the summation range which is understood to be wide enough to include in the sum all nonzero terms when $\tau^q$ is applied to an element of $\Omega^q$.

PROPOSITION 3.1. — If $q > 0$, then $\tau^q d_{T^\alpha} = 0$ for each $\alpha = 1, \ldots, p$.

(2) For definitions of algebraic topology terms used in this and the following sections, see reference [5].
Proof:  
\[ \tau^q d_{T_k} = \frac{1}{q} \sum_{m} (-1)^m (m!)^{-1} d_{T_k}^m \frac{d_{T_k}^m}{d_{T_k}^{m+e^0}} i_{\varphi_m} \]

\[ = \frac{1}{q} \sum_{m} (-1)^m (m!)^{-1} (d_{T_k}^{m+e^0} i_{\varphi_m} + d_{T_k}^{m+e^0} [i_{\varphi_m}, d_{T_k}]) \]

\[ = \frac{1}{q} \sum_{m} (-1)^m (m!)^{-1} d_{T_k}^{m+e^0} i_{\varphi_m} \]

\[ + \frac{1}{q} \sum_{m > e^0} (-1)^m ((m-e^0)!)^{-1} d_{T_k}^{m+e^0} i_{\varphi_m-e^0} = 0. \]

It follows from proposition 3.1, that \( \tau \tau = \tau \) and \( \tau d \tau = \tau d \).

Proposition 3.2. — The graded linear mapping \( \tau d = \{ \tau^{q+1} d^q \} \) is a differential of degree 1.

Proof. — \( \tau d \tau d = \tau d d = 0 \) and degree \( \tau d \) = degree \( \tau + d \) = 1.

We introduce the graded linear space \( \Lambda = \{ \Lambda^q \} \), where \( \Lambda^q = \text{im} \tau^q \).

The differential \( \tau d \) can be restricted to \( \Lambda \) due to \( \tau d \tau = \tau d \).

The restriction of \( \tau d \) to \( \Lambda \) is a differential of degree 1 denoted by \( \delta \).

Definition 3.1. — The differential \( \delta = \{ \delta^q \} \) is called the Lagrange differential, and the cochain complex \( \{ \Lambda^q, \delta^q \} \) is called the Lagrange complex.

Theorem 3.1 (\( \delta \)-Poincaré lemma). — If the manifold \( M \) is contractible then the Lagrange complex \( \{ \Lambda^q, \delta^q \} \) is acyclic for \( q > 0 \).

Let \( R \) denote the subspace of \( \Lambda^0 = \Omega^0 \) consisting of equivalence classes of constant functions and let \( \gamma : G \to \Lambda^0 \) be the canonical injection of the subspace \( G = R \oplus (d_{T_k} (\Omega^0) + \ldots + d_{T_k} (\Omega^0)) \).

Theorem 3.2. — The mapping \( \gamma : G \to \Lambda^0 \) is an augmentation of the Lagrange complex and the sequence

\[ 0 \to G \xrightarrow{\gamma} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{q-1}} \Lambda^q \xrightarrow{\delta^q} \ldots \]

is a resolution of \( G \).

We give proofs of the two theorems in the following section after having constructed a resolution of the graded linear space \( \Lambda' = \{ \Lambda^q \}_{q > 0} \).

4. A resolution of \( \Lambda' \)

Let \( K \) be the simplicial complex with vertices \( 1, \ldots, p \), and let \( \Delta_r (K) \) denote the free abelian group generated by the ordered \( r \)-simplexes of \( K [5] \).
We introduce a bigraded linear space $\Phi = \{ \Phi_r^q \}$, where $\Phi_r^q = \Delta_{q-1} (K) \otimes \Omega^q$ for $r > 0$, $\Phi_0^0 = \Omega^0$, and $\Phi_0^p = 0$ for $r < 0$. Elements of $\Phi_r^q$ are said to be of bidegree $(q, r)$. The exterior differential in $\Omega$ is extended to a bigraded linear mapping $d = \{ d_r^q \}$ of bidegree $(1, 0)$ by the formula

$$d_r^q((\alpha_1, \ldots, \alpha_r) \otimes \mu) = (\alpha_1, \ldots, \alpha_r) \otimes d\mu,$$

where $(\alpha_1, \ldots, \alpha_r)$ is an ordered $r+1$-simplex and $\mu \in \Omega^q$. A bigraded linear mapping $\partial = \{ \partial_r^q \}$ of bidegree $(0, -1)$ is defined by

$$\partial_r^q((\alpha_1, \ldots, \alpha_r) \otimes \mu) = \sum_{i=1}^{r} (-1)^{i-1} (\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_r) \otimes d\mu_i.$$

