Bulletin de la S. M. F.

Anthony Joseph

Symplectic structure in the enveloping algebra of a Lie algebra

Bulletin de la S. M. F., tome 102 (1974), p. 75-83

<http://www.numdam.org/item?id=BSMF_1974__102__75_0>

Bull. Soc. math. France,
102, 1974, p. 75-83.

SYMPLECTIC STRUCTURE IN THE ENVELOPING ALGEBRA OF A LIE ALGEBRA

BY

ANTHONY JOSEPH
[Tel-Aviv University]

ABSTRACT. — It is shown that the enveloping algebra of a Lie algebra satisfies a condition which implies a weakened form of the Gel'fand-Kirillov conjecture. This condition leads to a generalization of a commutant property previously derived for the Weyl algebra, which has its origins in a classical theorem on function groups. This provides a dimensionality estimate which is central to a proof of the Gel'fand-Kirillov conjecture for solvable algebraic Lie algebras.

RÉSUMÉ. — Il est démontré que l'algèbre enveloppante d'une algèbre de Lie satisfait une condition qui implique une forme affaiblie de la conjecture Gel'fand-Kirillov. Cette condition amène à une généralisation d'une propriété commutante précédemment dérivée pour l'algèbre de Weyl, qui a ses origines dans un théorème classique en groupes fonctionnels. Ceci fournit une estimation dimensionnelle qui est centrale pour la preuve de la conjecture Gel'fand-Kirillov pour les algèbres de Lie algébriques résolubles.

1. Introduction

Let g be a finite dimensional Lie algebra over a commutative field K of characteristic zero. Let $U\,g$ denote the enveloping algebra of g and $D\,g$ the quotient field of $U\,g$. Let $D_{n,k}$ denote the quotient field of the Weyl algebra $A_{n,k}$ of degree n over K and extended by k indeterminates. GEL'FAND and KIRILLOV ([2]-[4]) have suggested that $D\,g$ should depend rather weakly on g and for g algebraic have conjectured that $D\,g$ is isomorphic to one of the standard fields $D_{n,k}$. $A_{n,k}$ is itself related to a polynomial algebra over the Poisson bracket (essentially equivalent to a manifold with symplectic structure) which has been subjected to considerable analysis. We wish to exploit these interrelationships in studying $U\,g$. In this it is often sufficient to establish a correspondence of leading order terms. This is illustrated by Theorem 2.3, the second part of which represents a weak form of the Gel'fand-Kirillov conjecture. The first part leads to an important dimensionality estimate contained in the theorem stated below.

Let g^* denote the dual of g. To each $f \in g$ define an antisymmetric bilinear form B_f on $g \times g$ through

$$B_f(x, y) = (f, [x, y]). \tag{1.1}$$

Recall that B_f must have even rank and set

$$m = \dim g, \qquad n = \frac{1}{2} \sup_{f \in g^*} \operatorname{rank} B_f. \tag{1.2}$$

Let Dim_K denote the dimensionality introduced by GEL'FAND and KIRILLOV [2]. It is shown in section 3 that :

THEOREM 1.1. — *Let A be a subalgebra of $U\, g$ and denote by A' its commutant in $U\, g$. Then*

$$\mathrm{Dim}_K A + \mathrm{Dim}_K A' \leqq 2(m - n),$$

with m, n given by (1.2).

We remark that $\mathrm{Dim}_K\, U\, g = \dim g$ for all g. If further g is either nilpotent or semisimple $\mathrm{Dim}_K\, C(U\, g) = m - 2n$ (where C denotes centre). It follows that the above bound is saturated in either of these two cases. This is also true if g is solvable and algebraic. Indeed for g solvable NGHIÊM [11] has constructed a maximal commutative subalgebra A of $U\, g$ and it is shown in [9] by use of the above theorem that $\mathrm{Dim}_K A = m - n$. This equality motivated the proof of the Gel'fand-Kirillov conjecture for g solvable given in [9]. We remark that Theorem 1.1 does not follow in any obvious fashion from the truth of this conjecture. This is because the corresponding dimensionality estimates are more difficult to make in $D\, g$.

