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COMMUTATION THEOREMS AND GENERALIZED COMMUTATION RELATIONS

BY

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RÉSUMÉ. — On démontre un théorème de commutation pour une structure qui généralise les algèbres quasi hilbertiennes de Dixmier de telle façon qu'elle peut manier des paires d'algèbre de von Neumann de tailles différentes. On se sert de ce théorème de commutation pour donner une démonstration directe de la relation de commutation généralisée, de Takesaki pour la représentation régulière d'un groupe (étendue au cas non séparable par NIELSEN), évitant des réductions par représentations induites et intégrales directes.

SUMMARY. — A commutation theorem is proved for a structure which generalizes Dixmier's quasi-Hilbert algebras in such a way that it can handle pairs of von Neumann algebras of different size. This commutation theorem is then applied to give a direct proof of Takesaki's generalized commutation relation for the regular representations of groups (as extended to the non-separable case by NIELSEN), thus avoiding reductions by induced representations and direct integrals.

Let G be a locally compact group, and let H and K be closed subgroups of G. Let $L^2(G)$ be the usual Hilbert space of square-integrable functions on G with respect to left Haar measure. Let $M(K, G/H)$ be the von Neumann algebra of operators on $L^2(G)$ generated by the left translations by elements of K together with the pointwise multiplications by bounded continuous functions on G which are constant on left cosets of H. Similarly, let $M(H, K\backslash G)$ be the von Neumann algebra generated by the right translations by elements of H together with the pointwise multiplications by bounded continuous functions which are constant on right cosets of K. TAKESAKI [19] showed that, when G is separable, these two von Neumann algebras are each other's commutants. He called this theorem a generalized commutation relation, since, as he showed, it is closely related to the Heisenberg commutation relations.

Takesaki's proof was restricted to the separable case because of the use he made of direct integral theory. NIELSEN [10], by using the theory of liftings of measures, extended Takesaki's theorem to non-separable groups. In both papers, the strategy of the proof consists of using the theory of induced representations to make a reduction to a situation in which Dixmier's commutation theorem for quasi-Hilbert algebras [5] can be invoked. In the present paper, we give a proof of Takesaki's theorem (in the general case) whose strategy consists of first proving a commutation theorem for a structure which generalizes Dixmier's quasi-Hilbert algebras, and then showing that this commutation theorem is directly applicable to Takesaki's situation. Our commutation theorem is directly applicable in part because, unlike Dixmier's, it is able to handle pairs of von Neumann algebras which are of different size, such as those which arise in Takesaki's situation. For this reason, our theorem may also be useful in other contexts. Because our commutation theorem is basically algebraic, and applies directly, our proof of Takesaki's theorem avoids both induced representations and most of the measure-theoretic difficulties in which the proofs of Takesaki and Nielsen become embroiled.

In the course of Nielsen's proof, he obtained a generalization of a theorem of MACKEY [8] and BLATTNER [2] concerning intertwining operators for induced representations. We will show elsewhere [15] that this generalization has a natural interpretation and proof within the version of the theory of induced representations developed in [12], and that, in fact, it is a major part of the statement that certain $C^{\star}$-algebras associated with the situation are strongly Morita equivalent [13]. These results are independent of the present paper. Nevertheless, the structure we introduce here to generalize Dixmier's quasi-Hilbert algebras was motivated by the "imprimitivity bimodules" [12] used in discussing Morita equivalence, and some of the formulas used here in applying our commutation theorem to Takesaki's situation were motivated by the formulas used in [15] to discuss Nielsen's generalization of the Mackey-Blattner theorem.

The reader is referred to [19] for a discussion of the connection between the generalized commutation relations and the Heisenberg commutation relations, and to [10] for the connection with the Takesaki-Tatsuuma duality theory for locally compact groups [20]. Finally, we remark that, as Takesaki points out at the end of [19], essentially the same generalized commutation relation has been obtained by ARAKI [1] for the case in which G is a Hilbert space, and H and K are closed subspaces, all acting on the Fock representation. It would be interesting to see how to fit Araki's theorem into the framework of the present paper, but so far this has eluded me. However, it was in studying this question that I did find a fairly simple proof [14] of Araki's theorem whose lines are rather parallel to those of the present paper.

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1. The Commutation Theorems

In this section, we introduce a structure which generalizes the quasi-Hilbert algebras of DIXMIER [5], and we then prove the analogue for this structure of Dixmier's commutation theorem for quasi-Hilbert algebras. The axioms for this structure are motivated by the needs of the next section, and by formulas and structures appearing in [12], [13] and [15].

All the algebras and vector spaces we consider will be defined over the complex numbers. If C and D are algebras (not assumed to have identity elements), then by a C-D-bimodule we mean a vector space, X, on which C and D act on the left and right respectively, with the action of C commuting with that of D, and with both actions being compatible with the action of the complex numbers. If C has an involution, then by a $\star$-representation of C we mean a $\star$-homomorphism of C into the pre-$C^\star$-algebra of those bounded operators on a pre-Hilbert space, V, which have adjoints defined on V. We will say that such a representation is non-degenerate if CV is dense in V (using module notation). Analogous definitions are made for a $\star$-representation on the right, that is, an anti-$\star$-representation. A representation is said to be faithful if its kernel consists only of 0.

We now introduce our generalization of Dixmier's quasi-Hilbert algebras [5]. Because we will not introduce until later the analogue of axiom (v) of Dixmier's definition of a quasi-Hilbert algebra, we will use the prefix "semi" instead of "quasi".

