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SYMPLECTIC STRUCTURE
IN THE ENVELOPING ALGEBRA OF A LIE ALGEBRA

BY

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ABSTRACT. — It is shown that the enveloping algebra of a Lie algebra satisfies a condition which implies a weakened form of the Gel'fand-Kirillov conjecture. This condition leads to a generalization of a commutant property previously derived for the Weyl algebra, which has its origins in a classical theorem on function groups. This provides a dimensionality estimate which is central to a proof of the Gel'fand-Kirillov conjecture for solvable algebraic Lie algebras.

RESUME. — Il est démontré que l’algèbre enveloppante d’une algèbre de Lie satisfait une condition qui implique une forme affaiblie de la conjecture Gel’fand-Kirillov. Cette condition amène à une généralisation d’une propriété commutante précédemment dérivée pour l’algèbre de Weyl, qui a ses origines dans un théorème classique en groupes fonctionnels. Ceci fournit une estimation dimensionnelle qui est centrale pour la preuve de la conjecture Gel’fand-Kirillov pour les algèbres de Lie algébriques résolubles.

1. Introduction

Let \( g \) be a finite dimensional Lie algebra over a commutative field \( K \) of characteristic zero. Let \( Ug \) denote the enveloping algebra of \( g \) and \( Dg \) the quotient field of \( Ug \). Let \( Dn,k \) denote the quotient field of the Weyl algebra \( An,k \) of degree \( n \) over \( K \) and extended by \( k \) indeterminates. GEL’FAND and KIRILLOV ([2]-[4]) have suggested that \( Dg \) should depend rather weakly on \( g \) and for \( g \) algebraic have conjectured that \( Dg \) is isomorphic to one of the standard fields \( Dn,k \). \( An,k \) is itself related to a polynomial algebra over the Poisson bracket (essentially equivalent to a manifold with symplectic structure) which has been subjected to considerable analysis. We wish to exploit these interrelationships in studying \( Ug \). In this it is often sufficient to establish a correspondence of leading order terms. This is illustrated by Theorem 2.3, the second part of which represents a weak form of the Gel’fand-Kirillov
conjecture. The first part leads to an important dimensionality estimate contained in the theorem stated below.

Let \( g^* \) denote the dual of \( g \). To each \( f \in g \) define an antisymmetric bilinear form \( B_f \) on \( g \times g \) through

\[
B_f(x, y) = (f, [x, y]).
\]

(1.1)

Recall that \( B_f \) must have even rank and set

\[
m = \dim g, \quad n = \frac{1}{2} \sup_{f \in g^*} \text{rank } B_f.
\]

(1.2)

Let \( \dim_k \) denote the dimensionality introduced by Gel'fand and Kirillov [2]. It is shown in section 3 that:

**Theorem 1.1.** — Let \( A \) be a subalgebra of \( U g \) and denote by \( A' \) its commutant in \( U g \). Then

\[
\dim_k A + \dim_k A' \leq 2(m - n),
\]

with \( m, n \) given by (1.2).

We remark that \( \dim_k U g = \dim g \) for all \( g \). If further \( g \) is either nilpotent or semisimple \( \dim_k C(U g) = m - 2n \) (where \( C \) denotes centre). It follows that the above bound is saturated in either of these two cases. This is also true if \( g \) is solvable and algebraic. Indeed for \( g \) solvable Ngőižem [11] has constructed a maximal commutative subalgebra \( A \) of \( U g \) and it is shown in [9] by use of the above theorem that \( \dim_k A = m - n \). This equality motivated the proof of the Gel'fand-Kirillov conjecture for \( g \) solvable given in [9]. We remark that Theorem 1.1 does not follow in any obvious fashion from the truth of this conjecture. This is because the corresponding dimensionality estimates are more difficult to make in \( D g \).

