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IAN G. CRAW

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AXIOMATIC COHOMOLOGY OF OPERATOR ALGEBRAS

BY

IAN G. CRAW

[Aberdeen]

RÉSUMÉ. — Recently, JOHNSON, KADISON and RINGROSE have considered the normal cohomology groups of a fixed operator algebra A, with coefficients from the class of dual normal A-modules. In the case where the coefficient modules are actually modules over the weak operator closure of A, they prove that the normal cohomology groups and the continuous cohomology groups coincide. A shorter proof of this result is obtained based on an axiomatic characterisation of the normal cohomology groups with coefficients in such modules. This characterisation also gives a short proof of the isomorphism between the normal cohomology groups of A, and of its weak operator closure.

1. Introduction

In a recent paper [1], the author showed that it was possible to axiomatise a cohomology theory for Banach modules over a fixed Banach algebra. As well as being of interest in its own right, this result enabled simplifications to be made in the proofs of known isomorphism theorems. In this paper, we show that a similar theorem remains valid when we consider the normal cohomology groups defined for modules over a fixed operator algebra. One consequence of this result is a considerable simplification of work of JOHNSON, KADISON and RINGROSE [4] in which it is shown that, for a wide class of modules, norm continuous and normal cohomology coincide.

A cohomology theory for an arbitrary associative linear algebra was first introduced by HOCHSCHILD, in 1945 [2]. If $\mathfrak{A}$ is an algebra, and $\mathcal{M}$ a two sided $\mathfrak{A}$-module, let $\mathcal{L}^n(\mathfrak{A}, \mathcal{M})$ denote the space of all n-linear maps from $\mathfrak{A} \times \ldots \times \mathfrak{A}$ to $\mathcal{M}$; we call this the space of *n-cochains*. We define a *coboundary operator* $\Delta : \mathcal{L}^n(\mathfrak{A}, \mathcal{M}) \to \mathcal{L}^{n+1}(\mathfrak{A}, \mathcal{M})$ by

$$\begin{aligned}\Delta\rho(A_1, \ldots, A_{n+1}) = {} & A_1 \rho(A_2, \ldots, A_{n+1}) \\ & + \textstyle\sum_{i=1}^{n} (-1)^i \rho(A_1, \ldots, A_i A_{i+1}, \ldots, A_{n+1}) \\ & + (-1)^{n+1} \rho(A_1, \ldots, A_n) A_{n+1},\end{aligned}$$

$$(A_1, \ldots, A_{n+1} \in \mathfrak{A}, \ \rho \in \mathcal{L}^n(\mathfrak{A}, \mathcal{M})).$$

Conventionally, we put $\mathcal{L}^0(\mathfrak{A}, \mathcal{M}) = \mathcal{M}$, and for $m \in \mathcal{M}$, define $\Delta m(A) = Am - mA$ $(A \in \mathfrak{A})$. A calculation shows that, for $n \geq 0$, $\Delta^2 : \mathcal{L}^n(\mathfrak{A}, \mathcal{M}) \to \mathcal{L}^{n+2}(\mathfrak{A}, \mathcal{M})$ always vanishes; thus if we define

$$B^n(\mathfrak{A}, \mathcal{M}) = \{ \Delta\rho : \rho \in \mathcal{L}^{n-1}(\mathfrak{A}, \mathcal{M}) \} \qquad (n \geq 1),$$

the space of *n-coboundaries,* and

$$Z^n(\mathfrak{A}, \mathcal{M}) = \{ \rho \in \mathcal{L}^n(\mathfrak{A}, \mathcal{M}) : \Delta\rho = 0 \} \qquad (n \geq 0),$$

the space of *n cocycles,* then $B^n(\mathfrak{A}, \mathcal{M}) \subseteq Z^n(\mathfrak{A}, \mathcal{M})$ $(n \geq 1)$. Each of these has the structure of a linear space, and so in particular of an abelian group. Conventionally, we write $B^0(\mathfrak{A}, \mathcal{M}) = \{0\}$, the zero group. We may thus define the *n-dimensional cohomology group* of $\mathfrak{A}$, with coefficients in $\mathcal{M}$, by

$$H^n(\mathfrak{A}, \mathcal{M}) = Z^n(\mathfrak{A}, \mathcal{M})/B^n(\mathfrak{A}, \mathcal{M}) \qquad (n \geq 0).$$

The continuous, and normal cohomology groups, in which we shall be interested, are obtained in the same way as the Hochschild groups, except that we allow as n-cochains, only those multilinear maps which satisfy certain continuity conditions. Let $\mathfrak{A}$ be a complex Banach algebra, and $\mathcal{M}$ a *Banach* $\mathfrak{A}$*-module*; i. e., $\mathcal{M}$ is a two-sided $\mathfrak{A}$-module, a complex Banach space, and

$$\| A.m \| \leq \| A \| . \| m \|, \qquad \| m.A \| \leq \| m \| . \| A \| \qquad (A \in \mathfrak{A}, m \in \mathcal{M}).$$