For each fixed $r$, $\{ \Phi_r^q, d_r^q \}$ is a cochain complex, and for each fixed $q$, $\{ \Phi_r^q, \partial_r^q \}$ is a chain complex. Since $\partial_r^{q+1} d_r^q = d_{r-1}^q \partial_r^q$, for each fixed $r$ the collection $\{ \partial_r^q : \Phi_r^q \rightarrow \Phi_{r-1}^q \}$ is a cochain mapping, and for each fixed $q$ the collection $\{ d_r^q : \Phi_r^q \rightarrow \Phi_{r+1}^q \}$ is a chain mapping.

**Proposition 4.1.** For each fixed $q > 0$ the chain complex $\{ \Phi_r^q, \partial_r^q \}$ is acyclic for $r > 0$.

**Proof.** For each $\alpha = 1, \ldots, p$, let a graded linear mapping

$$\sigma_\alpha = \{ \sigma_\alpha^q : \Omega^q \rightarrow \Omega^q \}$$

be defined by $\sigma_\alpha^0 = 0$ and

$$\sigma_\alpha^q = -\frac{1}{q} \sum_{m \in I_\alpha} (-1)^{m-1} (m!)^{-1} d_r^m \iota_{\mu_m}, \quad \text{where} \quad q > 0,$$

$I_\alpha = \{ m \in \mathbb{N}^p; m_\alpha > 0, m_\beta = 0 \text{ for } \beta > \alpha \}$ and the summation range is governed by a convention similar to the one used in the definition of $\tau$ in Section 3. From Proposition 2.1, it follows easily for $q > 0$ that $\sigma_\alpha^q d_{r, r} = 0$ if $\beta < \alpha$, $\sigma_\alpha^q d_{r, r} = 1 - \sum_{\gamma < \alpha} d_{r, r} \sigma_\gamma^q$, and $\sigma_\alpha^q d_{r, r} = d_{r, r} \sigma_\alpha^q$ if $\beta > \alpha$. A bigraded linear mapping $D = \{ D_r^q \}$ is defined by $D_0^q \mu = \sum_{\beta} (\beta) \otimes \sigma_\beta^q \mu$ and

$$D_r^q((\alpha_1, \ldots, \alpha_r) \otimes \mu) = \sum_{\beta < \alpha_1} (\beta, \alpha_1, \ldots, \alpha_r) \otimes \sigma_\beta^r \mu,$$

where $\mu \in \Omega^q$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. Relations $\partial_r^{q+1} D_r^q + D_{r-1}^q \partial_r^q = 1$ for $r > 0$, $q > 0$ are readily verified using the above stated properties of $\sigma_\alpha$. It follows that for each fixed $q > 0$ the graded mapping $D^q = \{ D_r^q \}$ defines a chain contraction of $\{ \Phi_r^q, \partial_r^q \}$ for $r > 0$. Hence $\{ \Phi_r^q, \partial_r^q \}$ is acyclic for $r > 0$.

**Proposition 4.2.** For each $q > 0$, the mapping $\tau^q : \Phi_0^q \rightarrow \Lambda^q$ is an augmentation of the chain complex $\{ \Phi_r^q, \partial_r^q \}$ and the sequence

$$\ldots \rightarrow \Phi_r^q \xrightarrow{\delta_r^q} \Phi_{r-1}^q \xrightarrow{\delta_{r-1}^q} \ldots \xrightarrow{\delta_1^q} \Phi_0^q \xrightarrow{\tau^q} \Lambda^q \rightarrow 0$$

is a resolution of $\Lambda^q$. 

TOME 105 — 1977 — N° 4
Proof. — The mapping $\tau^q : \Omega^q \to \Lambda^q$ is an epimorphism, and $\tau^q \partial^q_1 = 0$ follows from Proposition 3.1. Further $\tau^q + \partial^q_1 D_0^q = 1$, where $D_0^q$ is the mapping defined in the proof of Proposition 4.1. Hence $\tau^q \mu = 0$ implies $\mu = \partial^q_1 D_0^q \mu$ for each $\mu \in \Omega^q$. It follows that $\ker \tau^q = \im \partial^q_1$.