1.1. *Definition.* — Let C and D be algebras, each equipped with involutions, which we denote by $^\sharp$ and $^\flat$ respectively. By a *Hilbert semi-C-D-birigged space* we mean a C-D-bimodule, X, equipped with an ordinary inner product, $[,]$, and with C and D-valued sesquilinear forms, $\langle\ ,\ \rangle_C$ (conjugate linear in the second variable) and $\langle\ ,\ \rangle_D$ (conjugate linear in the first variable), such that:

1° The representations of C and D on the left and right of X are faithful $\star$-representations.

2° $\langle x, y \rangle_C\, z = x \langle y, z \rangle_D$ for all $x, y, z \in X$.

3° $\langle X, X \rangle_C$ is a self-adjoint set, that is, for any $x, y \in X$ there exist x_1, $y_1 \in X$ such that $\langle x, y \rangle_C^\# = \langle x_1, y_1 \rangle_C$.

4° $\langle X, X \rangle_C X$ (the linear span) is dense in X.

We will say that X is a *Hilbert C-D-birigged space* if in addition:

5° $\langle x, y \rangle_C^\# = \langle y, x \rangle_C$ and $\langle x, y \rangle_D^\flat = \langle y, x \rangle_D$ for $x, y \in X$.

6° For any $x \in X$ both $\langle x, x \rangle_C$ and $\langle x, x \rangle_D$ act as non-negative operators on X.

We remark that axiom 4 implies that the action of C is non-degenerate. From axiom 2 it then follows that the action of D is non-degenerate. We further remark that if $x, y \in X$, $c \in C$ and $d \in D$, then

$$c\langle x, y \rangle_C = \langle cx, y \rangle_C \quad \text{and} \quad \langle x, y \rangle_D d = \langle x, yd \rangle_D.$$

To see this, let $z \in X$. Then

$$(c\langle x, y \rangle_C) z = c(x\langle y, z \rangle_D) = (cx)\langle y, z \rangle_D = \langle cx, y \rangle_C z,$$

and we can now use the hypothesis that the representation of C on X is faithful. From this we see that the linear span of the range of $\langle\, , \rangle_C$ is a left ideal in C. But by axiom 3 this linear span is self-adjoint, so that it is, in fact, a two-sided ideal in C. Since in the definition of a Hilbert semi-C-D-birigged space there is no hypothesis relating $\langle\, , \rangle_D$ and the involution on D, we can not draw a similar conclusion about the linear span of $\langle\, , \rangle_D$. However, the commutation theorem will concern itself with the von Neumann algebra generated by D, and thus, in particular, with the $\star$-algebra generated by this linear span. Finally, we remark that there is no real loss of generality in the assumption that the representations of C and D are faithful, since if they are not, then it is easily seen that one can factor by their kernels.

1.2. *Example.* — Every quasi-Hilbert algebra can be viewed as a Hilbert semi-birigged space. Specifically, if A is a quasi-Hilbert algebra, let C, D and X all be A, with the actions of C and D on X being just the left and right regular representations. Using the notation from page 66 of [5], we define involutions on C and D by

$$a^\# = a^{\star\wedge}, \qquad a^\flat = a^{\wedge\star}$$

for $a \in C$ and $a \in D$. It is then clear that axiom 1 above is satisfied. Define C and D-valued sesquilinear forms on X by

$$\langle a, b \rangle_C = ab^\#, \qquad \langle a, b \rangle_D = a^\# b$$

for $a, b \in X$. Then it is clear that axioms 2 and 3 are satisfied. To see that axiom 4 is satisfied, we note that from the hypothesis that A^2 is dense in A it follows by von Neumann's double commutant theorem [5] that there is a net of elements of A which, as operators on the completion of A, converge in the strong operator topology to the identity operator on A. It follows that A^3 (as AA^2) is dense in A. But this is axiom 4 above. If A happens to be a Hilbert algebra, then it is easily seen that axioms 5 and 6 above are also satisfied. In the verification just sketched, no use was made of axiom (v) of Dixmier's definition of a quasi-Hilbert algebra. We will see that this axiom corresponds to the Coupling Condition which is used in the commutation theorem we will consider shortly.

Of course, for us the important example of a Hilbert semi-C-D-birigged space is that given in the next section in connection with the generalized commutation relations.

If X is a Hilbert semi-C-D-birigged space, we will let $\overline{X}$ denote its Hilbert space completion, and our notation will not distinguish between the elements of C and D viewed as operators on X and their extensions by uniform continuity acting as operators on $\overline{X}$. We will let E' denote the commutant of any set of operators, E, on $\overline{X}$. Thus $C'' \subseteq D'$ (and $D'' \subseteq C'$). We will say that C and D generate each other's commutants on X if in fact $C'' = D'$ (so that $D'' = C'$). The commutation theorem gives a necessary and sufficient condition, which we call the Coupling Condition, for C and D to generate each other's commutants. This condition is the appropriate analogue for the present situation of axiom (v) in Dixmier's definition of a quasi-Hilbert algebra, and our proof that the condition is sufficient is obtained by trying to imitate the proof of Dixmier's commutation theorem for quasi-Hilbert algebras [5]. We will see later that the Coupling Condition is automatically satisfied if X is, in fact, a Hilbert C-D-birigged space. This generalizes the commutation theorem for Hilbert algebras ([5], [11]).

1.3. *The commutation theorem for Hilbert semi-birigged spaces.* — Let X be a Hilbert semi-C-D-birigged space. Then C and D generate each other's commutants on $\overline{X}$ if and only if the following condition is satisfied:

Coupling Condition: If $m, n \in X$ and $x, y \in X$, and if

$$[m \langle x, z \rangle_D, w] = [z, n \langle y, w \rangle_D] \quad \textit{for all } z, w \in X,$$

then for any fixed $z, w \in X$ *there is a net,* $\{ c_k \}$, *of elements of* C *such that*

$$c_k z \quad \textit{converges to} \quad m \langle x, z \rangle_D,$$

and

$$c_k^{\natural} w \quad \textit{converges to} \quad n \langle y, w \rangle_D.$$

Proof. — We first prove the sufficiency of the Coupling Condition. Imitating the proof for quasi-Hilbert algebras, we set:

1.4. *Definition.* — An element, m, of $\bar{X}$ is said to be *D-bounded* if for every $x \in X$ the linear map $y \mapsto m \langle x, y \rangle_D$ from X to $\bar{X}$ is bounded, and so extends by uniform continuity to a bounded operator on $\bar{X}$, which we will denote by $L_{\langle m, x \rangle}$. We will let X_L denote the linear manifold of D-bounded elements in $\bar{X}$.

