DAN BURGHELEA
REINHARD SCHULTZ

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ON THE SEMISIMPLE DEGREE OF SYMMETRY

BY

DAN BURGHELEA and REINHARD SCHULTZ [1]

[Bucuresti], [Lafayette, Ind.]

ABSTRACT. — One defines the semisimple degree of symetry $S_s(M)$ for a compact manifolds M^n, as the heighest dimension of all compact semisimple Lie groups acting effectively on M^n; one recognise the manifolds M^n with small $S_s(M^n)$ in terms of the cup-length.

RÉSUMÉ. — On introduit le degré semisimple de symétrie $S_s(M)$ d'une variété compacte M, comme la plus grande dimension des groupes de Lie compacts et semisimples qui agissent effectivement sur M^n; on donne des conditions en termes de longueur des cup-produits permettant de majorer $S_s(M)$.

For any compact topological (differentiable) manifold one defines the topological (differentiable) *degree of symmetry* as in [3] to be $S(M) = \sup \{ \dim G;\ G \text{ compact Lie group acting topologically and effectively on } M^n \}$ (resp. $S_d(M)$), and the *semisimple degree of symmetry* is similarly defined: $S^s(M) = \sup \{ \dim G;\ G \text{ compact semi simple Lie group acting topologically and effectively on } M \}$ (resp., $S^s_d(M)$).

If M^n is differentiable, then $S^s_d(M^n) \leqslant S^s(M^n)$ and $S_d(M^n) \leqslant S(M^n)$. By [7] (p. 243), $0 \leqslant S^s(M^n) \leqslant S(M^n) \leqslant n(n+1)/2$. These numbers are further related as follows:

PROPOSITION 1. — $S(M^n) - S^s(M^n) \leqslant n$ *(resp.* $S_d(M^n) - S^s_d(M^n) \leqslant n$*), and if* $S(M^n) - S^s(M^n) = n$ *then* M^n *is homeomorphic (diffeomorphic) to the torus* T^n.

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This is a consequence of the following three observations:

1° Every compact connected Lie group has a finite cover of the form $S \times T^k$ with S semisimple.

2° If a group acts almost effectively, so does every subgroup.

3° It T^k acts almost effectively on a connected manifold M, the principal orbit theorem implies that almost all orbits are k-dimensional and (hence) $\dim M \geqslant k$.

The following facts suggest some differential-geometric interest in the computation of $S^s(M^n)$:

(*a*) If M^n is a differentiable manifold with $S^s_d(M^n) = 0$ then for any riemannian metric on M^n two infinitesimal isometries X, Y have $[X, Y] = 0$ (for the connected component of the isometry group is a torus).

(*b*) If $S^s_d(M^n) \neq 0$, M^n admits à riemannian metric with positive scalar curvature.

Part (*a*) is obvious, and Part (*b*) is due to Lawson and Yau [5]. The purpose of this note is to give simple criteria in terms of rational (or real) cup length to estimate $S^s(M^n)$, and in particular recognize manifolds with $S^s(M^n) = 0$. These criteria are furnished by proposition 2. The cup length criteria parallel those of [3] (theorem 2), but the results obtained here are somewhat different.

DEFINITION. — We say M^n *has abelian symmetry* if $S^s(M^n) = 0$, and M^n *has strong abelian symmetry* if $S^s(M^n) = 0$ and for any effective action of a compact Lie group the isotropy groups are finite.

THEOREM A. — *Let M^n be a compact connected manifold and suppose there exist $W_1, \ldots, W_n \in H^1(M; R)$ with $W_1 \cup \ldots \cup W_n \neq 0$. Then $S^s(M^n) = 0$; moreover M^n has strong abelian symmetry.*

Theorem A generalizes some results of S.-T. Yau [9], who proved the result for smooth actions (using harmonic forms) and for topological S^1 actions having fixed points (using the Gysin-Smith sequence [2]). Proofs of Theorem A have also been independently obtained by J.-P. BOURGUIGNON and F. RAYMOND.

Example. — $T^n \# M^n$ (M^n any orientable manifold) has strong abelian symmetry. Thus n even and $\chi(M \# T^n) \neq 0$ imply $S(M \# T^n) = 0$.

THEOREM B. — *Let M^n be a compact connected manifold and suppose there exist $W_1, \ldots, W_{n-2} \in H^1(M; R)$ and $C \in H^2(M; R)$ with W_1 $W^1 \cup \ldots \cup W_{n-2} \cup C \neq 0$ and*

$$(\star) \qquad H^*(M; R) \neq H^*(S^i; R) \otimes B, \qquad i = 2,3,$$

with B a Poincaré duality cohomology algebra. Then $S^n(M^n) = 0$.

