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THE LERAY-SCHAUDER INDEX AND THE FIXED POINT THEORY FOR ARBITRARY ANRs (1)

BY

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1. Introduction

The Leray-Schauder theory of the degree or the equivalent notion of the fixed point index ([18], [19]) has played a basic role in non-linear functional analysis. In this note, we intend to show that the suitably modified and supplemented theory of the Leray-Schauder index belongs to topology and occupies in fact the central place in topological fixed point theory.

Let U be an open subset of a normed space E, and $f: U \to E$ be a compact map with a compact set of fixed points. To every such f, we assign an integer Ind (f), *the Leray-Schauder index of f*, which satisfies a number naturally expected properties; among those that supplement the classical ones the following two are of especial importance : (i) the Leray-Schauder index Ind (f) is topologically invariant, and (ii) when $f: U \to U$, it is equal to the (generalized) Lefschetz number $\Lambda(f)$ of f [and hence $\Lambda(f) \neq 0$ implies that f has a fixed point].

Now let X be a space which is r-dominated by an open set V in E [= a metric ANR (2)], $r : V \to X$, $s : X \to V$ a pair of maps with $rs = 1_X$. Let U be open in X and $f: U \to X$ be a compact map with a compact set of fixed points. Then the map $sfr : r^{-1}(U) \to V$ is also compact, and we define Ind (f) to be the Leray-Schauder index of sfr. Properties of the Leray Schauder index imply that Ind (f) is the extension of the

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(2) ANR = Absolute Neighbourhood Retract.

former over the larger category of spaces and thus we obtain the topological fixed point index theory for compact maps of arbitrary ANR-s. This theory contains several basic known results in topology (for example, the well-known theory of the fixed-point index for compact ANR-s) and non-linear functional analysis (e. g. the Schauder fixed point theorem, the Birkhoff-Kellogg theorem). It contains also the authors generalization [11] of the Lefschetz fixed point theorem to compact maps of arbitrary ANR-s.

The treatment of the fixed point index theory presented in this note has as its starting point the fixed point index in R^n due to A. Dold [8] and depends also on the notion of the generalized trace as given by J. Leray in [16]. A part of results presented here was announced earlier in some detail in [12].

The author whishes to thank R. Knill for several helpful discussions.

2. The Leray trace

In what follows an essential use will be made of the notion of the generalized trace and the Lefschetz number as given by J. Leray in [16]. We shall consider vector spaces only over the field of rational numbers Q.

A graded vector space $E = \{ E_q \}$ is of a *finite type* provided : (i) $\dim E_q < \infty$ for all q, and (ii) $E_q = 0$ for almost all q. If $f = \{ f_q \}$ is an endomorphism of such a space, then the Lefschetz number $\lambda(f)$ of f is given by

$$\lambda(f) = \Sigma_q (-1)^q \operatorname{tr}(f_q),$$

where tr stands for the ordinary trace function.

Let $f : E \to E$ be an endomorphism of an arbitrary vector space E. Let us put

$$N(f) = \cup_{n \geq 1} \ker f^{(n)}, \qquad \tilde{E} = E / N(f)$$

where $f^{(n)}$ is the n-th iterate of f. Since $f(N(f)) \subset N(f)$, we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. Assume that $\dim \tilde{E} < \infty$; in this case, we define the *generalized trace* $\operatorname{Tr}(f)$ of f by putting $\operatorname{Tr}(f) = \operatorname{tr}(\tilde{f})$.

Now let $f = \{ f_q \} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{ E_q \}$. Call f the *Leray endomorphism* provided the graded space $\tilde{E} = \{ \tilde{E}_q \}$ if of a finite type. For such an f, we define the (generalized) *Lefschetz number* $\Lambda(f)$ by putting

$$\Lambda(f) = \Sigma_q (-1)^q \operatorname{Tr}(f_q).$$

The following important property of the Leray endomorphisms [18] is a consequence of the well-known formula $\operatorname{tr}(uv) = \operatorname{tr}(vu)$ for the ordinary trace :

(2.1) LEMMA. — *Assume that in the category of graded vector spaces the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{u} & E'' \\ {\scriptstyle f'}\uparrow & \nwarrow^{v} & \uparrow{\scriptstyle f''} \\ E' & \xrightarrow{u} & E'' \end{array}$$

Then, if f' or f'' is a Leray endomorphism, then so is the other and in that case $\Lambda(f') = \Lambda(f'')$.