We write $\mathcal{L}_c^n(\mathfrak{A}, \mathcal{M})$ for the space of all continuous n-cochains, and then define the continuous n-coboundaries $B_c^n(\mathfrak{A}, \mathcal{M})$, the continuous n-cocycles $Z_c^n(\mathfrak{A}, \mathcal{M})$ and finally the *continuous cohomology groups* $H_c^n(\mathfrak{A}, \mathcal{M})$ $(n \geq 0)$ in the same way as was done above. These groups have been studied by Johnson [3], and Kadison and Ringrose [6], and we refer to these papers for a detailed discussion of both their properties and utility. In fact, it proves convenient to restrict the class of Banach $\mathfrak{A}$-modules slightly; we say that a Banach $\mathfrak{A}$-module $\mathcal{M}$ is a *dual* $\mathfrak{A}$*-module* if $\mathcal{M}$ is the (continuous) dual of some Banach space $\mathcal{M}_*$, and if the maps

$$m \mapsto A.m, \qquad m \mapsto m.A \qquad (A \in \mathfrak{A}, m \in \mathcal{M})$$

are weak-$\star$-continuous $(= \sigma(\mathcal{M}, \mathcal{M}_*)$-continuous). This class of modules coincides with $\{ \mathfrak{X}^* : \mathfrak{X}$ is a Banach $\mathfrak{A}$-module $\}$, for which the main results in Johnson [3] were obtained.

Our main concern in this paper is when $\mathfrak{A}$ is a C^*-algebra concretely represented as an algebra of bounded linear operators on some Hilbert space $\mathcal{H}$. If $\mathcal{M}$ is a dual $\mathfrak{A}$-module, we say that $\mathcal{M}$ is *normal* if the maps

$$A \mapsto A.m, \qquad A \mapsto m.A \qquad (A \in \mathfrak{A}, m \in \mathcal{M})$$

are ultraweak-weak-★-continuous. In this situation, we consider $\mathcal{L}_w^n(\mathfrak{A}, \mathcal{M})$, the space of continuous n-cochains which are separately ultraweak-weak-★-continuous, which we call the space of *normal* n-cochains. In the same way as before, we define normal n-coboundaries $B_w^n(\mathfrak{A}, \mathcal{M})$, normal n-cocycles $Z_w^n(\mathfrak{A}, \mathcal{M})$, and thus the normal cohomology groups $H_w^n(\mathfrak{A}, \mathcal{M})$ $(n \geq 0)$.

We shall consider the relationship between the normal cohomology groups for $\mathfrak{A}$, and those for $\mathfrak{A}^-$, the weak operator closure of $\mathfrak{A}$. Clearly, a dual normal $\mathfrak{A}^-$-module becomes a dual normal $\mathfrak{A}$-module when the multiplication is restricted. Even if $\mathfrak{A}$ is not unital, $\mathfrak{A}^-$ contains an identity E, the principal identity, which is the projection onto the space $\mathcal{H}_1 = \{A x : A \in \mathfrak{A}, x \in \mathcal{H}\}$. It thus makes sense to demand that a dual normal $\mathfrak{A}^-$-module $\mathcal{M}$ be unital, i. e. that $E.m = m = m.E$ $(m \in \mathcal{M})$. Finally, we note that since in general E is not the identity operator on $\mathcal{H}$, $\mathfrak{A}^-$ is not in general a von Neumann algebra. Thus to appeal to general theory for results about $\mathfrak{A}^-$, we must first restrict the representation to act on $\mathcal{H}_1$.

2. Construction of a dual normal $\mathfrak{A}$-module

As in [7], [4], our main tool is the existence of the universal representation of a C^*-algebra. This is described in Kadison ([5], p. 181-182); we review the main results here. Let $\mathfrak{A}$ be a given (abstract) C^*-algebra; then among all the concrete representations of $\mathfrak{A}$ as an algebra of bounded linear operators on a Hilbert space there is a distinguished representation, the universal representation, which dominates every other representation. Precisely, if φ is an arbitrary representation of $\mathfrak{A}$, and ψ the universal representation, there is a central projection $P \in \psi(\mathfrak{A})^-$, and an algebra isomorphism $\alpha : \psi(\mathfrak{A})^- P \to \varphi(\mathfrak{A})^-$ such that if $A \in \mathfrak{A}$, $\alpha(\psi(A).P) = \varphi(A)$. If we are given $\mathfrak{A}$ simply as an abstract C^*-algebra then it is notationally simpler to identify the isometrically isomorphic algebras $\mathfrak{A}$ and $\psi(\mathfrak{A})$. We then have a representation φ of $\mathfrak{A}$, a central projection $P \in \mathfrak{A}^-$ and an algebra isomorphism $\alpha : \mathfrak{A}^- P \to \varphi(\mathfrak{A})^-$ such that

$$\alpha(A.P) = \varphi(A) \qquad (A \in \mathfrak{A}).$$

Restricting $\mathfrak{A}$ to be a representation on the range of the principal identity, α becomes an isomorphism of von Neumann algebras, and so automatically ultraweakly continuous. It follows that α is ultraweakly continuous when $\mathfrak{A}$ is in its universal representation; thus φ has an ultraweakly continuous extension to a representation of $\mathfrak{A}^-$, and $\varphi(\mathfrak{A}^-) = \varphi(\mathfrak{A})^-$. We note also that a similar argument shows α^{-1} to be ultraweakly continuous.