2. Weighted filtrations

Let \( n, k \) be integers with \( n \) non-negative and \( k \) positive. Let \( g_{n,k} \) denote the Lie algebra over \( \mathbb{K} \) with basis \( \{ x_i, y_j, z_j; i = 1, 2, \ldots, n; j = 0, 1, \ldots, k - 1 \} \) where \( [x_i, y_j] = z_0 \) and all other brackets vanish. Let \( I \) denote the two-sided ideal in \( U g_{n,k} \) generated by \( z_0 - 1 \). Set \( A_{n,k} = U g_{n,k+1}/I \). Observe that \( U g_{n,k} \) is isomorphic to a subalgebra of \( A_{n,k} \) (divide the \( x_i \) by \( z_0 \)) and that \( D g_{n,k} = D_{n,k} \) [2].

For arbitrary \( g \), let the subspaces \( \{ U^{(i)}; i = 0, 1, 2, \ldots \} \) define a filtration of \( U g \). Set \( U_i = U^{(i)} U^{(i-1)} \) and \( G(U g) = \bigoplus_{i \geq 0} U_i \).

Only filtrations making \( G(U g) \) integral are considered.
In the remainder of this section we assume $K$ algebraically closed.

**Lemma 2.1.** — Suppose $g$ is either nilpotent or semisimple. Define $m, n$ by (1.2) and set $k = m - 2n$. Then $Ug$ admits a filtration such that $G(Ug) = U_{g_{n,k}}$.

**Proof.** — Take $g$ nilpotent. Recalling (1.1) and (1.2) choose $f \in g^*$ such that rank $B_f = 2n$. Set $g_0 = \{ x \in g; f(x) = 0 \}$.

Let $B_f$ denote the restriction of $B_f$ to $g_o$. We wish to show that rank $B_f = 2n$. Let $N_{g_0}, N_{g'}$ respectively denote the null spaces of $B_f$ and $B_f$. By [1], Lemma 5, it suffices to show that $N_{g} \subset N_{g'}$. Now given $x \in N_{g'}$,

\[(f, [x, y]) = B_f(x, y) = 0, \quad \text{for all } y \in g_0.\]

Hence $(ad x) g_0 \subset g_0$. Let $z_0 \in g$, $z_0 \notin g_0$. Since dim $g - \dim g_0 = 1$, we may write $(ad x) z_0 = x z_0 + y$, for some $x \in K$, $y \in g_0$. Then for each positive integer $r$,

\[\text{rank } B_f = 2n. \quad \text{Let } N_{B}, N_{B'} \text{ respectively denote the null spaces of } B_f \text{ and } B_f. \quad \text{By [1], Lemma 5, it suffices to show that } N_B \subset N_{B'}. \quad \text{Now given } x \in N_{B'}, \]

\[f([x, y]) = B_f(x, y) = 0, \quad \text{for all } y \in g_0.\]

Hence $(ad x) g_0 \subset g_0$. Let $z_0 \in g$, $z_0 \notin g_0$. Since dim $g - \dim g_0 = 1$, we may write $(ad x) z_0 = x z_0 + y$, for some $x \in K$, $y \in g_0$. Then for each positive integer $r$, \[
(\text{ad}^r x) z_0 = x^r z_0 + y_r; \quad y_r \in g_0.
\]

Since $g$ is nilpotent ; $x^r z_0 + y_r = 0$ for some $r$ and hence $x = 0$. It follows that $(ad x) g \subset g_0$ which implies that $x \in N_{g_0}$, as required.

Define a filtration on $Ug$ by setting $U^{(0)} = K$, $g_0 \subset U^{(1)}$, $z_0 \in U^{(2)}$, $z_0 \notin U^{(1)}$. To show that $G(Ug)$ has the asserted property it suffices to show that the generators of $g$ satisfy the commutation relations of $g_{n,k}$ in $G(Ug)$. Scale $z_0$ so that $f(z_0) = 1$. Then for all $x, y \in g$, we have

\[xy - yx = (f, [x, y]) z_0 \mod g_0\]

in $Ug$. Hence by choice of filtration we obtain, for all $x, y \in g_0$,

\[xy - yx = B_f(x, y) z_0, \quad x z_0 - z_0 x = 0\]

in $G(Ug)$. Finally bringing $B_f$ to canonical form exhibits the defining basis for $g_{n,k}$.