1.5. Lemma. — *Let* $m \in X_L$. *Then* $L_{\langle m, x \rangle}$ *is in* D' *for every* $x \in X$. *Furthermore, if* $T \in D'$, *then* $T m \in X_L$, *and*

$$L_{\langle Tm, x \rangle} = T L_{\langle m, x \rangle} \quad \textit{for every} \quad x \in X.$$

Finally, $X \subseteq X_L$, *and* $L_{\langle y, x \rangle} = \langle y, x \rangle_C$ *for all* $y, x \in X$.

This lemma is verified by routine calculations.

1.6. Lemma. — *Let* $J = \{ L_{\langle m, x \rangle} ; m \in X_L, x \in X \}$, *and let* $K = J \cap J^*$, *where* $\star$ *denotes the adjoint operation on operators on* $\bar{X}$. *Note that* $L_{\langle x, y \rangle} \in K$ *for all* $x, y \in X$. *Then* $K'' = D'$.

Proof. — It is axiom 3 which ensures that $L_{\langle x, y \rangle} \in K$ for all $x, y \in X$. Now let $T \in D'$ and $x, y, z, w \in X$. Then from Lemma 1.5 we see that $L^{\star}_{\langle x, y \rangle} T L_{\langle z, w \rangle}$ is in J. But equally well its adjoint is also in J, so that it is, in fact, in K. Thus if we let E be the linear span of the range of $\langle \, , \rangle_C$ (which we saw earlier is a 2-sided ideal in C), then it follows that the linear span of K, and so K'', contains $e\, T f$ for every $e, f \in E$ and $T \in D'$. Now from axiom 4 we see that $\overline{EX}$ is dense in $\bar{X}$, so that by von Neumann's double commutant theorem there is a net of elements of E which converges in the strong operator topology to the identity operator on $\bar{X}$. If in the expression $e\, T f$ we let first e, and later f, range over this net, we find that $T \in K''$. Thus $D' \subseteq K''$. But $D' \supseteq K''$ by Lemma 1.5. Thus $D' = K''$.

Q. E. D.

Now let $m \in X_L$ and $x \in X$, and suppose that $L_{\langle m, x \rangle} \in K$. Then from the definition of K it follows that there exist $n \in X_L$ and $y \in X$ such that

$L^{\star}_{\langle m,\, x\rangle} = L_{\langle n,\, x\rangle}$, that is $[m \langle x, z \rangle_D, w] = [z, n \langle y, w \rangle_D]$ for all z, $w \in X$. But then the Coupling Condition is applicable, and so we can find, for any given z, $w \in X$, a net $\{ c_k \}$ having the convergence properties of that condition. Suppose now that $S \in C'$. Then for the given z and w, we have

$$\begin{aligned}\langle L_{\langle m,\, x\rangle} S z, w \rangle &= \langle S z, L_{\langle n,\, y\rangle} w \rangle \\ &= \langle S z, n \langle y, w \rangle_D \rangle = \lim_k \langle S z, c_k^{\#} w \rangle \\ &= \lim_k \langle S c_k z, w \rangle = \langle S L_{\langle m,\, x\rangle} z, w \rangle.\end{aligned}$$

Since z and w are arbitrary elements of X, it follows that S commutes with all elements of K. Thus $C' \subseteq K'$, so that $C'' \supseteq K'' = D'$. But $C' \supseteq D$ so that $C'' \subseteq D'$, and consequently $C'' = D'$. Thus the sufficiency of the Coupling Condition has been shown.

We now prove the necessity of the Coupling Condition. Most likely a proof of this which uses only bounded operators can be given, along the lines of the necessity proofs for the commutation theorems in [14] and [16]. Partly because of the connections with the developments in the next section, we prefer to give here a proof which is shorter but which involves unbounded operators. It is easily seen that a proof along the lines presented here can also be used in [14] and [16].

Assume that $C' = D''$, and let m, $n \in X$ and x, $y \in \bar{X}$ be such that $[m \langle x, z \rangle_D, w] = [z, n \langle y, w \rangle_D]$ for all z, $w \in X$. Define (possibly unbounded) operators $L_{\langle m,\, x\rangle}$ and $L_{\langle n,\, y\rangle}$ on $\bar{X}$, with domain X, by $L_{\langle m,\, x\rangle} z = m \langle x, z \rangle_D$ for $z \in X$, and similarly for $L_{\langle n,\, y\rangle}$. Then the above relation says that each of these operators is contained in the adjoint of the other. It follows that both operators are closeable. We will denote their closures by $\bar{L}_{\langle m,\, x\rangle}$ and $\bar{L}_{\langle n,\, y\rangle}$. Now routine calculations show that the domains of $L_{\langle m,\, x\rangle}$ and $L_{\langle n,\, y\rangle}$ are invariant under the action of D, and that these operators commute with the action of D. Further routine calculations show that this is also true with respect to the action of the strong operator closure, D'', of D. In other words, these operators are affiliated [5] with D'.