From our point of view, the most important property of the universal representation is that any bounded linear functional on $\mathfrak{A}$ is necessarily ultraweakly continuous. One consequence of this, although the deduction is by no means trivial, is the following extension theorem of JOHNSON, KADISON and RINGROSE ([4], Theorem 2.3). Although we only require the extension in one variable, this does not enable the proof given in [4] to be significantly simplified.

LEMMA 2.1. — *Let $\mathfrak{A}$ be a* C*-*algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{M}$ the dual space of a complex Banach space $\mathcal{M}_*$, and $\rho : \mathfrak{A}\times\ldots\times\mathfrak{A}\to\mathcal{M}$ a bounded multilinear mapping which is separately ultraweak-weak-★-continuous. Then ρ extends uniquely without change of norm to a bounded multilinear mapping $\tilde{\rho} : \overline{\mathfrak{A}}\times\mathfrak{A}\times\ldots\times\mathfrak{A}\to\mathcal{M}$, which is separately ultraweak-weak-★-continuous in each variable.*

The main result, in this section, is that the " relatively injective " modules which played a crucial role in the axiomatic continuous cohomology theory of CRAW [1] are, in certain cases, actually dual normal $\mathfrak{A}^-$-modules.

Let $\mathfrak{A}$ be a Banach algebra, and $\mathcal{M}$ an arbitrary Banach space. We may consider $\mathcal{L}_c^1(\mathfrak{A}\hat{\otimes}\mathfrak{A}, \mathcal{M})$ as a Banach $\mathfrak{A}$-module by defining $A.\rho(A_1\otimes A_2) = \rho(A_1 A\otimes A_2)$,

$$\rho.A(A_1\otimes A_2) = \rho(A_1\otimes AA_2) \qquad (A, A_1, A_2\in\mathfrak{A}, \rho\in\mathcal{L}_c^1(\mathfrak{A}\hat{\otimes}\mathfrak{A}, \mathcal{M})).$$

Then $A.\rho$, and $\rho.A$ both extend, by linearity and continuity, to linear maps on $\mathfrak{A}\hat{\otimes}\mathfrak{A}$, and

$$\| A.\rho \| \leqq \| A \|.\| \rho \|, \qquad \| \rho.A \| \leqq \| A \|.\| \rho \|.$$

Further, if $\mathcal{M}$ is the dual of a Banach space $\mathcal{M}_*$, then

$$\mathcal{L}_c^1(\mathfrak{A}\hat{\otimes}\mathfrak{A}, \mathcal{M}) = (\mathfrak{A}\hat{\otimes}\mathfrak{A}\hat{\otimes}\mathcal{M}_*)^*,$$

so that $\mathcal{L}_c^1(\mathfrak{A}\hat{\otimes}\mathfrak{A}, \mathcal{M})$ is a dual Banach $\mathfrak{A}$-module.

LEMMA 2.2. — *Let $\mathfrak{A}$ be a C*-algebra acting, in its universal representation, on a Hilbert space $\mathcal{H}$, and let $\mathcal{M}$ be the dual space of a complex Banach space $\mathcal{M}_*$; then $\mathcal{L}_c^1(\mathfrak{A}^-\hat{\otimes}\mathfrak{A}^-, \mathcal{M})$ is a dual normal $\mathfrak{A}^-$-module.*

Proof. — The remarks above show that $\mathcal{L}_c^1(\mathfrak{A}^-\hat{\otimes}\mathfrak{A}^-, \mathcal{M})$ is a dual Banach $\mathfrak{A}^-$-module. Fix $\rho\in\mathcal{L}_c^1(\mathfrak{A}^-\hat{\otimes}\mathfrak{A}^-, \mathcal{M})$; we must show that the maps

$$R\mapsto R.\rho, \qquad R\mapsto\rho.R \qquad (R\in\mathfrak{A}^-)$$

are ultraweak-weak-★-continuous, or equivalently, that for each $x \in \mathfrak{A}^- \hat{\otimes} \mathfrak{A}^- \hat{\otimes} \mathfrak{M}_*$, the linear functionals

$$f_\rho(R) = (R.\rho)(x), \qquad {}_\rho f(R) = (\rho.R)(x)$$

are ultraweakly continuous on $\mathfrak{A}^-$. Since $\mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})$ is a Banach-module, both ${}_\rho f$ and f_ρ are bounded linear functionals; ultraweak continuity now follows automatically since $\mathfrak{A}$ is in its universal representation ([5], p. 181).

Lemma 2.3. — *Let φ be a faithful representation of a C*-algebra $\mathfrak{A}$, and let $\mathfrak{M}$ be the dual space of a complex Banach space. Then $\mathcal{L}_c^1(\varphi(\mathfrak{A})^- \hat{\otimes} \varphi(\mathfrak{A})^-, \mathfrak{M})$ is a dual normal $\varphi(\mathfrak{A})^-$-module.*

Proof. — There is no loss in assuming that the (abstract) C^*-algebra $\mathfrak{A}$ is in fact given in its universal representation. Then there is a central projection $P \in \mathfrak{A}^-$, and an algebraic isomorphism $\alpha : \mathfrak{A}^- P \to \varphi(\mathfrak{A})^-$ such that $\alpha(R.P) = \varphi(R)\ (R \in \mathfrak{A}^-)$. Let

$$\mathfrak{X} = \mathcal{L}_c^1(\varphi(\mathfrak{A}^-) \hat{\otimes} \varphi(\mathfrak{A}^-), \mathfrak{M}),$$
$$\mathfrak{Y} = \{\sigma \in \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M}) : \sigma = P\sigma P\},$$

and define

$$\theta(\sigma)(\varphi(R_1) \otimes \varphi(R_2)) = \sigma(R_1 P \otimes PR_2) \qquad (\sigma \in \mathfrak{Y}; R_1, R_2 \in \mathfrak{A}^-).$$

Since α is an isomorphism, θ is well defined, and extends by linearity and continuity to a norm decreasing map $\theta : \mathfrak{Y} \to \mathfrak{X}$.