Take $g$ semisimple. As is well-known, $k = \text{rank } g$, and $n$ is the number of positive roots. Let $h$ be a Cartan subalgebra for $g$, and $\Delta$ the set of all non-zero roots. Each root subspace $g^\alpha$ is one-dimensional, and $g$ is a direct sum of $h$ and the $g^\alpha$; $\alpha \in \Delta$. Let $B$ denote the Killing form. To each $\alpha \in \Delta$ define $H_\alpha \in h$ (cf. [5], Theorem 4.2) through $B(H, H_\alpha) = \alpha(H)$ for all $H \in h$. Let $H_0$ be a regular element ([5], p. 137) of $h$. Then $\alpha(H_0) \neq 0$ for all $\alpha \in \Delta$. Define $f \in g^*$ through $f(H) = B(H, H_0)$ for all $H \in h$ and the condition that it vanish on each
root subspace. Set \( g_0 = \{ x \in g; f(x) = 0 \} \). For each \( \alpha \in \Delta \) choose \( E_\alpha \in g^a \) such that \( B(E_\alpha, E_{-\alpha}) = 1 \). Then through [5], Theorem 5.5,

\[
E_\alpha E_{-\alpha} - E_{-\alpha} E_\alpha = \frac{x(H_0)}{B(H_0, H_0)} H_0 \mod g_0
\]

in \( U \ g \) and all other commutators vanish mod \( g_0 \). Setting \( H_0 = z_0 \) the proof is completed with the filtration defined in the nilpotent case.

The conclusion of the Lemma fails on general \( g \). For example, consider the two dimensional (solvable) Lie algebra with relation \([x, z] = z\). Yet through [1], Lemma 5, \( \text{rank } B' = \text{rank } B_f - 2; B', B_f \) as above. It follows that we always have the weaker result, namely that

\[
G(U \ g) = U g_{n-1, k+2},
\]

for a suitable filtration of \( U \ g \).

**Lemma 2.2.** — Let \( B \) a non-degenerate antisymmetric bilinear form on \( V \times V \). Let \( a \) be a linear transformation on \( V \) such that

\[
B(ax, y) + B(x, ay) = 0
\]

for all \( x, y \in V \). Then there exists a basis \( \{ x_i; i = 1, 2, \ldots, 2 \ell \} \) for \( V \) such that

\[
B(x_i, x_{2\ell-j}) = \delta_{ij} (-1)^j,
\]

\( i, j = 1, 2, \ldots, 2 \ell \), and \( a \) is upper triangular.

**Proof.** — Recalling [10], p. 398, choose a basis \( \{ y_i \} \) for \( V \) such that

\[
y_i = x_i y_i + \gamma, y_{i+1},
\]

\( \alpha, \beta \in K \) with \( \alpha_i \leq \alpha_j \) for \( i \leq j \). Since \( B \) is non-degenerate there exists a second basis \( \{ z_i \} \) for \( V \) such that \( B(y_i, z_j) = \delta_{ij} \), for all \( i, j \). Substitution in (2.1) and (2.3) gives

\[
\alpha z_i = - \alpha_i z_i - \beta z_{2\ell-i} - z_{2\ell-1}.
\]

Let \( V_i \) denote the eigenspace belonging to eigenvalue \( \alpha_i \). By (2.2), \( B(V_i, V_j) = 0 \), unless \( \alpha_i + \alpha_j = 0 \). Further when this holds \( B \) non-degenerate implies \( \dim V_i = \dim V_j \). Let \( V_0 \) denote the zero eigenspace and \( V' \) the direct sum of the \( V_i \) omitting \( V_0 \). On \( V' \) set

\[
x_i = \begin{cases} y_i; & \alpha_i < 0, \\ (-1)^i z_{2\ell-i}; & \alpha_i > 0. \end{cases}
\]

By (2.3) and (2.4), this determines the required basis on \( V' \). It remains to determine a basis on \( V_0 \). Equivalently we can assume \( a \) of the lemma nilpotent.