3. Lefschetz maps

Let H be *the singular homology functor* (with the rational coefficients) from the category of topological spaces and continuous mappings to the category of graded vector spaces and linear maps of degree 0. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional singular homology group of X. For a continuous mapping $f: X \to Y$, $H(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$.

A continuous map $f: X \to X$ is called a *Lefschetz map* provided $f_* : H(X) \to H(X)$ is a Leray endomorphism. For such f, we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$.

Clearly, if f and g are homotopic, $f \sim g$, then $\Lambda(f) = \Lambda(g)$.

(3.1) LEMMA. — *Assume that in the category of topological spaces the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi}\uparrow & \nwarrow^{g} & \uparrow{\scriptstyle \psi} \\ X & \xrightarrow{f} & Y \end{array}$$

Then :

(*a*) *if one of the maps φ or ψ is a Lefschetz map, then so is the other and in that case* $\Lambda(\varphi) = \Lambda(\psi)$;

(*b*) *φ has a fixed point if and only if ψ does.*

Proof. — The first part follows (by applying the homology functor to the above diagram) from lemma 2.1. The second part is obvious.

The following are the two instances in which the above lemma is used :

(3.2) *Example.* — Let $f: X \to X$ be a map such that $f(X) \subset K \subset X$. Then we have the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\subset} & X \\ {\scriptstyle f_K}\uparrow & \nwarrow & \uparrow{\scriptstyle f} \\ K & \xrightarrow{\subset} & X \end{array}$$

with the obvious contractions (³).

(3.3) *Example.* — Let $r: Y \to X$, $s: X \to Y$ be a pair of continuous mappings such that $rs = 1_X$. In this case, X is said to be r-dominated by Y and r is said to be an r-map. In this situation, given a map $\varphi: X \to X$, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle \varphi}\uparrow & \nwarrow{\scriptstyle \varphi r} & \uparrow{\scriptstyle \psi} \\ X & \xrightarrow{s} & Y \end{array}$$

with $\psi = s \varphi r$.

4. Compact maps

A continuous map $f: X \to Y$ between topological spaces is called *compact* provided it maps X into a compact subset of Y. Let $h_t: X \to Y$ be a homotopy and $h: X \times I \to Y$ be defined by $h(x, t) = h_t(x)$ for $(x, t) \in X \times I$; then h_t is said to be a *compact homotopy* provided the map h is compact. Two compact maps $f, g: X \to Y$ are *compactly homotopic* provided there is a compact homotopy $h_t: X \to Y$ with $h_0 = f$ and $h_1 = g$. If Y *is a linear space then* f (resp. h_t) is said to be *finite dimensional* provided it is compact and the image $f(X)$ [resp. $h(X)$] is contained in a finite dimensional subspace of Y.

In what follows, we shall combine the Schauder approximation theorem [20] and a result of P. ALEKSANDROV concerning the maps of compacta into the polyhedra.

(4.1) THEOREM (*cf.* [11], [18]). — *Let U be an open subset of a normed space E and let $f: X \to U$ be a compact mapping. Then for every suffi-*

(³) Let $f: X \to Y$ be a map such that $f(A) \subset B$, where $A \subset X$ and $B \subset Y$. By the *contraction* of f to the pair (A, B), we understand a map $f': A \to B$ with the same values as f. A contraction of f to the pair (A, Y) is simply the restriction $f \mid A$ of f to A.

ciently small $\varepsilon > 0$ *there exists a finite polyhedron* $K_\varepsilon \subset U$ *and a mapping* $f_\varepsilon : X \to U$, *called an* ε*-approximation of* f, *such that :*

(i) $\| f(x) - f_\varepsilon(x) \| < \varepsilon$ *for all* $x \in X$;

(ii) $f_\varepsilon(X) \subset K_\varepsilon$;

(iii) *the formula* $h_t(x) = t f_\varepsilon(x) + (1-t) f(x)$ *defines a compact homotopy* $h_t : X \to U$ *joining* f_ε *with* f.

Proof. — Given $\varepsilon > 0$ (which we may assume to be sufficiently small) $f(X)$ is contained in the union of a finite number of open balls $V(y_i, \varepsilon) \subset U$ $(i = 1, 2, \ldots, k)$. Putting for $x \in X$,

$$f_\varepsilon(x) = (\Sigma_i \lambda_i(x) y_i)/(\Sigma_j \lambda_j(x)), \qquad 1 \leqq i \leqq k, \qquad 1 \leqq j \leqq k,$$

where

$$\lambda_i(x) = \max \{ 0, \varepsilon - \| f(x) - y_i \| \},$$

we obtain the map f_ε satisfying (i). Clearly, the values of f_ε are in a finite polyhedron $K_\varepsilon \subset U$ with vertices $y_1, y_2, \ldots, y_k$. Property (iii) is evident.