Similarly, by defining

$$\psi(\rho)(R_1 \otimes R_2) = \rho(\varphi(R_1) \otimes \varphi(R_2)) \qquad (\rho \in \mathfrak{X}, R_1, R_2 \in \mathfrak{A}^-),$$

we obtain a norm decreasing map $\psi : \mathfrak{X} \to \mathfrak{Y}$. Since $\theta\psi = 1$, $\psi\theta = 1$, in particular θ is isometric. Further, since P is in the centre of $\mathfrak{A}^-$, it is clear that

(1) $\theta(R.\sigma) = \sigma(R).\theta(\sigma); \qquad \theta(\sigma.R) = \theta(\sigma).\sigma(R) \qquad (R \in \mathfrak{A}^-, \sigma \in \mathfrak{Y}).$

Now fix $\rho \in \mathfrak{X}$; we show that the map $\varphi(R) \mapsto \varphi(R).\rho\ (R \in \mathfrak{A}^-)$ is ultraweak-weak-★-continuous. Continuity of right multiplication can be obtained in exactly the same way, and since only the normality of $\mathfrak{X}$ was in doubt, this will prove the lemma.

Let $\sigma = \theta^{-1}(\rho)$, and define $m_\sigma : \mathfrak{A}^- \to \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})$ by

$$m_\sigma(R) = R\sigma \qquad (R \in \mathfrak{A}^-).$$

Then, using (1),

$$\theta \circ m_\sigma \circ \alpha^{-1}(\varphi(R)) = \theta(RP_\sigma) = \varphi(R)\,\rho \qquad (R \in \mathfrak{A}^-).$$

As remarked earlier, α^{-1} is automatically ultraweakly continuous. By lemma 2.2, m_σ is ultraweak-weak-★-continuous on $\mathfrak{A}^-$, and since the ultraweak topology on $\mathfrak{A}^- P$ is the restriction to $\mathfrak{A}^- P$ of the ultraweak topology on $\mathfrak{A}^-$, m_σ is an ultraweak-weak-★-continuous map from $\mathfrak{A}^- P = \alpha^{-1}(\varphi(\mathfrak{A}^-))$ to $\mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})$. Since $\sigma \in \mathfrak{Y}$, m_σ in fact has its range in $\mathfrak{Y}$, and since $\mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})$ is a dual Banach $\mathfrak{A}^-$-module, $\mathfrak{Y}$ is weak-★-closed in $\mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})$ and so is itself the dual of a Banach space. Since the weak-★-topology on $\mathfrak{Y}$ then coincides with its relative weak-★-topology as a subspace of $\mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})$, it follows that $m_\sigma : \mathfrak{A}^- P \to \mathfrak{Y}$ is ultraweak-weak-★-continuous. Finally, since $\theta : \mathfrak{Y} \to \mathfrak{X}$ is isometric, it is necessarily weak-★-weak-★-continuous. We now see that

$$\theta \circ m_\sigma \circ \alpha^{-1} : \quad \varphi(R) \mapsto \varphi(R).\rho \qquad (R \in \mathfrak{A}^-)$$

is ultraweak-weak-★-continuous, and the lemma follows.

3. A characterisation of normal cohomology

We now derive a series of properties which, we prove in theorem 3.5, characterise the normal cohomology of an algebra of operators.

LEMMA 3.1. — *Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space, and let $\mathcal{M}$ be a dual normal $\mathfrak{A}$-module. Then*

$$H_w^0(\mathfrak{A}, \mathcal{M}) = \{ m \in \mathcal{M} : m.A = A.m \ (A \in \mathfrak{A}) \}.$$

Proof :

$$\begin{aligned} H_w^0(\mathfrak{A}, \mathcal{M}) &= \{ m \in \mathcal{M} : \Delta(m) = 0 \} \\ &= \{ m \in \mathcal{M} : Am - mA = 0 \ (A \in \mathfrak{A}) \}. \end{aligned}$$

LEMMA 3.2. — *Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space. Let*

$$0 \to \mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}'' \to 0$$

be an exact sequence of dual normal $\mathfrak{A}$-modules, in which the maps are continuous $\mathfrak{A}$-module homomorphisms, and suppose that f has a continuous linear left inverse (not necessarily an $\mathfrak{A}$-module homomorphism). Then group homomorphisms can be defined in such a way that the sequence

$$\ldots \to H_w^n(\mathfrak{A}, \mathcal{M}') \xrightarrow{f^n} H_w^n(\mathfrak{A}, \mathcal{M}) \xrightarrow{g^n} H_w^n(\mathfrak{A}, \mathcal{M}'') \xrightarrow{\delta^n} H_w^{n+1}(\mathfrak{A}, \mathcal{M}') \to \ldots$$

is exact, and f^0, g^0 are restrictions of f and g.