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Let \( r \) be the least positive integer such that \( a^r V_0 = \{0\} \). Set

\[
W^{(i)} = \{x \in V_{i}; \ a^{i+1} x = 0\} \quad \text{and} \quad W_{i} = W^{(i)}/W^{(i-1)}.
\]

We have

\[
V_0 = \bigoplus_{i=0}^r W_i; \quad a W_i \subset W_{i-1} \quad \text{for all} \ i.
\]

Hence to prove the lemma it suffices to exhibit a basis for \( V_0 \) on which \( B \) is antidiagonal and which, for each \( i \), contains as a subbasis a basis for \( W_i \).

Set \( U = a^r V_0 \). Then \( \dim U = \dim W_r \), and by (2.2):

\[
(2.5) \quad B(U, W_i) = 0; \quad s < r.
\]

Hence there exists a basis \( \{y_i; i = 1, 2, \ldots, t\} \) for \( W_r \) and a basis \( \{y'_i; i = 1, 2, \ldots, t\} \) for \( U \) such that \( B \) is antidiagonal on their linear span \( U' \). Further since \( B \) in non-degenerate we may assume that \( B(y_i, y'_{i-j}) = (-1)^j \). Set

\[
x_j = y_j, \quad x_{t-j} = y'_{i-j}; \quad j = 1, 2, \ldots, t.
\]

Observe that \( U \subset W_0 \) and set \( V'_0 = W^{(r-1)}/U \). Let \( \{z_i\} \) be a basis for \( W^{(r-1)} \). Set

\[
z'_i = z_i - \sum_{j=1}^t (-1)^j B(y_j, z_i)y'_{i-j}.
\]

Recalling (2.5) it follows that \( B(x, z_i) = 0 \), for all \( x \in U' \) and all \( i \). On the other hand \( z'_i = z_i \) on \( V'_0 \). Induction provides the required basis. The lemma is proved.

**Theorem 2.3.** — Define \( m, n \) by (1.2) and set \( k = m - 2n \). Then \( U g \) admits a filtration such that \( G(U g) \) is isomorphic to a subalgebra of \( A_{n,k} \) and \( D(G(U g)) = D_{n,k} \).

**Proof.** — Let \( f, B_f, B'_f, g_0, N_B, N_B' \), be as in the proof of lemma 2.1. Given rank \( B'_f = \text{rank} \ B_f \), the conclusion of lemma 2.1 holds and the theorem follows easily. Otherwise by [1], Lemma 5, \( N_B \) is of codimension 1 in \( N_B' \). Choose \( x \in N_B, \ x \notin N_B' \). Then as before (ad \( x \)) \( g_0 \subset g_0 \) and given \( z_0 \in g, \ z \notin g_0 \),

\[
(\text{ad} \ x) z_0 = z_0 + y; \quad y \in g_0; \quad x \neq 0.
\]

By definition of \( B_f \) and the Jacobi identity:

\[
(2.6) \quad B_f([x, y], z) = B_f([x, z], y) + B_f(x, [y, z])
\]

for all \( x, y, z \in g \). Choosing \( x \) as above, \( y \in N_B, \ z \in g_0 \), it follows from (2.6) that (ad \( x \)) \( N_B \subset N_B' \).