2. Weighted filtrations

Let n, k be integers with n non-negative and k positive. Let $g_{n,k}$ denote the Lie algebra over K with basis $\{ x_i, y_i, z_j ; i = 1, 2, \ldots, n ; j = 0, 1, \ldots, k-1 \}$ where $[x_i, y_i] = z_0$ and all other brackets vanish. Let I denote the two-sided ideal in $U\, g_{n,k}$ generated by $z_0 - 1$. Set $A_{n,k} = U\, g_{n,k+1}/I$. Observe that $U\, g_{n,k}$ is isomorphic to a subalgebra of $A_{n,k}$ (divide the x_i by z_0) and that $D\, g_{n,k} = D_{n,k}$ [2].

For arbitrary g, let the subspaces $\{ U^{(i)} ; i = 0, 1, 2, \ldots \}$ define a filtration of $U\, g$. Set $U_i = U^{(i)}/U^{(i-1)}$ and $G(U\, g) = \bigoplus_{i=0}^{\infty} U_i$.

Only filtrations making $G(U\, g)$ integral are considered.

In the remainder of this section we assume K algebraically closed.

LEMMA 2.1. — *Suppose g is either nilpotent or semisimple. Define m, n by* (1.2) *and set $k = m - 2n$. Then $U\,g$ admits a filtration such that $G(U\,g) = U\,g_{n,k}$.*

Proof. — Take g nilpotent. Recalling (1.1) and (1.2) choose $f \in g^*$ such that rank $B_f = 2n$. Set $g_0 = \{ x \in g ; f(x) = 0 \}$.

Let B'_f denote the restriction of B_f to g_0. We wish to show that rank $B'_f = 2n$. Let N_B, $N_{B'}$ respectively denote the null spaces of B_f and B'_f. By [1], Lemma 5, it suffices to show that $N_{B'} \subset N_B$. Now given $x \in N_{B'}$,

$$(f, [x, y]) = B_f(x, y) = 0, \quad \text{for all } y \in g_0.$$

Hence $(\operatorname{ad} x)\, g_0 \subset g_0$. Let $z_0 \in g$, $z_0 \notin g_0$. Since $\dim g - \dim g_0 = 1$, we may write $(\operatorname{ad} x)\, z_0 = \alpha z_0 + y$, for some $\alpha \in K$, $y \in g_0$. Then for each positive integer r,

$$(\operatorname{ad}^r x)\, z_0 = \alpha^r z_0 + y_r; \qquad y_r \in g_0.$$

Since g is nilpotent ; $\alpha^r z_0 + y_r = 0$ for some r and hence $\alpha = 0$. It follows that $(\operatorname{ad} x)\, g \subset g_0$ which implies that $x \in N_B$, as required.

Define a filtration on $U\,g$ by setting $U^{(0)} = K$, $g_0 \subset U^{(1)}$, $z_0 \in U^{(2)}$, $z_0 \notin U^{(1)}$. To show that $G(U\,g)$ has the asserted property it suffices to show that the generators of g satisfy the commutation relations of $g_{n,k}$ in $G(U\,g)$. Scale z_0 so that $f(z_0) = 1$. Then for all x, $y \in g$, we have

$$xy - yx = (f, [x, y])\, z_0 \qquad \operatorname{mod} g_0$$

in $U\,g$. Hence by choice of filtration we obtain, for all x, $y \in g_0$,

$$\begin{aligned} xy - yx &= B'_f(x, y)\, z_0, \\ x z_0 - z_0 x &= 0 \end{aligned}$$

in $G(U\,g)$. Finally bringing B'_f to canonical form exhibits the defining basis for $g_{n,k}$.

Take g semisimple. As is well-known, $k = \operatorname{rank} g$, and n is the number of positive roots. Let h be a Cartan subalgebra for g, and Δ the set of all non-zero roots. Each root subspace g^α is one-dimensional, and g is a direct sum of h and the g^α; $\alpha \in \Delta$. Let B denote the Killing form. To each $\alpha \in \Delta$ define $H_\alpha \in h$ (*cf.* [5], Theorem 4.2) through $B(H, H_\alpha) = \alpha(H)$ for all $H \in h$. Let H_0 be a regular element ([5], p. 137) of h. Then $\alpha(H_0) \neq 0$ for all $\alpha \in \Delta$. Define $f \in g^*$ through $f(H) = B(H, H_0)$ for all $H \in h$ and the condition that it vanish on each root subspace. Set $g_0 = \{ x \in g; f(x) = 0 \}$. For each $\alpha \in \Delta$ choose $E_\alpha \in g^\alpha$ such that $B(E_\alpha, E_{-\alpha}) = 1$. Then through [5], Theorem 5.5,

$$E_\alpha E_{-\alpha} - E_{-\alpha} E_\alpha = \frac{\alpha(H_0)}{B(H_0, H_0)} H_0 \qquad \text{mod } g_0$$

in $U\,g$ and all other commutators vanish mod g_0. Setting $H_0 = z_0$ the proof is completed with the filtration defined in the nilpotent case.