In the differentiable category we can replace $(\star)$ by

$(\star\star)$ *At least one rational characteristic number is nonzero.*

This also true for locally smooth actions [2], but we shall not prove this generalization.

Example. — $T^{n-2} \times S^2 \# M^n$ with M^n an oriented manifold and $H^*(M^n; R) \neq H^*(S^n; R)$ obviously satifies the hypotheses and has *abelian symmetry* but not necessarily *strong abelian symmetry.* For example $T^{n-2} \times S^2 \# T^{n-2} \times S^2$ carries an effective S^1-action with nonempty fixed point set. To construct this action, let S^1 act on $T^{n-2} \times S^2$ *via* $1 \times \mu$, where μ is the standard linear action of $S^1 = SO_2$ on S^2; since the fixed point set of this actions is $T^{n-2} \times \{\pm N\}$, where N is the north pole of S^2, we can form the equivariant connected sum of two copies of this action at $(1, N)$. This gives a smooth S^1-action on $T^{n-2} \times S^2 \# T^{n-2} \times S^2$ which has fixed points.

THEOREM C. — *Let M^n be a compact connected manifold. Furthermore, assume $\chi(M^n)$ is odd and there exists $W_1, \ldots, W_{n-3} \in H^1(M; R)$ and $C \in H^3(M; R)$ with $W_1 \cup \ldots \cup W_{n-3} \cup C \neq 0$.* Then $S^s(M) = 0$.

Proofs. — The proofs of A, B, and C, are pleasant consequences of the following:

PROPOSITION 2 : [*(a) Let $\mu : S^1 \times M^n \to M^n$ be a nontrivial action on a connected manifold and suppose that for any $x \in M^n$,*

$$\alpha_x : S^1 \to \mu(S^1, x) \subseteq M^n$$

induces the trivial homomorphism $H^1(M; R) \xrightarrow{\alpha_x^*} H^1(S^1; R)$. Then for any $W_1, \ldots, W_n \in H^1(M; R)$, we have $W_1 \cup \ldots \cup W_n = 0$.

(b) Let G be a compact semisimple Lie group and $\mu : G \times M^n \to M^n$ be an action on the connected manifold M^n. Let $k = \dim G/H_0$ where H_0 is the minimal isotropy group. Assume that, for any point $x \in M^n$, $\bar{\alpha}_x : G/G_x \to M$ induces the trivial homomorphism $H^k(M; R) \to H^k(G/G_x; R)$ then, for any $W_1, \ldots, W_{n-k} \in H^1(M; R)$ and $C \in H^k(M; R)$, we have

$$W_1 \cup \ldots \cup W_{n-k} \cup C = 0.$$

COROLLARY 3. — *The conclusions of proposition* 2 *hold if, for one* x, *the map* $\bar{\alpha}_x : G/G_x \to \mu(G, x) \subseteq M$ *induces the trivial homomorphism* $\bar{\alpha}_x^* : H^k(M; R) \to H^k(G/G_x; R)$. *In particular, this is true if* μ *has orbits of different dimensions.*

The corollary follows immediately using the connectedness of M, the principal orbit theorem, and the existence of slices (*see* [1], [2]).

We shall first derive theorems A-C from proposition 2 and corollary 3, after which we shall prove proposition 2.

Proof of theorem A. — If G is a compact semisimple Lie group, and $\mu: G \times M^n \to M^n$ is an action, then μ restricted to any $S^1 \times M^n \to M^n$ (S^1 is a subgroup in G) is trivial by proposition 2; for the homomorphism $\alpha_x^*: H^1(M; R) \to H^1(S^1; R)$ factors through $H^1(G; R)$, which is zero. Furthermore, if G is a torus then all orbits have the same dimension by corollary 3, and hence all isotropy subgroups must be finite if G is effective.

Proof of theorem B. — Because any compact semisimple Lie group contains a 3-dimensional compact semisimple Lie subgroup, it is enough to prove that any $\mu: S^3 \times M^n \to M^n$ is trivial provided M^n satisfies the hypotheses of theorem B.