Proof of Theorems 3.1 and 3.2. — We define a nonnegative graded linear space $C = \{ C_r \}$ by $C_0 = \mathbb{R}$ and $C_r = \Delta_{r-1} (K) \otimes \mathbb{R}$ for $r > 0$, and a collection $\eta = \{ \eta_r : C_r \to \Phi^0_{-r} \}$ by $\eta_r = 1 \otimes \eta_0$, where $\eta_0 : \mathbb{R} \to \Omega^0$ is the canonical injection of the space $\mathbb{R} \subset \Omega^0$ of equivalence classes of constant functions identified with the field $\mathbb{R}$ of constants. If the manifold $M$ is contractible, then all rows except the bottom row of the commutative diagram

\[ \begin{array}{ccccccccc}
0 & \to & C_p & \eta_p & \Phi^0_p & \Phi^1_p & \cdots & \Phi^q_p & \to & \cdots \\
\downarrow & & \downarrow \phi^0_p & & \downarrow \phi^1_p & & \cdots & & \downarrow \phi^q_p & \\
0 & \to & C_0 & \eta_0 & \Phi^0_0 & \Phi^1_0 & \cdots & \Phi^q_0 & \to & \cdots \\
\downarrow & & \downarrow \phi^0_0 & & \downarrow \phi^1_0 & & \cdots & & \downarrow \phi^q_0 & \\
0 & \to & G & \gamma & \Lambda^0 & \Lambda^1 & \cdots & \Lambda^q & \to & \cdots \\
\downarrow & & \downarrow \gamma & & \downarrow \gamma & & \cdots & & \downarrow \gamma & \\
0 & & 0 & & 0 & & \cdots & & 0 & \\
\end{array} \]

are known to be exact and all columns for $q > 0$ are exact. For each $q > 0$, the top statement in the sequence

\[ \ker (\partial^q_{p+1} d^q_{p+1}) = \im \partial^q_{p+1} d^q_{p+1}, \]
\[ \ker (\partial^q_{p-1} d^q_{p-2}) = \im \partial^q_{p-2} d^q_{p-1} + \im \partial^q_{p-1}, \]
\[ \ker (\partial^q_1 d^q_1) = \im \partial^q_1 d^q_1 + \im \partial^q_2, \]
\[ \ker (\tau^q d^q_0) = \im \eta_0 \otimes \im \partial^q_1, \]

is true, and each of the remaining statements follows from the one immediately above. Hence the bottom statement is true. The same holds for $q = 0$ if the bottom statement is replaced by

\[ \ker (\tau^1 d^0_0) = \im \eta_0 \otimes \im \partial^0_1. \]
If \( q > 0 \) and \( \mu \) is an element of \( \Lambda^q \subset \Omega^q \), then \( \tau^q \mu = \mu \), and \( \delta^q \mu = \tau^q d_0^q \mu \).
If \( \delta^q \mu = 0 \), then there are elements \( \kappa \in \Phi_0^{q-1} \) and \( \lambda \in \Phi_q^1 \) such that
\[
\mu = d_0^{q-1} \kappa + \delta^q \lambda.
\]
It follows that
\[
\mu = \tau^q \mu = \tau^q d_0^{q-1} \kappa = \tau^q d_0^{q-1} \tau^{q-1} q = \delta^{q-1} \tau^{q-1} q.
\]
Hence \( \ker \delta^q = \im \delta^{q-1} \) and the Lagrange complex is acyclic for \( q > 0 \).
We note that \( \delta^0 = \tau^1 d_0^0 \) and
\[
G = \mathbb{R} \otimes (d_{T_1}(\Omega^0) + \ldots + d_{T_p}(\Omega^0)) = \im \chi_0 \otimes \im \delta_1^0.
\]
Hence \( \ker \delta^0 = G \). It follows that the sequence
\[
0 \to G \to \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \to \Lambda^q \xrightarrow{\delta^q} \ldots
\]
is exact.