The conclusion of the Lemma fails on general g. For example, consider the two dimensional (solvable) Lie algebra with relation $[x, z] = z$. Yet through [1], Lemma 5, $\operatorname{rank} B'_f \geqq \operatorname{rank} B_f - 2$; B'_f, B_f as above. It follows that we always have the weaker result, namely that

$$G(U\,g) = U\,g_{n-1,\,k+2},$$

for a suitable filtration of $U\,g$.

LEMMA 2.2. — *Let B a non-degenerate antisymmetric bilinear form on $V \times V$. Let a be a linear transformation on V such that*

$$B(ax, y) + B(x, ay) = 0 \tag{2.1}$$

for all $x, y \in V$. Then there exists a basis $\{ x_i;\ i = 1, 2, \ldots, 2\,l \}$ for V such that

$$B(x_i, x_{2l-j}) = \delta_{ij}(-1)^j, \tag{2.2}$$

$i, j = 1, 2, \ldots, 2\,l$, and a is upper triangular.

Proof. — Recalling [10], p. 398, choose a basis $\{ y_i \}$ for V such that

$$a y_i = \alpha_i\, y_i + \beta_i\, y_{i+1}, \tag{2.3}$$

α_i, $\beta_i \in K$ with $\alpha_i \leqq \alpha_j$ for $i \leqq j$. Since B is non-degenerate there exists a second basis $\{ z_i \}$ for V such that $B(y_i, z_j) = \delta_{ij}$, for all i, j. Substitution in (2.1) and (2.3) gives

$$a z_i = -\alpha_i\, z_i - \beta_{i-1}\, z_{i-1}. \tag{2.4}$$

Let V_i denote the eigenspace belonging to eigenvalue α_i. By (2.2), $B(V_i, V_j) = 0$, unless $\alpha_i + \alpha_j = 0$. Further when this holds B non-degenerate implies $\dim V_i = \dim V_j$. Let V_0 denote the zero eigenspace and V' the direct sum of the V_i omitting V_0. On V' set

$$x_i = \begin{cases} y_i; & \alpha_i < 0, \\ (-1)^i z_{2l-i}; & \alpha_i > 0. \end{cases}$$

By (2.3) and (2.4), this determines the required basis on V'. It remains to determine a basis on V_0. Equivalently we can assume a of the lemma nilpotent.

Let r be the least positive integer such that $a^{r+1} V_0 = \{0\}$. Set

$$W^{(s)} = \{ x \in V_0;\ a^{s+1} x = 0 \} \qquad \text{and} \qquad W_s = W^{(s)}/W^{(s-1)}.$$

We have

$$V_0 = \oplus_{i=0}^{r} W_i; \qquad a W_i \subset W_{i-1} \quad \text{for all } i.$$

Hence to prove the lemma it suffices to exhibit a basis for V_0 on which B is antidiagonal and which, for each i, contains as a subbasis a basis for W_i.

Set $U = a^r V_0$. Then $\dim U = \dim W_r$, and by (2.2) :

$$(2.5) \qquad B(U, W_s) = 0; \qquad s < r.$$

Hence there exists a basis $\{ y_i;\ i = 1, 2, \ldots, t \}$ for W_r and a basis $\{ y'_i;\ i = 1, 2, \ldots, t \}$ for U such that B is antidiagonal on their linear span U'. Further since B in non-degenerate we may assume that $B(y_j, y'_{t-j}) = (-1)^j$. Set

$$x_j = y_j, \qquad x_{2t-j} = y'_{t-j}; \qquad j = 1, 2, \ldots, t.$$

Observe that $U \subset W_0$ and set $V'_0 = W^{(r-1)}/U$. Let $\{ z_i \}$ be a basis for $W^{(r-1)}$. Set

$$z'_i = z_i - \Sigma_{j=1}^{t} (-1)^j B(y_j, z_i)\, y'_{t-j}.$$

Recalling (2.5) it follows that $B(x, z'_i) = 0$, for all $x \in U'$ and all i. On the other hand $z'_i = z_i$ on V'_0. Induction provides the required basis. The lemma is proved.