Suppose that there exists \( z \in N_B' \), such that (ad \( x \)) \( z = \beta z, \ \beta \neq x, 0 \).
Then for all $\gamma \in K$,

$$
(2.7) \quad (\text{ad } x)(z + \gamma z_0) = \beta (z + \gamma z_0) + (x - \beta) \gamma z_0 + \gamma y.
$$

Since $\beta \neq 0$, $x, z$ are linearly independent. Let $\{x_i\}$ be a basis for $g_0$ with $x_1 = z$, $x_2 = x$. Set $g_0(\gamma) = \text{lin span} \{z + \gamma z_0, x_1, x_2, \ldots, x_{n-1}\}$. Define $f_\gamma \in g^*$ such that

$$
g_0(\gamma) = \{x \in g; f_\gamma(x) = 0\}.
$$

Denote by $g_{00}$ a maximal subspace of $g_0$ on which $B^r$ is non-degenerate. Since $z \in N_B$, we may choose a fixed $g_{00}$ such that $g_{00} \subset g_0(\gamma)$ for all $\gamma$. Let $B_{f_\gamma}^r$ denote the restriction of $B_{f_\gamma}^r$ to $g_{00} \times g_{00}$. By choice of $g_{00}$,

$$
\text{rank } B_{f_\gamma}^r = \text{rank } B_f^r = \text{rank } B_f - 2.
$$

Hence except for finitely many values of $\gamma$,

$$
\text{rank } B_{f_\gamma}^r = \text{rank } B_f - 2.
$$

Since $x \neq \beta$, it follows by [1], Lemma 5 and the above that we may choose $\gamma$ such that $\text{rank } B_{f_\gamma}^r = \text{rank } B_f^r$. Then the conclusion of Lemma 2.1 holds and the theorem is proved in this case. We conclude that there is no loss of generality in assuming that $N_B$ admits a basis $\{z_i\}$ such that

$$
(2.8) \quad (\text{ad } x) z_i = x_i z_i + \beta_i z_{i+1},
$$

where $x_i = x$, $0$, for all $i = 1, 2, \ldots, k$, with $x = z_k$.

Set $V = g_0/N_B$, and let $B$ denote the restriction of $B_f$ to $V$. Use of (2.6) shows that $\text{ad } x - (x/2)$ is a linear transformation on $V$ satisfying (2.1). Further $B$ is non-degenerate on $V$, so Lemma 2.2 applies. Let $\{x_i; i = 1, 2, \ldots, 2l\}$ be a basis satisfying its conclusion. Since $\text{rank } B = \text{rank } B_f - 2$, we have $l = n - 1$. Define a filtration on $U g$ as follows.

Set $U^{(0)}$ equal the tensor algebra generated by $x$. Let

$$
\begin{align*}
z_0 & \in U^{(0)}, & z_0 & \in U^{(0)-1}, \\
x_i & \in U^{(2m+n-i-1)}, & x_i & \in U^{(2m+n-i-2)}, \\
z_j & \in U^{(2m-n+i-1)}, & z_j & \in U^{(2m-n-1)};
\end{align*}
$$

$$
i = 1, 2, \ldots, 2n - 2; \quad j = 1, 2, \ldots, k - 1.
$$
Recalling that $k = m - 2n$ and that ad $x$ is upper triangular on $V$ and on $N_B$, we obtain the following bracket relations in $G(Ug)$:

\[
[x_{ij}, x_{i-2j}] = \delta_{ij} z_0,
\]

\[
[x_{ij}, z_r] = 0,
\]

\[
[x, z_r] = x_r x_r,
\]

\[
[x_r, z_s] = 0,
\]

for all $i, j = 1, 2, \ldots, 2n - 2; r, s = 0, 1, \ldots, k - 1$, where $x'_0 = x, x'_r = 0, x; x_i + x_{2n-2-i} = x$. Set

\[
y_i = x_{2n-1-i}; \quad i = 1, 2, \ldots, n - 1
\]

and

\[
x'_n = (x - \sum_{i=1}^{n-1} (x - x_i) (x_i z_0^{-1}) y_i) z_0.
\]

Then for all $i = 1, 2, \ldots, n - 1, r = 0, 1, 2, \ldots, k$,

\[
[x_i, y_i] = z_s, \quad [x'_n, z_r] = x'_r z_0 z_r
\]

and all remaining brackets vanish. Set $x_n = x'_n z_0^{-1}, y_n = z_0$. The proof is completed by noting that $x_i z_0^{-1}, y_i; i = 1, 2, \ldots, n; z_r z_0^{-1}; r = 0, x'_r = x$ and $z_r; x'_r = 0$ generate $A_{n,k}$.