5. Applications of the \( \delta \)-Poincaré lemma in the calculus of variations

A smooth mapping \( \chi : \mathbb{R}^{p+1} \to M : (s, t_1, \ldots, t_p) \mapsto \chi(s, t_1, \ldots, t_p) \) will be called a homotopy. For each \( s \in \mathbb{R} \), we denote by \( \chi_s \) the mapping
\[
\chi_s : \mathbb{R}^p \to M : (t_1, \ldots, t_p) \mapsto \chi(s, t_1, \ldots, t_p).
\]
The mapping \( \gamma = \chi_0 \) will be called the base of the homotopy \( \chi \). We say that the homotopy \( \chi \) is constant on \( A \subset \mathbb{R}^p \) if \( \chi(s, t_1, \ldots, t_p) = \chi(0, t_1, \ldots, t_p) \) for each \( s \in \mathbb{R} \) and each \( (t_1, \ldots, t_p) \in A \). For each mapping \( \varphi : \mathbb{R}^p \to M : (t_1, \ldots, t_p) \mapsto \varphi(t_1, \ldots, t_p) \),
we denote by \( \varphi^{(k)} \) the mapping
\[
\varphi^{(k)} : \mathbb{R}^p \to T^{(k)} M : (t_1, \ldots, t_p) \mapsto j^{(k)}_{(t_1, \ldots, t_p)}(\chi).
\]
For each homotopy \( \chi \), we denote by \( \chi^{(k)} \) the mapping
\[
\chi^{(k)} : \mathbb{R}^p \to TT^{(k)} M : (t_1, \ldots, t_p) \mapsto j^{(1,k)}_{(t_1, \ldots, t_p)}(\chi),
\]
where \( j^{(1,k)}_{(0,t_1,\ldots,t_p)}(\chi) \) is a jet-like object similar to \( j^0_{(1,k)}(\chi) \) defined in terms of partial derivatives at \( (0, t_1, \ldots, t_p) \) instead of \( (0, 0, \ldots, 0) \) and identified with an element of \( TT^{(k)} M \).

Each element \( L \in \Omega^0_{(k)} \) gives rise to a family of functions
\[
\gamma \mapsto \int_V L \circ \gamma^{(k)},
\]
TOME 105 — 1977 — N° 4
defined on the set of smooth mappings of $\mathbb{R}^p$ into $M$ for each domain $V \subset \mathbb{R}^p$.

**Definition 5.1.** A mapping $\gamma : \mathbb{R}^p \rightarrow M$ is called an extremal of the family of functions

$$\gamma \mapsto \int_V L \circ \gamma^{(k)} \quad \text{if} \quad \frac{d}{ds} \int_V L \circ \chi_s^{(k)} \bigg|_{s=0} = 0,$$

for each domain $V \subset \mathbb{R}^p$ and each homotopy $\chi$ with base $\gamma$ constant on the boundary $\partial V$ of $V$.

**Definition 5.2.** A form $\lambda \in \Omega^1_{(k)}$ is called an Euler-Lagrange form associated with $L \in \Omega^0_{(k)}$ if $i_{\gamma_m} \lambda = 0$ for each $m > 0$ and if

$$\int_V \langle \chi^{(k)}', dL \rangle = \int_V \langle \chi^{(k)}', \lambda \rangle$$

for each domain $V \subset \mathbb{R}^p$ and each homotopy $\chi$ constant on $\partial V$. It is clear from the definition of $F_m$ that if $\lambda \in \Omega^1_{(k)}$ satisfies $i_{\gamma_m} \lambda = 0$ for each $m > 0$, then $\lambda$ can be interpreted as a mapping $\lambda : T^{(k)} M \rightarrow T^* M$. If $\lambda$ is an Euler-Lagrange form associated with $L$ then

$$\frac{d}{ds} \int_V L \circ \chi_s^{(k)} \bigg|_{s=0} = \int_V \langle \chi^{(k)}', dL \rangle$$

for each homotopy $\chi$ with base $\gamma$ constant on $\partial V$. It follows that $\gamma : \mathbb{R}^p \rightarrow M$ is an extremal of the family

$$\gamma \mapsto \int_V L \circ \gamma^{(k)},$$

if, and only if, $\gamma$ satisfies the equation $\lambda \circ \gamma^{(k)} = 0$ called the Euler-Lagrange equation.

We show that $\lambda = \delta^0 L$ is the unique Euler-Lagrange form associated with $L \in \Omega^0$. We also show that $i_{\gamma_m} \lambda = 0$ for each $m > 0$ means that $\lambda \in \Omega^1$ is in $\Lambda^1$. These statements imply applications of the $\delta$-Poincaré lemma. A form $\lambda \in \Omega^1$ is an Euler-Lagrange form if, and only if, $\lambda \in \Lambda^1$ and $\delta^1 \lambda = 0$. Euler-Lagrange forms associated with two elements $L$ and $L'$ of $\Omega^0$ are the same if, and only if, $L' - L \in \mathbb{R} \oplus (d_T \Omega^0) + \ldots + d_{T^p} (\Omega^0)$. 

**BULLETIN DE LA SOCIETE MATHEMATIQUE DE FRANCE**
PROPOSITION 5.1. — A form $\lambda \in \Omega^1$ belongs to $\Lambda^1$ if, and only if, $i_{T_m} \lambda = 0$ for each $m > 0$.