THEOREM 2.3. — *Define m, n by* (1.2) *and set $k = m - 2n$. Then Ug admits a filtration such that $G(Ug)$ is isomorphic to a subalgebra of $A_{n,k}$ and $D(G(Ug)) = D_{n,k}$.*

Proof. — Let f, B_f, B'_f, g_0, N_B, $N_{B'}$, be as in the proof of lemma 2.1. Given $\operatorname{rank} B'_f = \operatorname{rank} B_f$, the conclusion of lemma 2.1 holds and the theorem follows easily. Otherwise by [1], Lemma 5, N_B is of codimension 1 in $N_{B'}$. Choose $x \in N_{B'}$, $x \notin N_B$. Then as before $(\operatorname{ad} x)\, g_0 \subset g_0$ and given $z_0 \in g$, $z \notin g_0$,

$$(\operatorname{ad} x)\, z_0 = \alpha z_0 + y; \qquad y \in g_0; \qquad \alpha \neq 0.$$

By definition of B_f and the Jacobi identity :

$$(2.6) \qquad B_f([x, y], z) = B_f([x, z], y) + B_f(x, [y, z])$$

for all $x, y, z \in g$. Choosing x as above, $y \in N_{B'}$, $z \in g_0$, it follows from (2.6) that $(\operatorname{ad} x)\, N_{B'} \subset N_{B'}$.

Suppose that there exists $z \in N_{B'}$, such that $(\operatorname{ad} x)\, z = \beta z$; $\beta \neq \alpha, 0$.

Then for all $\gamma \in K$,

(2.7) $$(\operatorname{ad} x)(z + \gamma z_0) = \beta (z + \gamma z_0) + (\alpha - \beta) \gamma z_0 + \gamma y.$$

Since $\beta \neq 0$; x, z are linearly independent. Let $\{ x_i \}$ be a basis for g_0 with $x_1 = z$, $x_2 = x$. Set $g_0(\gamma) = \text{lin span} \{ z + \gamma z_0, x_2, x_3, \ldots, x_{m-1} \}$. Define $f_\gamma \in g^*$ such that

$$g_0(\gamma) = \{ x \in g;\ f_\gamma(x) = 0 \}.$$

Denote by g_{00} a maximal subspace of g_0 on which B'_f is non-degenerate. Since $z \in N_{B'}$, we may choose a fixed g_{00} such that $g_{00} \subset g_0(\gamma)$ for all γ. Let B''_{f_γ} denote the restriction of B'_{f_γ} to $g_{00} \times g_{00}$. By choice of g_{00},

$$\operatorname{rank} B''_{f_0} = \operatorname{rank} B'_f = \operatorname{rank} B_f - 2.$$

Hence except for finitely many values of γ,

$$\operatorname{rank} B''_{f_\gamma} = \operatorname{rank} B_f - 2.$$

Since z, $x \in N_{B'}$, there exists, for all but finitely many values of γ, $y_\gamma \in g_{00}$ such that $B'_{f_\gamma}(x, y_\gamma) = O(\gamma)$ and setting $z_\gamma = z + \gamma z_0 + \gamma y_\gamma$ that $B'_{f_\gamma}(a, z_\gamma) = 0$ for all $a \in g_{00}$. Again $f_\gamma(y) = O(\gamma)$ so from (2.7) we obtain

$$B'_{f_\gamma}(x, z_\gamma) = (\alpha - \beta)\gamma + O(\gamma^2).$$

Since $\alpha \neq \beta$, it follows by [1], Lemma 5 and the above that we may choose γ such that $\operatorname{rank} B'_{f_\gamma} = \operatorname{rank} B_f$. Then the conclusion of Lemma 2.1 holds and the theorem is proved in this case. We conclude that there is no loss of generality in assuming that $N_{B'}$ admits a basis $\{ z_i \}$ such that

(2.8) $$(\operatorname{ad} x) z_i = \alpha'_i z_i + \beta'_i z_{i+1},$$

where $\alpha'_i = \alpha$, 0, for all $i = 1, 2, \ldots, k$, with $x = z_k$.