Now let $\bar{L}_{\langle m,\, x\rangle} = PT$ be the polar decomposition of $\bar{L}_{\langle m,\, x\rangle}$, where P is a partial isometry, and T is a positive self-adjoint operator (*see* p. 1249 of [6]). Then it is not difficult to show (*see* lemma 4.4.1 of [9]) that $P \in D'$ and that T is affiliated with D'. Let $\{ E(r) \}$ be the spectral resolution for T, so that $E(r) \in D'$ for each r. Then $PTE(r) \in D'$ for each r.

Now let z, $w \in X$ be given. Since C is assumed to generate D', it is dense in D' in the strong-$\star$-operator topology [17] by the Kaplansky density theorem [5], and so for each $r > 0$ we can find $c_r \in C$ such that

$$\| c_r z - PTE(r) z \| \leqslant 1/r \qquad \text{and} \qquad \| c_r^{\#} w - E(r)\, TP^{\star} w \| \leqslant 1/r.$$

But $PTE(r) z = PE(r) T z$, which converges to $PTz = m \langle x, z \rangle_D$. Similarly $E(r)\, TP^{\star} w$ converges to $TP^{\star} w$, which equals

$$L_{\langle n, y \rangle} w = n \langle y, w \rangle_D,$$

since $L_{\langle n, y \rangle}$ is contained in the adjoint of $L_{\langle m, x \rangle}$. Thus $c_r z$ and $c_r^{\#} w$ converge to $m \langle x, z \rangle_D$ and $n \langle y, w \rangle_D$ as desired.

Q. E. D.

The above theorem is indeed a generalization of the commutation theorem for quasi-Hilbert algebras [5], since we have:

1.7. COROLLARY. – *If A is a quasi-Hilbert algebra, then its left and right regular representations generate each other's commutant.*

Proof. – If we view A as a Hilbert semi-birigged space, as in Example 1.2, then it is easily seen that axiom (v) in the definition of a quasi-Hilbert algebra is essentially the Coupling Condition.

Q. E. D.

We remark that the Coupling Condition can be reformulated in a more spatial form similar to the forms used in [14] and [16], and about to be used in the next theorem. But the reformulation seems quite cumbersome, involving taking certain linear combinations of the operators $L_{\langle m, x \rangle}$, and it does not seem to facilitate the application considered in the next section.

We turn next to showing that if X is, in fact, a Hilbert C-D-birigged space, then the Coupling Condition is automatically satisfied, so that we obtain a generalization of the commutation theorem for Hilbert algebras ([5], [11]). To do this we first consider another coupling condition which is closer in form to the coupling condition used in [14]. This new coupling condition may conceivably also be useful in other situations, but it does not seem to hold in general in Hilbert semibirigged spaces in which the commutation property holds. I do not have an example to show this, but I suspect one can be found among the Hilbert semibirigged spaces which will be considered in the next section.

1.8. SECOND COMMUTATION THEOREM. — *Let X be a Hilbert semi-C-D-birigged space which also satisfies axiom* 5 *of Definition* 1.1. *Suppose that the following condition is satisfied:*

Second Coupling Condition:

If $m, n \in \overline{X}$ and $x, y \in X$, and if

(1) $$[m, cx] = +[c^{\#} y, n] \quad \text{for all } c \in C,$$

(2) $$[m, yd] = -[xd^{\flat}, n] \quad \text{for all } d \in D,$$

then $m \langle x, z \rangle_D = 0 = n \langle y, z \rangle_D$ for all $z \in X$.

If $m, n \in \overline{X}$, and $x, y \in X$, and if $[m \langle x, z \rangle_D, w] = [z, n \langle y, w \rangle_D]$ for all $z, w \in X$, then there is a sequence $\{ d_k \}$ of elements of D such that for all $z \in X$:

$$\langle y d_k, x \rangle_C z \quad \text{converges to} \quad m \langle x, z \rangle_D,$$

and

$$\langle x d_k^{\flat}, y \rangle_C z \quad \text{converges to} \quad n \langle y, z \rangle_D.$$

In particular, the Coupling Condition of Theorem 1.3 *is satisfied, so that C and D generate each other's commutants on $\overline{X}$.*

We remark that the first conclusion of this theorem says, more or less, that the (unbounded) operator $L_{\langle m, x \rangle}$ can be approximated in the strong-$\star$-operator topology [17] from X (not $\overline{X}$) by operators of the form $\langle y d, x \rangle_C$ for $d \in D$.

Proof. — The proof is similar to the first part of the proof of the commutation theorem in [14]. Let $\overline{X}^{\sim}$ denote $\overline{X}$ but with complex conjugate structure (p. 9 of [5]), and form the Hilbert space $\overline{X} \oplus \overline{X}^{\sim}$. If $z \in \overline{X}$, then $\tilde{z}$ will denote z viewed as an element of $\overline{X}^{\sim}$. Let x and y be given as in the statement of the theorem, and let G denote the set of pairs $(m, \tilde{n})$ in $\overline{X} \oplus \overline{X}^{\sim}$ such that

$$[m \langle x, z \rangle_D, w] = [z, n \langle y, w \rangle_D] \quad \text{for all } z, w \in X.$$

Let $H = \{ (y d, (x d^{\flat})^{\sim}); \ d \in D \}$. Because we are assuming that axiom 5 is satisfied, it is easily seen that $H \subseteq G$. Of course, G and H are submanifolds of $\overline{X} \oplus \overline{X}^{\sim}$.