If the action has at least two orbits of different dimension, then by proposition 2 the action has to be trivial. So assume that all the orbits of the action $\mu : S^3 \times M^n \to M^n$ are 2-dimensional. *If* $\bar{\alpha}_x : S^3/S_x^3 \to M^n$ induces the trivial homomorphism $\bar{\alpha}_x : H^2(M^n; R) \to H^2(S^3/S_x^3; R)$ the action is trivial by proposition 2. If not $\bar{\alpha}_x^* : H^*(M^n; R) \to H^*(S^3/S_x^3; \mathrm{R})$ is surjective for all x, and every isotropy subgroup is conjugate to S^1. In this case, $M \to M/S^3$ is a sphere bundle whose fiber is totally nonhomologous to zero. Therefore

$$H^*(M; R) = (S^3/S_x^3; R) \otimes H^*(M/S^3; R) = H^*(S^2; R) \otimes H(M/S^3; R).$$

which by ($\star$) is not possible. Also, M bounds $M \times_{SO_3} D^3$, which contradicts ($\star\star$). In the case when all orbits have dimension 3, a similar analysis (2) shows

$$H^*(M; R) = H^3(S^3; R) \otimes H^*(M/S^3; R)$$

contradicting ($\star$). If M is differentiable, since M/S^3 is suitably triangu-

(2) Although $M\ W\ M/S^3$ is not a fiber bundle, it behaves in real cohomology as if it were one since the orbits are all real cohomology spheres.

lable it follows that the mapping cyclinder of $f : M \to M/S^3$ is a triangulated rationnal cohomology manifold with boundary M. Hence all rationnal characteristic numbers of M must vanish; but thiscont radicts ($\star\star$). Hence μ must be trivial.

Proof of theorem C. – Let $\mu : S^3 \times M^n \to M^n$ be an action. If μ has a 3-dimensional orbit, then by the cup product hypothesis proposition 2 and corollary 3 all orbits must be 3-dimensional. But in this case, the Leray spectral sequence implies $\chi(M) = \chi(M/S^3)\,\chi(S_3) = 0$ as before; since $\chi(M)$ is odd, μ has no 3-dimensional orbits.

Since all orbits have dimension 0 or 2, the classification of subgroups of S^3 implies that the S^3 action reduces to an SO_3 action, each orbit of which is either a fixed point, RP^2, or S^2. If μ is nontrivial, the orientability of M^n implies that S^2 is the principal orbit type. On the other hand, if μ is nontrivial there must also be nonprincipal orbits; for otherwise, M would be an S^2 bundle over M/SO_3, so that $\chi(M) = \chi(M/SO_3)\,\chi(S^2)$ would be even.

Let F be the fixed point set of SO_3, and let E be the union of the RP^2 orbits. We claim that

$$\chi(E; Z_2) = \chi(F; Z_2) = 0,$$

and F contains no limit points of E (assuming μ is nontrivial). If this is true, then of course

$$\chi(M) = \chi(M; Z_2) = \chi(M, E \cup F; Z_2);$$

on the other hand, $(M, (E \cup F))$ is a relative S^2 bundle and hence $\chi(M, E \cup F; Z^2)$ is even, a contradiction since $\chi(M)$ is odd. Hence μ is trivial if the above claim is true.

If μ is a smooth action, we may prove the claim as follows: By the principal orbit theorem the local representations of SO_3 at points of F all have SO_2 as their principal isotropy subgroup. Since the only such representation (up to equivalence) is the usual one on R^3, it follows that every component of F has codimension 3 and F contains no limit points of E. Since $\chi(M^n)$ is nonzero, n must be even and $n-3$ must be odd; hence $\chi(F) = 0$. On the other hand, by the differentiable slice theorem, an RP^2 orbit has an invariant tubular neighborhood of the form $S^2 \times_{Z_2} V$ for some Z_2-representation V; let V_0 be its fixed point set. Since RP^2 is nonorientable but M^n is orientable, dim V-dim V_0 must be odd. It follows that E is also a union of closed submanifolds each having odd codimension (hence odd dimension), so that $\chi(E) = 0$.

A similar sort of argument applies to topological actions. The local analog of some results due to W.-Y. HSIANG ([4], p. 346-349) shows that F is a union of closed Z-cohomology $(n$-$3)$-manifolds, so that

$$\chi(F) = \chi(F; Z_2) = 0$$

still holds. In addition, the proof of [4] (Proposition 3, p. 348) shows that F contains no limit points of E. The topological slice theorem implies that every RP^2 orbit has a neighborhood of the form $S^2 \times_{Z_2} V$ for some Z-cohomology manifold V; let V_0 be its fixed point set. The cohomological characterization of orientability, and P.A. Smith theory again imply V_0 is a Z_2-cohomology manifold and coh dim V-coh dim V_0 is odd. Thus E is again a union of closed odd-dimensional Z_2-cohomology manifolds, so that $\chi(E; Z_2)$ is still zero. This completes the proof of theorem C.