Set $V = g_0 / N_{B'}$, and let B denote the restriction of B'_f to V. Use of (2.6) shows that $\operatorname{ad} x - (\alpha/2)$ is a linear transformation on V satisfying (2.1). Further B is non-degenerate on V, so Lemma 2.2 applies. Let $\{ x_i;\ i = 1, 2, \ldots, 2l \}$ be a basis satisfying its conclusion. Since $\operatorname{rank} B = \operatorname{rank} B_f - 2$, we have $l = n - 1$. Define a filtration on $U g$ as follows.

Set $U^{(0)}$ equal the tensor algebra generated by x. Let

$$\begin{array}{ll} z_0 \in U^{(4m)}, & z_0 \notin U^{(4m-1)}, \\ x_i \in U^{(2m+n-i-1)}, & x_i \notin U^{(2m+n-i-2)}, \\ z_j \in U^{(2m-n+1-j)}, & z_j \notin U^{(2m-n-j)}; \\ i = 1, 2, \ldots, 2n-2; & j = 1, 2, \ldots, k-1. \end{array}$$

Recalling that $k = m - 2n$ and that $\operatorname{ad} x$ is upper triangular on V and on $N_{B'}$, we obtain the following bracket relations in $G(U\,g)$:

$$
\begin{aligned}
[x_i, x_{2n-2-j}] &= \delta_{ij}\, z_0,\\
[x_i, z_r] &= 0,\\
[x, x_i] &= \alpha_i\, x_i,\\
[x, z_r] &= \alpha'_r\, z_r,\\
[z_r, z_s] &= 0,
\end{aligned}
$$

for all $i, j = 1, 2, \ldots, 2n-2$; $r, s = 0, 1, \ldots, k-1$, where $\alpha'_0 = \alpha$, $\alpha'_r = 0,\ \alpha$; $\alpha_i + \alpha_{2n-2-i} = \alpha$. Set

$$y_i = x_{2n-1-i}; \qquad i = 1, 2, \ldots, n-1$$

and

$$x'_n = (x - \Sigma_{i=1}^{n-1}(\alpha - \alpha_i)(x_i\, z_0^{-1})\, y_i)\, z_0.$$

Then for all $i = 1, 2, \ldots, n-1$, $r = 0, 1, 2, \ldots, k$,

$$[x_i, y_i] = z_0, \qquad [x'_n, z_r] = \alpha'_r\, z_0\, z_r$$

and all remaining brackets vanish. Set $x_n = x'_n\, z_0^{-1}$, $y_n = z_0$. The proof is completed by noting that $x_i\, z_0^{-1}$, y_i; $i = 1, 2, \ldots, n$; $z_r\, z_0^{-1}$; $r = 0$, $\alpha'_r = \alpha$ and z_r; $\alpha'_r = 0$ generate $A_{n,k}$.

We remark that the proof and consequently the filtration simplifies should g be almost algebraic [6], p. 98. In this case $\operatorname{ad} x$ may be assumed semisimple.

Given $x \in U^{(s)}$, $x \notin U^{(s-1)}$, we write $f(x)$ for the leading term of x. Theorem 2.3 has the following easy corollary which illustrates the symplectic structure associated with the enveloping algebra of a Lie algebra.

COROLLARY 2.4. — *There exists a filtration of* $U\,g$ *with* $U^{(0)} = K$, *such that* $G(U\,g)$ *is isomorphic to a subalgebra of* $K[x_i, y_i, z_j]$; $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k$. *Furthermore given* $x \in U^{(r)}, x \notin U^{(r-1)}$, $y \in U^{(s)}$, $y \notin U^{(s-1)}$; *then either*

$$[x, y] \in U^{(k+s-2)} \qquad \textit{and} \qquad \{ f(x), f(y) \} = 0,$$

or

$$f([x, y]) = \{ f(x), f(y) \},$$

where

$$\{ f(x), f(y) \} = \Sigma_{i=1}^{n} \left(\frac{\partial f(x)}{\partial x_i} \frac{\partial f(y)}{\partial y_i} - \frac{\partial f(x)}{\partial y_i} \frac{\partial f(y)}{\partial x_i} \right).$$

Proof. — Given $x \in g$ suppose with respect to the filtration of $U\,g$ defined in Theorem 2.3 that $x \in U^{(s)}$, $x \notin U^{(s-1)}$. Define a new filtration in $U\,g$ by setting $U^{(0)} = K$ and defining $x \in U^{(s+1)}$, $x \notin U^{(s)}$. Computation shows that the new graded algebra $G(U\,g)$ has the asserted properties.