Let $H^{\perp}$ denote the orthogonal complement of H in $\overline{G}$, and suppose that $(m_0, n_0) \in H^{\perp}$. Then, for all $d \in D$,

$$0 = [(m_0, n_0), (y d, (x d^{\flat})^{\sim})] = [m_0, y d] + [x d^{\flat}, n_0].$$

But since $(m_0, \tilde{n}_0) \in \bar{G}$, there is a sequence, $(m_k, \tilde{n}_k)$, of elements of G which converges to $(m_0, \tilde{n}_0)$. Then for any $z, w \in X$ a routine calculation shows that

$$[m_0, \langle w, z \rangle_C x] = [\langle w, z \rangle_C^{\#} y, n_0].$$

But we have seen that $\langle X, X \rangle_C$ (the linear span) is an ideal in C which has the identity operator in its strong operator closure (*see* the proof of Lemma 1.6), and so is strong operator dense in C. It follows that $[m_0, cx] = [c^{\#} y, n_0]$ for all $c \in C$. We can thus apply the Second Coupling Condition to conclude that

$$m_0 \langle x, z \rangle_D = 0 = n_0 \langle y, z \rangle_D \quad \text{for all } z \in X.$$

Now suppose that $(m, \tilde{n}) \in G$. Then we have

$$(m, \tilde{n}) = (m_0, \tilde{n}_0) + (m_1, \tilde{n}_1)$$

where $(m_0, \tilde{n}_0) \in H^{\perp}$ and $(m_1, \tilde{n}_1) \in \bar{H}$. In particular, there is a sequence $\{ d_k \}$ in D such that

$$(y\, d_k, (x\, d_k^{\flat})^{\sim}) \quad \text{converges to } (m_1, \tilde{n}_1).$$

Furthermore, from the previous paragraph we have

$$m_0 \langle x, z \rangle_D = 0 = n_0 \langle y, z \rangle_D \quad \text{for all } z \in X.$$

It follows that

$$m \langle x, z \rangle_D = m_1 \langle x, z \rangle_D = \lim y\, d_k \langle x, z \rangle_D,$$
$$n \langle y, z \rangle_D = n_1 \langle y, z \rangle_D = \lim y\, d_k^{\flat} \langle y, z \rangle_D,$$

for all $z \in X$, which, upon using axiom 2, gives the desired result.

Q. E. D.

1.9. THEOREM. — *Let X be a Hilbert D-C-birigged space. Then the Second Coupling Condition is automatically satisfied, so that C and D generate each other's commutants.*

Proof. — Let $m, n \in \bar{X}$ and $x, y \in X$, and suppose that they satisfy the two equations of the Second Coupling Condition. In those equations set $c = \langle y, z \rangle_C$ and $d = \langle z, x \rangle_D$, where z is an arbitrary element of X. Then we obtain

$$[m, \langle y, z \rangle_C x] = [\langle z, y \rangle_C y, n],$$
$$[m, y \langle z, x \rangle_D] = -[x \langle x, z \rangle_D, n]$$

for all $z \in X$. Using axiom 2 and rearranging, we find that

$$[z\langle y, y\rangle_D, n] = [m, y\langle z, x\rangle_D] = -[\langle x, x\rangle_C z, n]$$

for all $z \in X$. By continuity this is also true for $z = n$, that is,

$$[n\langle y, y\rangle_D, n] = -[\langle x, x\rangle_C n, n].$$

Since the operators $\langle x, x\rangle_C$ and $\langle y, y\rangle_D$ are assumed to be non-negative, it follows that

$$[n\langle y, y\rangle_D, n] = 0.$$

Now if D is viewed as a pre-$C^\star$-algebra with the $C^\star$-norm as operators on $\bar{X}$, then we see that X with the right action of D and the D-valued inner product is a right D-rigged space in the sense of definition 2.8 of [12] (except for the density of the range of the inner-product, which is irrelevant here as in [13]). Consequently the generalized Cauchy-Schwarz inequality of Proposition 2.9 of [12] is applicable, so that for any $w \in X$ we have

$$\langle y, w\rangle_D \langle y, w\rangle_D^{b} \leqslant \|\langle w, w\rangle_D\| \langle y, y\rangle_D,$$

as an inequality for positive operators. Thus

$$[n\langle y, w\rangle_D, n\langle y, w\rangle_D] \leqslant \|\langle w, w\rangle_D\| [n\langle y, y\rangle_D, n] = 0,$$

so that $n\langle y, w\rangle_D = 0$ for all $w \in X$. Now for any $z \in X$:

$$[m\langle x, z\rangle_D, w] = [z, n\langle y, w\rangle_D] = 0$$

so that $m\langle x, z\rangle_D = 0$ for all $z \in X$. Thus the conclusion of the Second Coupling Condition is satisfied.

Q. E. D.

That the question of when two algebras of operators generate each other's commutants can be fairly delicate may be seen by considering the "factorizations which are not coupled factors" found by MURREY and von NEUMANN (sections 3.1 and 13.4 of [9]). This question is closely related to the subject of normalcy in von Neumann algebras (*see* references in [21]).

2. Generalized Commutation Relations

In this section we show how to apply the commutation theorem of the last section to obtain Takesaki's generalized commutation relations.

Some of the formulas we introduce are suggested by the developments in [12] and [15].

Let G be a locally compact group, and let H and K be closed subgroups of G. We equip G, H and K with left Haar measures, whose modular functions we denote by Δ, δ, and δ_K, respectively. Let $C_c(G)$ denote the space of continuous complex-valued functions of compact support on G.