We remark that the proof and consequently the filtration simplifies should $g$ be almost algebraic [6], p. 98. In this case ad $x$ may be assumed semisimple.

Given $x \in U^{(s)}, x \notin U^{(s-1)}$, we write $f(x)$ for the leading term of $x$. Theorem 2.3 has the following easy corollary which illustrates the symplectic structure associated with the enveloping algebra of a Lie algebra.

**Corollary 2.4.** — There exists a filtration of $Ug$ with $U^{(0)} = K$, such that $G(Ug)$ is isomorphic to a subalgebra of $K[x_i, y_i, z_i]; i = 1, 2, \ldots, n; j = 1, 2, \ldots, k$. Furthermore given $x \in U^{(r)}, x \notin U^{(r-1)}, y \in U^{(s)}, y \notin U^{(s-1)}$; then either

\[
\{ [x, y] \} = 0,
\]

or

\[
f ([x, y]) = \{ f(x), f(y) \}.
\]

where

\[
\{ f(x), f(y) \} = \sum_{i=1}^{n} \left( \frac{\partial f(x)}{\partial x_i} \frac{\partial f(y)}{\partial y_i} - \frac{\partial f(x)}{\partial y_i} \frac{\partial f(y)}{\partial x_i} \right).
\]

**Proof.** — Given $x \in g$ suppose with respect to the filtration of $Ug$ defined in Theorem 2.3 that $x \in U^{(s)}, x \notin U^{(s-1)}$. Define a new filtration in $Ug$ by setting $U^{(0)} = K$ and defining $x \in U^{(s+1)}, x \notin U^{(s)}$. Computation shows that the new graded algebra $G(Ug)$ has the asserted properties.
3. The commutant theorem

In this section, we consider only filtrations on $U$ such that $U^{(o)} = K$ and $G(U)$ is commutative. Given a subalgebra $A$ of $U$, set $f(A) = \{ f(a); a \in A \}$. $f(A)$ is isomorphic to a subalgebra of polynomials in $m$ variables: $m = \dim g$. Set $df(A) = \{ df(a); f(a) \in f(A) \}$. Let $\dim df(A)$ denote the dimension of $df(A)$ considered as a module over $G(U)$.

**Lemma 3.1.** — Let $A$ be a subalgebra of $U$. Then

$$\dim_K A = \dim df(A).$$

**Proof.** — The proof follows that of [8], Theorem 3.3. Let $\{ x_i \}$ be a basis for $g$. We have $x_i \in U^{(n_i)}$, $x_i \notin U^{(n_i-1)}$, for each $i$, where the $n_i$ are positive integers. Set $y_i = x_i^{n_i}$. Let $f(A')$ denote $f(A)$ considered as an algebra of polynomials in the $y_i$. Clearly

$$\dim df(A') = \dim df(A).$$

That $\dim_K A \leq \dim df(A')$, follows from the dimensionality estimate of [7], Lemma 3.3. On the other hand choosing $a_1, a_2, \ldots, a_r \in A$ such that $\{ df(a_i); i = 1, 2, \ldots, r \}$ is a basis for $df(A)$ shows that $\dim_K A \geq \dim df(A)$.

Theorem 1.1 follows on application of [8], Lemma 2.1, and Corollary 2.4 and Lemma 3.1 to the algebraic closure of $K$.

**References**


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