3. The commutant theorem

In this section, we consider only filtrations on $U\,g$ such that $U^{(0)} = K$ and $G(U\,g)$ is commutative. Given a subalgebra A of $U\,g$, set $f(A) = \{ f(a);\ a \in A \}$. $f(A)$ is isomorphic to a subalgebra of polynomials in m variables : $m = \dim g$. Set $df(A) = \{ df(a);\ f(a) \in f(A) \}$. Let $\dim df(A)$ denote the dimension of $df(A)$ considered as a module over $G(U\,g)$.

LEMMA 3.1. — *Let A be a subalgebra of $U\,g$. Then*

$$\mathrm{Dim}_K A = \dim df(A).$$

Proof. — The proof follows that of [8], Theorem 3.3. Let $\{ x_i \}$ be a basis for g. We have $x_i \in U^{(n_i)}$, $x_i \notin U^{(n_i - 1)}$, for each i, where the n_i are positive integers. Set $y_i = x_i^{1/n_i}$. Let $f(A')$ denote $f(A)$ considered as an algebra of polynomials in the y_i. Clearly

$$\dim df(A') = \dim df(A).$$

That $\mathrm{Dim}_K A \leqq \dim df(A')$, follows from the dimensionality estimate of [7], Lemma 3.3. On the other hand choosing $a_1, a_2, \ldots, a_r \in A$ such that $\{ df(a_i);\ i = 1, 2, \ldots, r \}$ is a basis for $df(A)$ shows that $\mathrm{Dim}_K A \geqq \dim df(A)$.

Theorem 1.1 follows on application of [8], Lemma 2.1, and Corollary 2.4 and Lemma 3.1 to the algebraic closure of K.

REFERENCES

[1] DIXMIER (J.). — Sur les représentations unitaires des groupes de Lie nilpotents, II., *Bull. Soc. math. Fr.*, t. 85, 1957, p, 325-388.

[2] GEL'FAND (I. M.) and KIRILLOV (A. A.). — Fields associated with enveloping algebras of Lie algebras, *Soviet Mathematics*, t. 7, 1966, p. 407-409 [en russe]; *Doklady Akad. Nauk SSSR*, t. 167, 1966, p. 503-505.

[3] GEL'FAND (I. M.) and KIRILLOV (A. A.). — *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie.* — Paris, Presses universitaires de France, 1966 (*Institut des Hautes Études Scientifiques, Publications mathématiques*, 31, p. 5-19).

[4] GEL'FAND (I. M.) and KIRILLOV (A. A.). — Structure of the Lie field connected with a split semisimple Lie algebra [en russe], *Funkcional'nyj Analiz*, t. 3, 1969, fasc. 1, p. 7-26.

[5] HELGASON (S.). — *Differential geometry and symmetric spaces.* — New York, Academic Press, 1962 (*Pure and applied Mathematics, Academic Press*, 12).

[6] JACOBSON (N.). — *Lie algebras.* — New York, Interscience Publishers, 1962 (*Interscience Tracts in pure and applied Mathematics*, 10).

[7] JOSEPH (A.). — Commuting polynomials in quantum canonical operators and realizations of Lie algebras, *J. of math. Phys.*, t. 13, 1972, p. 351-357.
[8] JOSEPH (A.). — Gel'fand-Kirillov dimension for algebras associated with the Weyl algebra, *Ann. Inst. H. Poincaré*, série A, t. 17, 1973, p, 325-336.
[9] JOSEPH (A.). — *Proof of the Gel'fand-Kirillov conjecture for solvable Lie algebras*, *Proc. Amer. math. Soc.* (à paraître).
[10] LANG (S.). — *Algebra.* — New York, Addison-Wesley publishing Company, 1965 (*Addison-Wesley Series in Mathematics*).
[11] NGHIÊM (X. H.). — Sur certains sous-corps commutatifs du corps enveloppant d'une algèbre de Lie résoluble, *Bull. Sc. math.*, série 2, t. 96, 1972, p. 111-128.

(Texte reçu le 11 avril 1973.)

Anthony JOSEPH,
Institut des Hautes Études Scientifiques,
35, route de Chartres,
91440 Bures-sur-Yvette
et
Department of Physics and Astronomy,
Tel-Aviv University,
Ramat-Aviv, Israël.