Let G/H and $K\backslash G$ denote the left and right homogeneous spaces with respect to H and K respectively. Our notation will not distinguish between the points of G and the corresponding cosets, because our notation will not distinguish between functions on G/H or $K\backslash G$ and the corresponding functions on G which are constant on cosets. Now K acts as a transformation group by left translation on G/H, while H acts as a transformation group by right translation on $K\backslash G$. If we view $C_c(G/H)$ and $C_c(K\backslash G)$ as pre-$C^\star$-algebras under pointwise multiplication, then the above actions define an action of K as a group of automorphisms of $C_c(G/H)$ and an action of H as a group of automorphisms of $C_c(K\backslash G)$. We can then form the corresponding transformation group algebras as in [7] and [18]. Specifically, let $C = C_c(K \times G/H)$ with product defined as in 3.3 of [7] by

$$\Pi \star \Sigma(p, x) = \int_K \Pi(q, x)\Sigma(q^{-1}p, q^{-1}x)\,dq$$

for Π, $\Sigma \in C$, $p \in K$, $x \in G/H$, and define an involution on C as in 3.5 of [7] by

$$\Pi^{\#}(p, x) = \overline{\Pi}(p^{-1}, p^{-1}x)\,\delta_K(p^{-1}).$$

Similarly, let $D = C_c(H \times K\backslash G)$ with product defined by

$$\Omega \star \chi(s, x) = \int_H \Omega(t, x)\chi(t^{-1}s, xt)\,dt$$

for $\Omega, \chi \in D$, $s \in H$, $x \in K\backslash G$, and with involution defined by

$$\Omega^\star(s, x) = \overline{\Omega}(s^{-1}, xs)\,\delta(s^{-1}).$$

Let X denote $C_c(G)$ equipped with its usual inner-product,

$$[f, g] = \int_G f(x)\overline{g}(x)\,dx$$

for $f, g \in X$. Left translation on G by elements of K gives a unitary representation of K on the pre-Hilbert space X, while pointwise multiplication on X by elements of $C_c(G/H)$ gives a representation of $C_c(G/H)$ on X. These two representations together give a covariant representation [18] of the pair $(K, C_c(G/H))$, and so define a $\star$-representation of C on X, defined (following 3.20 of [7], or [18]) by

$$(\Pi \star f)(x) = \int_K \Pi(p, x) f(p^{-1} x)\, dp.$$

Let $\overline{X}$ denote the Hilbert space completion of X. Then routine arguments show that the von Neumann algebra generated on $\overline{X}$ by the representation of C is the same as the von Neumann algebra $M(K, G/H)$ defined in the introduction of this paper. Similarly we define an action of D on the right of X by

$$f \star \Omega(x) = \int_H f(xt^{-1}) \Delta(t^{-1}) \Omega(t, xt^{-1})\, dt.$$

We do not expect this action to preserve the involutions in general, since the action of H by right translation on X need not be unitary if G is not unimodular. But routine calculations show that this action is by bounded operators (since Ω has compact support), and that it commutes with the left action of C on X. Furthermore, if the adjoint of the action of an element Ω of D is calculated, it is easily seen to be given by the action of the element $\Omega^\flat$ of D defined by

$$\Omega^\flat(t, x) = \Delta(t) \Omega^\star(t, x).$$

Thus, if we define a new involution, $^\flat$, on D by this formula, we do obtain a $\star$-antirepresentation of D. As with C, routine arguments show that the von Neumann algebra generated on $\overline{X}$ by D is the same as the von Neumann algebra $M(H, K\diagdown G)$ defined in the introduction of this paper. Finally, it is easily seen that the representations of C and D on X are faithful. Thus axiom 1 in the definition of a Hilbert semi-birigged space is satisfied.

In 7.8 of [12] a sesquilinear form on X with values in $C_c(G \times G/H)$ was defined. If the values of this form are restricted to $K \times G/H$ we obtain a sesquilinear form on X with values in $C_c(K \times G/H)$, defined by

$$\langle f, g \rangle_C(p, x) = \int_H f(xt) g^\star(t^{-1} x^{-1} p)\, dt$$

for $f, g \in X$, where $g^\star(x) = \overline{g}(x^{-1}) \Delta(x^{-1})$ as usual. (The restriction map from $C_c(G \times G/H)$ to $C_c(K \times G/H)$ is essentially a generalized conditional expectation, as discussed in [15].) If axiom 2 in the definition of a Hilbert semi-birigged space is to be satisfied, this determines what the D-valued sesquilinear form must be. A straight forward calculation shows that we must set

$$\langle g, h \rangle_D (t, x) = \gamma(t)^2 \int_K g^\star (x^{-1} p)\, h(p^{-1} xt)\, dp$$

for $g, h \in X$ where $\gamma(t) = (\Delta(t)/\delta(t))^{1/2}$ as in 4.2 of [12], and that with this definition axiom 2 indeed holds. Similar formulas are used in [15].

We now consider axiom 3. A simple calculation shows that for $f, g \in X$ we have

$$(\langle f, g \rangle_C)^\# (p, x) = \beta(p)^2 \langle g, f \rangle_C (p, x),$$

where β is defined on K by $\beta(p) = (\Delta(p)/\delta_K(p))^{1/2}$, as in [15]. As shown on page 56 of [3] (or in [4]), we can find a strictly positive continuous function, ρ, on G such that $\rho(px) = \beta(p)^2 \rho(x)$ for all $p \in K$ and $x \in G$. Considering $\rho(p^{-1} x)$, we find that

$$\beta(p)^2 = \rho(x)/\rho(p^{-1} x).$$

A simple calculation using this shows that

$$\beta(p)^2 \langle g, f \rangle_C (p, x) = \langle \rho g, f/\rho \rangle_C (p, x).$$

Thus axiom 3 holds. We remark that $\langle , \rangle_C$ could have been made symmetric by introducing the factor β into its definition, as is done in [15], but this would complicate later calculations, and besides, it seems somewhat interesting that such symmetry is not of importance for our present purposes.

We consider next axiom 4. Now in [12] it was shown that the linear span of the range of the inner-product having values in $C_c(G \times G/H)$ is dense in $C_c(G \times G/H)$ for the inductive limit topology. The proof consists of lemma 7.10 and proposition 7.11 of [12] together with the easily seen fact that this range is an ideal in $C_c(G \times G/H)$, but the proof uses no other parts of [12]. Now $\langle , \rangle_C$ is obtained by restricting to $K \times G/H$ the elements of the range of this inner-product from [12]. It follows immediately that the linear span of the range of $\langle , \rangle_C$ is dense in C in the inductive limit topology. From this it follows that this linear span contains an approximate identity for the inductive limit topology of approximately the analogue of that described in lemma 7.10 of [12].