The proof of the following elementary fact is left as an easy exercise for the reader (*cf.* [10]).

(4.2) LEMMA. — *Let* V *be open in a normed space* E *and assume that* $f : \overline{V} \to E$ *is a compact map with no fixed points on the boundary* ∂V *of* V. *Then :*

(i) *the number* $\eta = \inf_{x \in \partial V} \| x - f(x) \|$ *is positive;*

(ii) *if* $\varepsilon < \eta$, *then any* ε*-approximation* f_ε *of* f *is fixed point free on* ∂V;

(iii) *given any two* ε*-approximations* f'_ε *and* f''_ε *of* f *with* $\varepsilon < (1/2)\eta$, *the formula*

$$h_t(x) = t f'_\varepsilon(x) + (1-t) f''_\varepsilon$$

defines a finite dimensional η*-homotopy* (4) *joining* f' *and* f'' *which has no fixed points on* ∂V.

5. The axioms for the fixed point index

Let $f : U \to X$ be a continuous map between topological spaces. Call f *admissible* provided U is an open subset of X and the fixed point set of f.

$$\chi_f = \{ x \in U, f(x) = x \} \subset U$$

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(4) A homotopy $h_t : X \to Y$ into a metric space (Y, ρ) is said to be an ε*-homotopy* provided $\rho(h_t(x), h_{t'}(x)) < \varepsilon$ for all $x \in X$ and $t, t' \in (0, 1)$. If $f, g : X \to Y$ can be joined by an ε-homotopy, then we say that f is ε-homotopic to g and write $f \tilde{\varepsilon} g$; clearly $f \tilde{\varepsilon} g$ implies, in particular, that $\rho(f(x), g(x)) < \varepsilon$ for all $x \in X$.

is compact. A homotopy $h_t : U \to X$ will be called *admissible* provided the set $\varkappa(\{h_t\}) = \cup_{0 \leq t \leq 1} \varkappa(h_t)$ is compact.

Definition (A. Dold [8]). — Let $\mathfrak{C}$ be a category of topological spaces in which a class $\mathfrak{A} = \mathfrak{A}(\mathfrak{C})$ of admissible maps and homotopies is distinguished. By a *fixed-point index* on $\mathfrak{A}$ we shall understand a function $\mathrm{Ind} : \mathfrak{A} \to Z$ which satisfies the following conditions :

(I) *Excision.* — If $U' \subset U$ and $\varkappa_f \subset U'$, then the restriction

$$f' = f \mid U' : U' \to X$$

is in $\mathfrak{A}$ and $\mathrm{Ind}(f) = \mathrm{Ind}(f')$.

(II) *Additivity.* — Assume that $U = \cup_i U_i$, $1 \leq i \leq k$, $f_i = f \mid U$ and the fixed point set $\varkappa_i = \varkappa_f \cap U_i$ are mutually disjoint, $\varkappa_i \cap \varkappa_j = \emptyset$ for $i \neq j$. Then

$$\mathrm{Ind}\, f = \Sigma_i \,\mathrm{Ind}\, f_i, \qquad 1 \leq i \leq k.$$

(III) *Fixed points.* — If $\mathrm{Ind}\, f \neq 0$, then $\varkappa_f \neq \emptyset$, i. e., the map f has a fixed point.

(IV) *Homotopy.* — Let $h_t : U \to X$, $0 \leq t \leq 1$, be an admissible homotopy in $\mathfrak{A}$. Then $\mathrm{Ind}(h_0) = \mathrm{Ind}(h_1)$.

(V) *Multiplicativity.* — *If $f_1 : U_1 \to X_1$ and $f_2 : U_2 \to X_2$ are in $\mathfrak{A}$ then so is the product map $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$ and*

$$\mathrm{Ind}(f_1 \times f_2) = \mathrm{Ind}(f_1) . \mathrm{Ind}(f_2).$$

(VI) *Commutativity.* — Let $U \subset X$, $U' \subset X'$ be open and assume $f : U \to X'$, $g : U' \to X$ are maps in $\mathfrak{C}$. If one of the composites

$$gf : V = f^{-1}(U') \to X \qquad \text{or} \qquad fg : V' = g^{-1}(U) \to X'$$

is in $\mathfrak{A}$, then so is the other and, in that case,

$$\mathrm{Ind}(gf) = \mathrm{Ind}(fg).$$

(VII) *Normalization.* — If $U = X$ and $f : X \to X$ is in $\mathfrak{A}$, then f is a Lefschetz map, and $\mathrm{Ind}(f) = \Lambda(f)$.