Proof. — Since f has a continuous left inverse, then also g has a continuous right inverse p (say). Now define maps f^n, g^n, δ $(n \geqq 0)$

$$\ldots \to \mathcal{L}_w^n(\mathfrak{A}, \mathfrak{M}') \xrightarrow{f^n} \mathcal{L}_w^n(\mathfrak{A}, \mathfrak{M}) \xrightarrow{g^n} \mathcal{L}_w^n(\mathfrak{A}, \mathfrak{M}'') \xrightarrow{\delta} \mathcal{L}_w^{n+1}(\mathfrak{A}, \mathfrak{M}') \to \ldots$$

by

$$f^n(\rho) = f \circ \rho \quad (\rho \in \mathcal{L}_w^n(\mathfrak{A}, \mathfrak{M}')), \qquad g^n(\rho) = g \circ \rho \quad (\rho \in \mathcal{L}_w^n(\mathfrak{A}, \mathfrak{M})),$$
$$\delta(\rho) = (\Delta p - p\,\Delta)\,\rho \qquad (\rho \in \mathcal{L}_w^n(\mathfrak{A}, \mathfrak{M}'')).$$

Then since Δ preserves separate ultraweak-weak-★-continuity, and since f, g, p are automatically weak-★-weak-★-continuous, it follows that f^n, g^n, δ map into the stated spaces. As in Johnson [3], Proposition 1.7], these induce maps between the corresponding cohomology groups, and it is clear that f^0 and g^0 are obtained by restriction. The proof that the long sequence is exact also follows that in [3].

We call a short exact sequence of the type considered above a *split exact sequence of dual normal* $\mathfrak{A}$*-modules.*

We saw in lemma 2.3 that certain " relatively injective " $\mathfrak{A}^-$-modules were indeed dual normal $\mathfrak{A}^-$-modules. It follows from this that the statement of the next lemma is meaningful.

Lemma 3.3. — *Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space, and let $\mathfrak{M}$ be the dual of a complex Banach space. Then*

$$H_w^n(\mathfrak{A}, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})) = \{0\} \qquad (n \geqq 1).$$

Proof. — Choose $n \geqq 1$, and let $\rho \in Z_w^n(\mathfrak{A}, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M}))$; then ρ extends by lemma 2.1 to a bounded multilinear map

$$\tilde{\rho} : \ \mathfrak{A}^- \times \mathfrak{A} \times \ldots \times \mathfrak{A} \to \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})$$

which is separately ultraweak-weak-★-continuous. Since $\Delta\rho = 0$, the ultraweak continuity shows that

$$\Delta\rho\,(R_1, A_1, \ldots, A_n)\,(E \otimes R_2) = 0 \qquad (R_1, R_2 \in \mathfrak{A}^-, A_1, \ldots, A_n \in \mathfrak{A}),$$

where E is the principal identity in $\mathfrak{A}^-$. Hence

$$(2)\quad \left\{ \begin{aligned} 0 = {} & \rho\,(A_1, \ldots, A_n)\,(R_1 \otimes R_2) - \tilde{\rho}\,(R_1 A_1, \ldots, A_n)\,(E \otimes R_2) \\ & + \textstyle\sum_{i=1}^{n-1} (-1)^{i+1}\, \tilde{\rho}\,(R_1, A_1, \ldots, A_i A_{i+1}, \ldots, A_n)\,(E \otimes R_2) \\ & + (-1)^{n+1}\, \tilde{\rho}\,(R_1, A_1, \ldots, A_{n-1})\,(E \otimes A_n R_2) \\ & \qquad (R_1, R_2 \in \mathfrak{A}^-, A_1, \ldots, A_n \in \mathfrak{A}). \end{aligned} \right.$$

Defining $\sigma(A_1, \ldots, A_{n-1})\ (R_1 \otimes R_2) = \tilde{\rho}\,(R_1, A_1, \ldots, A_{n-1})\,(E \otimes R_2)$, $(R_1, R_2 \in \mathfrak{A}^-, A_1, \ldots, A_{n-1} \in \mathfrak{A})$; then $\sigma(A_1, \ldots, A_{n-1})$ extends by linearity and continuity to an element of $\mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})$. Since $\tilde{\rho}$ is bounded and separately ultraweak-weak-★-continuous, so is σ, and we have $\sigma \in \mathcal{L}_w^{n-1}(\mathfrak{A}, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M}))$. We have

$$(3) \quad \begin{cases} \Delta\sigma(A_1, \ldots, A_n)(R_1 \otimes R_2) \\ \quad = \tilde{\rho}(R_1 A_1, \ldots, A_n)(E \otimes R_2) \\ \quad\quad + \sum_{i=1}^{n-1}(-1)^i \tilde{\rho}(R_1, A_1, \ldots, A_i A_{i+1}, \ldots, A_n)(E \otimes R_2) \\ \quad\quad + (-1)^n \tilde{\rho}(R_1, A_1, \ldots, A_{n-1})(E \otimes A_n R_2) \\ \qquad\qquad (R_1, R_2 \in \mathfrak{A}^-, A_1, \ldots, A_n \in \mathfrak{A}). \end{cases}$$

Adding (2) and (3) shows that $\Delta\sigma = \rho$, and the result follows.