From this it is easily seen that $\langle X, X \rangle_C X$ is dense in X in the inductive limit topology, and so in norm. Thus axiom 4 holds. We remark that a somewhat similar argument is given in lemma 2.5 of [15].

We have thus verified:

2.1. PROPOSITION. — *If G is a locally compact group, H and K are closed subgroups, and if X, C, D and their actions are defined as above, then X is a Hilbert semi-C-D-birigged space.*

It is natural to ask at this point under what conditions X will in fact be a Hilbert C-D-birigged space. From the calculation made above in the verification of axiom 3 it is clear that in order for axiom 5 to hold we must have $\beta \equiv 1$, which is just the condition under which there exists an invariant measure on $K \backslash G$. A small calculation shows that similarly we need $\Delta(s)/\delta(s) \equiv 1$ for $s \in H$ so that there is an invariant measure on G/H, and that we need $\Delta(s) \equiv 1$ for $s \in H$, so that the action of H on X is unitary. (These last two requirements imply that H must be unimodular.) Under these conditions axiom 5 will hold. Axiom 6 will then hold also, as can be shown by imitating the proof of theorem 4.4 (due to BLATTNER) of [12]. Thus under these conditions we can already conclude from Theorem 1.8 that C and D generate each other's commutants.

To show that C and D generate each other's commutants in general, which is Takesaki's generalized commutation relation, it suffices to show that the Coupling Condition of Theorem 1.3 holds. Thus suppose that $m, n \in \overline{X} = L^2(G)$, that $f, g \in X$, and that these satisfy

$$[m \langle f, h \rangle_D, k] = [h, n \langle g, k \rangle_D]$$

for all $h, k \in X$. Initially $m \langle f, h \rangle_D$ is defined only as the extension to m by continuity of the operator $\langle f, h \rangle_D$ on X. What we will show is that actually the operator $h \mapsto m \langle f, h \rangle_D$ is an integral operator defined by a kernel function on $K \times G/H$ whose adjoint is the kernel function for the operator $k \mapsto n \langle g, k \rangle_D$. We will then show that we can approximate these kernel functions in a suitable way by elements of C.

To describe the class of kernel functions we need to consider, we make the following comments. Let Π be a function on $K \times G/H$ which has σ-compact support, and which, when viewed as a function on $K \times G$, is locally integrable for the product Haar measure. Then in particular, if $h, k \in C_c(G)$, the function

$$(p, x) \mapsto \Pi(p, x) h(p^{-1} x) k(x)$$

is integrable, and so we can apply Fubini's theorem to conclude that

$$x \mapsto k(x) \int_K \Pi(p, x) h(p^{-1} x) \, dp$$

is integrable. Since k is arbitrary, it follows that the function $\Pi \star h$ defined by

$$(\Pi \star h)(x) = \int_K \Pi(p, x) h(p^{-1} x) \, dp$$

is determined except on a locally null set, and is locally integrable. Actually, a simple calculation shows that the support of $\Pi \star h$ is σ-compact. Thus it makes sense to ask whether $\Pi \star h$ is in $L^2(G)$. For Π of the kind we are considering, define $\Pi^{\#}$ by

$$\Pi^{\#}(p, x) = \overline{\Pi}(p^{-1}, p^{-1} x) \delta_K(p^{-1}).$$

Then it is readily seen that $\Pi^{\#}$ is again a function on $K \times G/H$ which is locally integrable as a function on $K \times G$. Finally, if we let $|\Pi|$ denote the absolute value of Π, it is readily seen that $|\Pi|$ has the same properties, and that $|\Pi^{\#}| = |\Pi|^{\#}$.

2.2. *Notation.* — Let Q denote the linear space of all those functions, Π, on $K \times G/H$ of σ-compact support, which are locally integrable on $K \times G$, and which have the property that for any $h \in C_c(G)$ both $|\Pi| \star h$ and $|\Pi|^{\#} \star h$ are in $L^2(G)$.

It is clear that if $\Pi \in Q$, then for any $h \in C_c(G)$ both $\Pi \star h$ and $\Pi^{\#} \star h$ will be in $L^2(G)$. If also $k \in C_c(G)$, then

$$(p, x) \to \Pi(p, x) h(p^{-1} x) k(x)$$

is integrable. By a small calculation involving Fubini's theorem, we conclude that

$$[\Pi \star h, k] = [h, \Pi^{\#} \star k].$$

Thus Π and $\Pi^{\#}$ define (probably unbounded) operators on $L^2(G)$, each of which is contained in the adjoint of the other. It is not difficult to show that the closures of these operators are affiliated with D', although we will not need this fact. But this situation should be compared with the part of the proof of Theorem 1.3 in which the necessity of the Coupling Condition was shown. We remark finally that if two functions in Q define the same operator on $L^2(G)$, then they differ only on a locally null set. In particular, if $\Sigma \in Q$ and if the operator defined by Σ is contained in the adjoint of Π, then Σ agrees with $\Pi^{\#}$ except on a locally null set.

