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Non-vanishing on the first cohomology


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NON-VANISHING
OF THE FIRST COHOMOLOGY

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RESUMÉ. — On démontre que, pour les réseaux $\Gamma$ du type fini dans les groupes semi simples sur les corps locaux de caractéristique positive, $H^1(\Gamma, \text{Ad})$ ne s'annule pas; ceci est bien différent de ce que passe dans le cas de caractéristique zéro.

ABSTRACT. — It is shown here that, for any finitely generated lattice $\Gamma$ in certain semi simple groups over local fields of positive characteristics, $H^1(\Gamma, \text{Ad})$ is non-vanishing; this is in sharp contrast with the situation in characteristic zero.

Let $K$ be a local field (i.e. a non-discrete locally compact field), and let $G$ be a connected semi simple algebraic group defined over $K$. Let $G = G(K)$, and let $r = K-\text{rank } G$. The topology on $K$ induces a locally compact Hausdorff topology on $G$; in the sequel, we assume $G$ endowed with this topology. $G$ is then a $K$-analytic group. Let $\Gamma$ be a lattice in $G$ i.e., a discrete subgroup of $G$ such that $G/\Gamma$ carries a finite $G$-invariant Borel measure. We assume that $\Gamma$ is irreducible, i.e. no subgroup of $\Gamma$ of finite index is a direct product of two infinite normal subgroups.

In case $K = \mathbb{R}$ and $G$ is not locally isomorphic to either $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$, it is known that $H^1(\Gamma, \text{Ad}) = 0$; where, as usual, $\text{Ad}$ denotes the adjoint representation of $G$ on its Lie algebra (see Weil [9], [10] for uniform lattices; for non-uniform lattices in groups of $R$-rank $> 1$, this vanishing theorem follows from the results of Raghunathan [8], combined with the results of Margulis [4] on arithmeticity; for non-uniform lattices in groups of $R$-rank $1$, it is contained in Garland-Raghunathan [2]).

It is also known, in view of a recent result of Margulis ([5], theorem 8), that in case $K$ is non-archimedean but of characteristic zero, $H^1(\Gamma, \text{Ad}) = 0$ when $r > 1$. 

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The object of this note is to show that when $K$ is of positive characteristic, then it is not in general true that $H^1(\Gamma, \text{Ad}) = 0$.

We shall in fact prove the following theorem.

**Theorem.** — Let $F$ be a finite field, and let $K$ be the local field $F((t))$. Let $G$ be a connected semi simple algebraic group, with trivial center, defined over $F$. Let $G = G(K)$, let $\Gamma$ be a finitely generated lattice in $G$. Then $H^1(\Gamma, \text{Ad}) \neq 0$.

**Remark.** — If $G$ has no $K$-rank 1 factors, then according to a well-known theorem of D.A. Kazhdan (see [1]), every lattice in $G$ is finitely generated.

For the proof of the theorem, we need to recall a result of Weil [10].

We introduce some notation and a definition.

Let $\Lambda$ be a finitely generated abstract group. We shall let $\mathcal{A}(\Lambda, G)$ denote the space of all homomorphisms of $\Lambda$ in $G$ with the topology of pointwise convergence. There is a natural action of $G$ on $\mathcal{A}(\Lambda, G)$ induced by the inner automorphism.

Now assume that $\Lambda$ is a finitely generated subgroup of $G$, and let $\iota : \Lambda \rightarrow G$ be the natural inclusion. Then $\Lambda$ is said to be locally (or infinitisimally) rigid if the orbit of $\iota$ under $G$ is open in $\mathcal{A}(\Lambda, G)$. According to a result of Weil [10], vanishing of $H^1(\Lambda, \text{Ad})$ implies local rigidity of $\Lambda$.

**Proof of the theorem.** — In view of the above result of Weil, to prove that $H^1(\Gamma, \text{Ad}) \neq 0$, it suffices to show that $\Gamma$ is not infinitisimally rigid.

For $i > 1$, $t \mapsto t + t^i$ extends uniquely to give a continuous automorphism $a_i$ of $F((t))/F$. It is evident that, for any fixed $x \in F((t))$, the sequence $\{ a_i(x) \}$ converges to $x$.

Now since $G$ is defined over $F$, $a_i$ induces a continuous automorphism $\alpha_i$ of $G$. Therefore, for all $i$, $\alpha_i, \iota$ is an embedding of $\Gamma$ in $G$; where $\iota : \Gamma \rightarrow G$ is the natural inclusion of $\Gamma$ in $G$. It is also obvious that the sequence $\{ \alpha_i, \iota \}$ converges to $\iota$ in $\mathcal{A}(\Gamma, G)$. We shall show that none of the $\alpha_i, \iota$ lie in the $G$-orbit of $\iota$. This will prove that $\Gamma$ is not locally rigid and hence $H^1(\Gamma, \text{Ad}) \neq 0$.

If possible, assume that, for some $i$, $\alpha_i, \iota = \text{Int} \ g_i, \iota$. Then $(\text{Int} \ g_i^{-1}, \alpha_i), \iota = \iota$, and the main theorem of Prasad [6] implies that $\text{Int} \ g_i^{-1}, \alpha_i$ is the identity automorphism of $G$. Hence, $\alpha_i = \text{Int} \ g_i$.

We now fix a 1-dimensional torus $T (\subset G)$ which is defined and split over the finite field $F$ (existence of such a torus follows from Lang's theorem [3]). Let $T = T(K)$. Then since $T$ is defined over $F$, $\alpha_i(T) = T$. Moreover,
for any rational character $\chi$ on $T$ and all $t \in T$,

$$\chi(\alpha_t(t)) = a_t(\chi(t)).$$

Since $\alpha_t = \text{Int} g_t$ and $\alpha_t(T) = T$, it follows that $g_t$ normalizes $T$ and hence also $T$. Therefore, for any rational character $\chi$ on $T$:

$$\chi(\alpha_t(t)) = \chi(g_t g_t^{-1}) = \chi^d(t),$$

where $d = +1$ or $-1$. Hence,

$$a_t(\chi(t)) = \chi^d(t), \quad \text{where} \quad d = +1 \quad \text{or} \quad -1.\tag{\ast}$$

Now take $\chi$ to be one of the generators of the group of rational characters on $T$. Then it follows from (\ast) that, for all $k \in K$, either

$$a_t(k) = k \quad \text{or} \quad a_t(k) = k^{-1}.$$  

But it is obvious from the definition of $a_t$ that this is not the case. Hence, none of the $\alpha_t, t$ lie in the $G$-orbit of $t$. This proves that $H^1(\Gamma, Ad) \neq 0$.

**Remark.** — As the above proof shows, $\Gamma$ is not locally rigid. However, in case $K$-rank $G > 1$ and $\Gamma$ is an irreducible uniform lattice, it is strongly rigid (see Prasad [7], § 8).

**REFERENCES**


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