DEFINITION 3.4. — *Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space. A* normal cohomology theory *for $\mathfrak{A}$ is a collection of groups $\{K_w^n(\mathfrak{A}, \mathfrak{M}) : n = 0, 1, \ldots\}$ defined for each unital dual normal $\mathfrak{A}^-$-module $\mathfrak{M}$, and satisfying the following conditions :*

(*a*) $K_w^0(\mathfrak{A}, \mathfrak{M}) = \{m \in \mathfrak{M} : Am = mA \ (A \in \mathfrak{A})\}$;

(*b*) *if*

$$0 \to \mathfrak{M}' \xrightarrow{f} \mathfrak{M} \xrightarrow{g} \mathfrak{M}'' \to 0$$

is a split exact sequence of unital dual normal $\mathfrak{A}^-$-modules, then group homomorphisms can be defined in such a way that the sequence

$$\ldots \to K_w^n(\mathfrak{A}, \mathfrak{M}') \xrightarrow{f^n} K_w^n(\mathfrak{A}, \mathfrak{M}) \xrightarrow{g^n} K_w^n(\mathfrak{A}, \mathfrak{M}'') \xrightarrow{\delta} K_w^{n+1}(\mathfrak{A}, \mathfrak{M}') \to \ldots$$

is exact, and f^0, g^0 are restrictions of f and g;

(*c*) $K_w^n(\mathfrak{A}, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})) = \{0\}$ $(n \geq 1)$ [*note that necessarily $\mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})$ is unital*].

We see from the preceeding three lemmas that $H_w^n(\mathfrak{A}, .)$ is a normal cohomology theory for $\mathfrak{A}$. We now show that it is essentially the only one.

THEOREM 3.5. — *Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space, and let $K_w^n(\mathfrak{A}, .)$ be a normal cohomology theory for $\mathfrak{A}$. Then for each unital dual normal $\mathfrak{A}^-$-module $\mathfrak{M}$, and $n = 0, 1, \ldots$ there is an isomorphism*

$$K_w^n(\mathfrak{A}, \mathfrak{M}) \cong H_w^n(\mathfrak{A}, \mathfrak{M}).$$

Proof. — Write $\mathfrak{X} = \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathfrak{M})$. For $m \in \mathfrak{M}$, $R_1, R_2 \in \mathfrak{A}^-$, define

$$f(m)(R_1 \otimes R_2) = R_1 m R_2;$$

then $f(m)$ extends by linearity and continuity to a map $f(m)$ on $\mathfrak{A}^- \otimes \mathfrak{A}^-$, so $f(m) \in \mathfrak{X}$. Further, $f : \mathfrak{M} \to \mathfrak{X}$ is a continuous $\mathfrak{A}$-module homomorphism and, since $\mathfrak{M}$ is unital, it is actually a monomorphism. Let $\mathfrak{M}'' = \mathfrak{X}/f(\mathfrak{M})$ and let $g : \mathfrak{X} \to \mathfrak{M}''$ be the canonical quotient map; since $\mathfrak{X}$ is unital so is $\mathfrak{M}''$.

For $\rho \in \mathfrak{X}$, define $h(\rho) = \rho(E \otimes E)$, where E is the principal identity in $\mathfrak{A}^-$. Then $\rho : \mathfrak{X} \to \mathfrak{M}$ is continuous, and $h\,f(m) = m \, (m \in \mathfrak{M})$. We thus have a split exact sequence

$$0 \to \mathfrak{M} \xrightarrow{f} \mathfrak{X} \xrightarrow{g} \mathfrak{M}'' \to 0$$

of unital dual normal $\mathfrak{A}^-$-modules. By hypothesis, we may define group homomorphisms to obtain a long exact sequence

$$\ldots \to K_w^n(\mathfrak{A}, \mathfrak{M}) \xrightarrow{f^n} K_w^n(\mathfrak{A}, \mathfrak{X}) \xrightarrow{g^n} K_w^n(\mathfrak{A}, \mathfrak{M}'') \to K_w^{n+1}(\mathfrak{A}, \mathfrak{M}) \to \ldots.$$

Since $K_w^n(\mathfrak{A}, \mathfrak{X}) = \{0\} \; (n \geqq 1)$ we have :

(4) $$K_w^{n+1}(\mathfrak{A}, \mathfrak{M}) \cong K_w^n(\mathfrak{A}, \mathfrak{M}'') \qquad (n \geqq 1),$$

(5) $$K_w^1(\mathfrak{A}, \mathfrak{M}) \cong K_w^0(\mathfrak{A}, \mathfrak{M})/g\,[K_w^0(\mathfrak{A}, \mathfrak{X})].$$

Since $H_w^n(\mathfrak{A}, .)$ is also a normal cohomology theory for $\mathfrak{A}$, we have corresponding equations

(6) $$H_w^{n+1}(\mathfrak{A}, \mathfrak{M}) \cong H_w^n(\mathfrak{A}, \mathfrak{M}'') \qquad (n \geqq 1),$$

(7) $$H_w^1(\mathfrak{A}, \mathfrak{M}) \cong H_w^0(\mathfrak{A}, \mathfrak{M}'')/g\,[H_w^0(\mathfrak{A}, \mathfrak{X})].$$

Since $H_w^0(\mathfrak{A}, .)$ and $K_w^0(\mathfrak{A}, .)$ are identical, it follows from (5) and (7) that $H_w^1(\mathfrak{A}, .)$ and $K_w^1(\mathfrak{A}, .)$ are isomorphic; the general result now follows by induction using (4) and (6).