Now initially $m\langle f, h\rangle_D$ is defined only by extending the operator $\langle f, h\rangle_D$ by continuity. We need to show that actually it is defined by "convolving" with $\langle f, h\rangle_D$. For this purpose, we can use any element, Ω, of D instead of $\langle f, h\rangle_D$. If $k \in C_c(G)$, then the function

$$(t, x) \mapsto k(xt)\, m(x)\, \Omega(t, x)$$

is easily seen to be integrable, so that Fubini's theorem can be applied to conclude, after making a translation, that

$$x \mapsto k(x)\int_H m(xt^{-1})\,\Delta(t^{-1})\,\Omega(t, xt^{-1})\,dt$$

is integrable for all k. Thus the function $m \star \Omega$ defined by

$$(m\star\Omega)(x) = \int_H m(xt^{-1})\,\Delta(t^{-1})\,\Omega(t, xt^{-1})\,dt$$

is locally integrable. It also clearly has σ-compact support. Furthermore, another calculation using Fubini's theorem shows that for any $k \in C_c(G)$:

$$\int_G (m\star\Omega)(x)\,\bar{k}(x)\,dx = [m, k\star\Omega^{\flat}] = [m\Omega, k],$$

where $m\Omega$ is defined by extending the operator corresponding to Ω by continuity. It follows both that $m \star \Omega$ is in $L^2(G)$, and that it equals $m\Omega$.

We find next the kernel of the operator $h \mapsto m\langle f, h\rangle_D$. If $F \in C_c(K\times G)$, then it is easily verified that the function on $G\times K\times H$ defined by

$$(x, p, t) \mapsto F(p, xt^{-1})\, m(x)\, (\Delta\bar{f})(p^{-1}x)\,\Delta(t^{-1})$$

is integrable. Applying Fubini's theorem and translating as before, we find that the function $\langle m, f\rangle_C$ defined on $K\times G/H$ by

$$\langle m, f\rangle_C(p, x) = \int_H m(xt)\, f^{\star}(t^{-1}x^{-1}p)\,dt \tag{2.3}$$

is locally integrable on $K\times G$ and has σ-compact support in $K\times G/H$. It follows from the discussion before 2.2 that if $h \in C_c(G)$, then $\langle m, f\rangle_C \star h$ is locally integrable. If $k \in C_c(G)$ and we let F just above be $h(p^{-1}x)\,k(x)$, then a routine calculation using Fubini's theorem and the results of the previous paragraph shows that

$$\int_G (\langle m, f \rangle_C \star h)(x) \bar{k}(x)\, dx = [m \star \langle f, h \rangle_D, k].$$

It follows that $\langle m, f \rangle_C \star h$ is in $L^2(G)$ and agrees with $m \star \langle f, h \rangle_D$ a. e. Since it is easily seen that

$$|\langle m, f \rangle_C| \leqslant \langle |m|, |f| \rangle_C,$$

the above considerations imply as well that $|\langle m, f \rangle_C| \star h$ is in $L^2(G)$ for any h in $C_c(G)$.

Now $\langle n, g \rangle_C$ is defined in the same way, and the Coupling Condition relation implies that $\langle n, g \rangle_C$ agree a. e. with $\langle m, f \rangle_C^{\#}$. Since, as above, $|\langle n, g \rangle| \star h$ is in $L^2(G)$ for any h in $C_c(G)$, it follows that $|\langle m, f \rangle_C^{\#}| \star h$ is in $L^2(G)$ for any $h \in C_c(G)$. We have thus obtained:

2.4. LEMMA. — *If $m, n \in \bar{X}$ and $f, g \in X$, and satisfy the relation of the Coupling Condition, then the functions $\langle m, f \rangle_C$ and $\langle n, g \rangle_C$ defined as in* 2.3 *are in Q, are each other's adjoints, and are the kernel functions for the operators*

$$h \mapsto m \langle f, h \rangle_D \quad \textit{and} \quad k \to n \langle g, k \rangle_D.$$

Thus the Coupling Condition will be verified once we have shown:

2.5. LEMMA. — *Let $\Pi \in Q$ and let $h, k \in X$. Then there is a sequence, $\{\Pi_j\}$, of elements of C such that*

$$\Pi_j \star h \quad \textit{converges to } \Pi \star h,$$

and

$$\Pi_j^{\#} \star k \quad \textit{converges to } \Pi^{\#} \star k.$$

Proof. — First find an increasing sequence, $\{E_i\}$, of compact subsets of $K \times G/H$ whose union contains the support of Π. Define Σ_i to have value $\Pi(p, x)$ if $(p, x) \in E_i$ and $|\Pi(p, x)| \leqslant i$, and value 0 otherwise. Thus $|\Sigma_i| \leqslant |\Pi|$, and Σ_i converges to Π a. e. on $K \times G$. It is easily seen that, in addition, $|\Sigma_i^{\#}| \leqslant |\Pi^{\#}|$ and that $\Sigma_i^{\#}$ converges to $\Pi^{\#}$ a. e. Then several applications of the Lebesgue dominated convergence theorem show that

$$\Sigma_i \star h \quad \text{converges to } \Pi \star h$$

and

$$\Sigma_i^{\#} \star h \quad \text{converges to } \Pi^{\#} \star k$$

in $L^2(G)$.

Thus it suffices to prove the Lemma when Π is bounded and of compact support in $K \times G/H$. In this case we can find $\Sigma \in C$ such that $|\Pi| \leqslant \Sigma$, and then we can find a sequence, $\{\Pi_j\}$, of elements of C such that Π_j converges to Π a. e. and $|\Pi_j| \leqslant \Sigma$ for all j. The $\Pi_j^{\#}$ will have similar properties. Once again several applications of the Lebesgue dominated convergence theorem show that

$$\Pi_j \star h \quad \text{converges to } \Pi \star h,$$

and

$$\Pi_j^{\#} \star k \quad \text{converges to } \Pi^{\#} \star k,$$

as desired.

Q. E. D.

This concludes the proof of the following theorem.

2.6. THEOREM (*Takesaki's generalized commutation relations*). — *With notation as above, C and D generate each other's commutants on $\bar{X}$.*

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