Proof. — If $i_{T_m} \lambda = 0$ for each $m > 0$, then

$$\tau^1 \lambda = \sum_m (-1)^m m! (m!)^{-1} d_{T_m}^m i_{T_m} \lambda = i_{T_0} \lambda = \lambda.$$ 

Hence $\lambda \in \text{im} \tau^1 = \Lambda^1$. From Proposition 2.1, it follows that

$$i_{T_m} \tau^1 = \sum_m (-1)^m m! (m!)^{-1} d_{T_m}^m i_{T_m} = i_{T_0} = 0.$$ 

if $m \geq e^a$ and $i_{T_m} d_{T_m} = d_{T_m} i_{T_m}$ in all other cases. Since $i_{T_m} i_{T_n} \mu = i_{T_{m+n}} \mu$ for each $\mu \in \Omega^1$, it follows that

$$i_{T_m} \tau^1 = \sum_m (-1)^m m! (m!)^{-1} d_{T}^m i_{T_m} = i_{T_0} = 0.$$ 

Consequently, $i_{T_m} \tau^1 = 0$ for each $m > 0$, and if $\lambda \in \Lambda^1$ then $i_{T_m} \lambda = 0$ for each $m > 0$.

PROPOSITION 5.2. — The space $\Omega^1$ is the direct sum of $\Lambda^1$ and

$$d_{T_1} (\Omega^1) + \ldots + d_{T_p} (\Omega^1).$$

Proof. — Let $\mu$ be an element of $\Omega^1$. Then $\mu = \lambda + \nu$, where $\lambda = \tau^1 \mu \in \Lambda^1$, and

$$\nu = -\sum_{m>0} (-1)^m m! (m!)^{-1} d_{T_m}^m i_{T_m} \mu = d_{T_1} (\Omega^1) + \ldots + d_{T_p} (\Omega^1).$$

It follows from $\tau^1 \tau^1 = \tau^1$ and $\tau^1 d_{T_m} = 0$ that this decomposition of $\mu$ into elements of $\Lambda^1$ and $d_{T_1} (\Omega^1) + \ldots + d_{T_p} (\Omega^1)$ is unique.

PROPOSITION 5.3. — Let $\mu$ be an element of $\Omega^1_{(k)}$. Then

$$\int_V \langle \chi''(k), \mu \rangle = 0,$$

for each domain $V \subset \mathbb{R}^p$ and each homotopy $\chi : \mathbb{R}^{p+1} \to M$ constant on $\partial V$ if, and only if, $\mu \in d_{T_1} (\Omega^1) + \ldots + d_{T_p} (\Omega^1)$.

Proof. — If $\mu = \sum \omega_{\nu}$ then

$$\int_V \langle \chi''(k), \omega \rangle = \sum \int_{\nu} \frac{\partial}{\partial t} \langle \chi''(k), \omega \rangle = \sum \int_{\nu} n_{\nu} \langle \chi''(k), \omega \rangle,$$

for each domain $V \subset \mathbb{R}^p$ and each homotopy $\chi : \mathbb{R}^{p+1} \to M$ constant on $\partial V$ if, and only if, $\mu \in d_{T_1} (\Omega^1) + \ldots + d_{T_p} (\Omega^1)$. 

TOME 105 — 1977 — N° 4
where \( n^z \) are the components of the normal vector. If \( \chi \) is constant on \( \partial V \),
then
\[
\int_V \langle \chi^{(k)}', \mu \rangle = 0.
\]
Let \( \mu = \lambda + \nu \) be the unique decomposition of \( \mu \in \Omega^1 \) used in the proof of
proposition 5.2. If \( \int_V \langle \chi^{(k)}', \mu \rangle = 0 \), then
\[
\int_V \langle \chi^{(k)}', \lambda \rangle = \int_V \langle \chi^{(0)}', \lambda \circ \gamma^{(k)} \rangle = 0,
\]
where \( \gamma \) is the base of \( \chi \), and \( \lambda \) is interpreted as a mapping \( \lambda : T^{(k)} M \to T^* M \).
It follows that \( \lambda = 0 \) and \( \mu = \nu \). Hence \( \mu \in d_{T^1} (\Omega^1) + \ldots + d_{T^p} (\Omega^1) \).

**COROLLARY.** — If \( L \) is an element of \( \Omega^0 \), then \( \lambda = \delta^0 L \) is the unique
element of \( \Lambda^1 \) such that \( dL - \lambda \in d_{T^1} (\Omega^1) + \ldots + d_{T^p} (\Omega^1) \). It follows that
\( \lambda \) is the unique Euler-Lagrange form associated with \( L \).

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Włodzimierz M. **Tulczyjew**
Dept of Math. and Statistics,
University of Calgary,
Calgary, Alberta T2N 1N4,
Canada.