REMARKS 3.6.

1º In fact, more is true; by regarding $K_w^n(\mathfrak{A}, .)$ as a functor between appropriate categories, the result can be strengthened to conclude that the isomorphisms appear as a natural equivalence. A statement of the theorem in this form is given ,for Hochschild cohomology, in MAC LANE [8].

2º The restriction to unital modules is not severe, as is shown in JOHNSON [3], 1.*c*. By adding the condition :

" (*d*) $K_w^n(\mathfrak{A}, \mathfrak{M}) = \{0\} \; (n \geqq 1)$ for any dual normal $\mathfrak{A}^-$-module $\mathfrak{M}$, in which the action of $\mathfrak{A}$ is trivial on one side of $\mathfrak{M}$ ";

to those in definition 3.4, it would be possible to establish theorem 3.5 without the restriction to unital modules. this condition seems to be independant of (*a*), (*b*), and (*c*), and, arguing as in JOHNSON [3], Proposition 1.5, is a property of $H_w^n(\mathfrak{A}, .)$. Alternatively, it is possible to adjoin an identity to $\mathfrak{A}^-$ in such a way that every module becomes unital. This approach was used in CRAW [1].

4. Isomorphism theorems

We show here how the uniqueness result of the previous section may by used to obtain the main results of JOHNSON-KADISON-RINGROSE [4]. The basis of this is the following lemma, which is of course a consequence of CRAW [1], Lemma 1, and [4], Theorem 6.1. However, by giving an independant proof, which works because we are only considering " relatively injective " modules, we are able to deduce the general result.

LEMMA 4.1. — *Let $\mathfrak{A}$ be a C*-algebra acting on a Hilbert space, and let $\mathcal{M}$ be the dual space of a complex Banach space. Then*

$$H_c^n(\mathfrak{A}, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})) = \{0\} \qquad (n \geq 1).$$

Proof. — It is notationally convenient to suppose that $\mathfrak{A}$ is in its universal representation, and that φ is the given representation; then writing $\mathfrak{X} = \mathcal{L}_c^1(\varphi(\mathfrak{A}^-) \hat{\otimes} \varphi(\mathfrak{A}^-), \mathcal{M})$, it is sufficient to show that $H_c^n(\varphi(\mathfrak{A}), \mathfrak{X}) = \{0\}$ $(n \geq 1)$. Let $\rho \in Z_c^n(\varphi(\mathfrak{A}), \mathfrak{X})$, and define

$$\rho_1(A_1, \ldots, A_n) = \rho(\varphi(A_1), \ldots, \rho(A_n)) \qquad (A_1, \ldots, A_n \in \mathfrak{A}).$$

Since φ is an isometry on $\mathfrak{A}$, ρ_1 is a bounded multilinear map. Defining $R.x = \varphi(R)x$ $(R \in \mathfrak{A}^-, x \in \mathfrak{X})$, $\mathfrak{X}$ becomes a dual Banach $\mathfrak{A}^-$-module, and with this multiplication, $\rho_1 \in Z_c^n(\mathfrak{A}, \mathfrak{X})$. Since $\mathfrak{A}$ is in its universal representation, ρ_1 is separately ultraweak-weak-★-continuous, and so extends by lemma 2.1 to a bounded multilinear mapping $\tilde{\rho}$: $\mathfrak{A}^- \times \mathfrak{A} \times \ldots \times \mathfrak{A} \to \mathcal{M}$.

For $n \geq 1$, define

$$\begin{aligned}\sigma_1(A_1, \ldots, A_{n-1})(\varphi(R_1) \otimes \varphi(R_2)) \\ = \tilde{\rho}_1(R_1 P, A_1, \ldots, A_{n-1})(\varphi(E) \otimes \varphi(R_2)),\end{aligned}$$

where E is the principal identity in $\mathfrak{A}^-$, P is the projection associated with the representation φ, as described in section 2, $A_1, \ldots, A_{n-1} \in \mathfrak{A}$, and $R_1, R_2 \in \mathfrak{A}^-$. This is well defined since the map $\varphi(R) \mapsto RP$ $(R \in \mathfrak{A}^-)$ is an isomorphism. Further, $\sigma(A_1, \ldots, A_{n-1})$ extends by linearity and continuity (since $\| \varphi(R_1) \| = \| R_1 P \|$) to the whole of $\varphi(\mathfrak{A}^-) \hat{\otimes} \varphi(\mathfrak{A}^-)$, and so $\sigma_1 \in \mathcal{L}_c^{n-1}(\mathfrak{A}, \mathfrak{X})$. Arguing as in lemma 3.3, we have $\Delta\sigma_1 = \rho_1$ on $\mathfrak{A} \times \ldots \times \mathfrak{A}$. Let

$$\sigma(\varphi(A_1), \ldots, \varphi(A_{n-1})) = \sigma_1(A_1, \ldots, A_{n-1}) \qquad (A_1, \ldots, A_{n-1} \in \mathfrak{A});$$

since φ is an isometry on $\mathfrak{A}$, σ is bounded, so $\sigma \in \mathcal{L}_c^{n-1}(\varphi(\mathfrak{A}), \mathfrak{X})$, and

$$\Delta\sigma(\varphi(A_1), \ldots, \varphi(A_n)) = \Delta\sigma_1(A_1, \ldots, A_n) \qquad (A_1, \ldots, A_n \in \mathfrak{A})$$

because of the $\mathfrak{A}$-module structure on $\mathfrak{X}$. Thus if $A_1, \ldots, A_n \in \mathfrak{A}$,

$$\Delta\sigma(\varphi(A_1), \ldots, \varphi(A_n)) = \rho_1(A_1, \ldots, A_n) = \rho(\varphi(A_1), \ldots, \varphi(A_n)).$$