Proof of proposition 2:

(*a*) Looking at the Leray spectral sequence $E_2^{p,q} \Rightarrow H^*(M; R)$ associated to $f : M \to M/S^1$ it is easy to see that any 1-class has filtration $\geqslant 1$ since $H^1(M; R) \to H^1(S^1/S^1_x; R)$ is trivial (because the edge map $H^k \to E_\infty^{0,k} \subseteq E_2^{0,k} = \Gamma(R^k f_* R)$ sends a class w into the section $S(x) = w \mid H^k(Gx))$. Thus the long cup product has filtration $\geqslant n$. Since $E_2^{p,q} = 0$ for $p \geqslant n-1+1 = n$, the cup product clearly vanishes (*See* [8], XIII, for the relevant multiplicative properties).

(*b*) If G is compact semisimple, the Leray spectral sequence $E_2^{p,q}$ has the line $q = 1$ consisting of trivial groups because $H^1(G/G_x; R) = 0$. Hence 1-dimensional classes in $H^1(M; R)$ have Leray spectral sequence filtration $\geqslant 1$. On the other hand, because $H^k(M; R) \to H^k(G/G_x; R)$ is zero every k-dimensional class also has filtration $\geqslant 1$ (look again at the edge map $H^k \to E_\infty^{0,k} \subseteq E_2^{0,k} = \Gamma(R^k(f_* R))$. Thus the cup product has filtration $\geqslant n-k+1$ $(n = \dim M)$. Since $E_2^{p,q} = 0$ for $p \geqslant \dim M/G+1 = n-k+1$, the cup product under consideration clearly vanishes.

Addendum on Massey products. – In [9], YAU points out the following strengthening of proposition 2 (i): If T^k acts effectively on M^n, and

$$H^1(M^n; R) \to H^1(T^k; R)$$

is trivial, then all higher order Massey products involving monomials in $H^1(M^n; R)$ are trivial provided the degrees of any two successive ones are $\geqslant n-k+1$. Our methods also yield this. For the strengthened hypothesis implies that $E_\infty^{0,1} = 0$ in the Leray spectral sequence for $M^n \to M^n/T^k$, and hence the edge map $H^1(M^n/T^k; R) \to H^1(M^n; R)$ is onto; but the corresponding Massey products in $H^*(M^n/T^k; R)$ are all defined and trivial since the latter cohomology vanishes above degree $n-k = \dim M^n/T^k$.

Of course, all Massey products in $H^n(M^n; R)$ are trivial since the indeterminacy is total by Poincaré duality. However, it does not seem that further triviality conditions for Massey products are obtainable, even when such products are always defined. For example, Poincaré duality and proposition 2 (i) imply that every product $W_1, \ldots, W_{n-1}$ is zero, and hence the Massey product

$$\langle W_1, W_2, \ldots, W_{n-1}, W_n \rangle \in H^{n-1}(M^n; R)$$

is always defined. However, one can construct M^n with a free S^1 action and a nontrivial Massey product of the above sort as follows: Let $n \geqslant 4$, and choose generators $e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}$ of $H^1(T^{n-1} \# T^{n-1}; R)$ so that the e's come from the first T^{n-1} summand, the f's come from the second, and

$$0 \neq \prod_{i=1}^{n-1} e_i = \prod_{i=1}^{n-1} f_i.$$

Take M^n to be the principal S^1 bundle with Euler class $f_1 f_2$, and let

$$\pi : M^n \to T^{n-1} \# T^{n-1}$$

be the projection. Then the Massey product

$$\langle \pi^* e_1, \pi^*(e_2, \ldots, e_{n-1}), \pi^* f_1 \rangle \in H^{n-1}(M^n; R)$$

is nontrivial; this follows easily from the fact that suitable Massey products in (E_2, d_2) of the Serre spectral sequence pass to Massey products in $H^*(M^n; R)$ (the argument of [6] (Theorem 4.1) is applicable).

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Dan Burghelea,
Institut de Mathématiques,
14, rue Academiei,
Bucuresti (Roumanie)

et

Reinhard Schultz,
Division of Mathematical Sciences
Purdue University,
Lafayette, Indiana 47907,
(États-Unis).