Hence $\Delta\sigma = \rho$ and $H_c^n(\varphi(\mathfrak{A}), \mathfrak{X}) = \{0\}$ $(n \geqq 1)$.

THEOREM 4.2 ([4], Theorem 5.6). — *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space, and $\mathcal{M}$ a dual normal $\mathfrak{A}$-module. Then*

$$H_w^n(\mathfrak{A}, \mathcal{M}) \cong H_c^n(\mathfrak{A}, \mathcal{M}) \qquad (n \geqq 0).$$

Proof. — We note first that from JOHNSON [3], 1 (c), and the corresponding result for normal cohomology (which follows in exactly the same way), it is sufficient to establish the isomorphism for all unital dual normal $\mathfrak{A}^-$-modules $\mathcal{M}$. For each such $\mathcal{M}$, let

$$K_w^n(\mathfrak{A}, \mathcal{M}) = H_c^n(\mathfrak{A}, \mathcal{M}) \qquad (n \geqq 0).$$

We show that $K_w^n(\mathfrak{A}, .)$ is a normal cohomology theory for $\mathfrak{A}$ in the sense of definition 3.4. The required isomorphism is then a consequence of our uniqueness result, theorem 3.5.

Clearly $K_w^0(\mathfrak{A}, \mathcal{M}) = \{m \in \mathcal{M} : Am = mA \ (A \in \mathfrak{A})\}$. Lemma 4.1 shows that $K_w^n(\mathfrak{A}, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})) = \{0\}$ $(n \geqq 1)$, and it remains to check the exactness axiom. Let

$$0 \to \mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}'' \to 0$$

be a split exact sequence of dual normal $\mathfrak{A}$-modules; then in particular we have a split exact sequence of Banach $\mathfrak{A}$-modules, and so JOHNSON [3], Proposition 1.7, applies. This guarantees the existence of a long exact sequence

$$\ldots \to H_c^n(\mathfrak{A}, \mathcal{M}') \xrightarrow{f^n} H_c^n(\mathfrak{A}, \mathcal{M}) \xrightarrow{g^n} H_c^n(\mathfrak{A}, \mathcal{M}'') \to H_c^{n+1}(\mathfrak{A}, \mathcal{M}') \to \ldots$$

and it may be checked that the first two non-trivial maps are indeed the restrictions of f and g. This long exact sequence is then precisely the one required.

THEOREM 4.3. — *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space, and $\mathcal{M}$ a dual normal $\mathfrak{A}^-$-module. Then*

$$H_w^n(\mathfrak{A}, \mathcal{M}) \cong H_w^n(\mathfrak{A}^-, \mathcal{M}) \qquad (n \geqq 0).$$

Proof. — As in theorem 4.2, it is sufficient to establish the isomorphism for each unital dual normal $\mathfrak{A}^-$-module $\mathcal{M}$. For such an $\mathcal{M}$, define

$$K_w^n(\mathfrak{A}^-, \mathcal{M}) = H_w^n(\mathfrak{A}, \mathcal{M}) \qquad (n \geqq 0).$$

Since $\mathcal{M}$ is a normal $\mathfrak{A}^-$-module, if $m \in \mathcal{M}$,

$$Am = mA \quad (A \in \mathfrak{A}) \iff Rm = mR \quad (R \in \mathfrak{A}^-),$$

so $K_w^0(\mathfrak{A}^-, \mathcal{M}) = H_w^0(\mathfrak{A}^-, \mathcal{M})$. Also, by lemma 3.3,

$$K_w^n(\mathfrak{A}^-, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})) = H_w^n(\mathfrak{A}, \mathcal{L}_c^1(\mathfrak{A}^- \hat{\otimes} \mathfrak{A}^-, \mathcal{M})) = \{0\} \qquad (n \geqq 1).$$

Since the exactness of $K_w^n(\mathfrak{A}, .)$ follows directly from the exactness of $H_w^n(\mathfrak{A}, .)$ given in lemma 3.2, the required isomorphism is now a consequence of theorem 3.5.

Corollary 4.4 ([4], Theorem 6.1). — *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space, and let $\mathcal{M}$ be a dual normal $\mathfrak{A}^-$-module. Then*

$$H_c^n(\mathfrak{A}, \mathcal{M}) \cong H_c^n(\mathfrak{A}^-, \mathcal{M}) \qquad (n \geqq 0).$$

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Ian G. Craw,
Department of Mathematics,
King's College,
University of Aberdeen,
Old Aberdeen AB 9 2 UB (Grande-Bretagne).