6. The fixed point index in R^n

In the following definition H is the singular homology over the integers Z. Let us fix for each n an orientation $1 \in H_n(S^n)$ of the n-th sphere $S^n = \{ x \in \mathbf{R}^{n+1};\ \| x \| = 1 \}$ and accordingly identify $H_n(S^n) \approx Z$ with the integers Z.

DEFINITION (*cf.* A. DOLD [8]). — Let $f: U \to R^n$ be an admissible map. Denote by $K = \varkappa_f$ the fixed point set for f and by

$$(i - f) : \quad (U, U - K) \to (R^n, R^n - \{0\})$$

the map given by $(i - f)(x) = x - f(x)$. The fixed point index Ind f of the map f is defined to be the image of 1 under the composite map

$$Z = H_n(S^n) \to H_n(S^n, S^n - K) \underset{\approx}{\to} H_n(U, U - K) \xrightarrow{(i-f)_*} H_n(R^n, R^n - 0) \approx Z.$$

The following theorem established by A. DOLD [8] represents a modernized version of the classical result due essentially to H. HOPF.

(6.1) (*The fixed point index in* R^n). *Let* $\mathfrak{C}$ *be the category of open subsets of euclidean spaces and* $\mathfrak{A}(\mathfrak{C})$ *the class of all continuous admissible maps in* $\mathfrak{C}$. *Then the integer valued function* $f \to \operatorname{Ind} f$ *defined above satisfies the properties* (I)-(VII). *In* (VII), *it is assumed that f is compact.*

We note that the excision and the commutativity implies the following property of the index :

(VIII) *Contraction.* — Let U be open in R^{n+1} and $f: U \to R^{n+1}$ be an admissible map such that $f(U) \subset R^n$. Denote by $f': U' \to R^n$ the contraction of f, where $U' = U \cap R^n$. Then Ind (f) = Ind (f').

7. The Leray-Schauder index

Let U be an open subset of a normed space E and let $f: U \to E$ be an admissible compact map. Take an open set $V \subset U$ such that $\partial V \subset U$ and $\varkappa_f \subset V$. Then the number $\eta = (1/2) \inf \| x - f(x) \|$ for $x \in \partial V$ is positive.

Let $g = f \mid V : V \to E$. From the definition of η and lemma (4.2) it follows that :

(i) if $\varepsilon < \eta$, then every ε-approximation $g_\varepsilon : V \to E$ of g is admissible;

(ii) given two ε-approximations $g'_\varepsilon, g''_\varepsilon : V \to E$ of g with $\varepsilon < \eta$ there exists an admissible finite dimensional compact homotopy $h_t : V \to E$ $0 \leq t \leq 1$, such that $h_0 = g'$, $h_1 = g''$.

Let $f: U \to E$ be an admissible compact map and $g_\varepsilon : V \to E$ be an ε-approximation of $g = f \mid V$ as above. Denote by E^n a finite dimensional subspace of E which contains $g_\varepsilon(V)$ and by $g'_\varepsilon : E^n \cap V \to E^n$ the evident contraction of g_ε.

Let us put Ind (f, V) = Ind (g'_ε). It follows from (i) and (ii) and the homotopy and contraction properties of the index in R^n, that Ind (f, V) does not depend on the choice g'_ε. Moreover, given V_1, V_2 with the same properties as V, we have

(7.1.1) $$\operatorname{Ind}(f, V_1) = \operatorname{Ind}(f, V_2).$$

For the proof of (7.1.1) we distinguish two cases :

(i) $V_1 \subset V_2$;

(ii) V_1 and V_2 are arbitrary.

In the first case, our assertion follows by the excision of the index in R^n and the second case reduces, évidently, to the first.

Definition. — For an admissible compact map $f\colon U \to X$, we define the *Leray-Schauder index* $\operatorname{Ind}(f)$ of f by putting

$$\operatorname{Ind}(f) = \operatorname{Ind}(f, V) = \operatorname{Ind}(g'_\varepsilon).$$

It follows from (7.1.1) that $\operatorname{Ind}(f)$ is well defined.

We may state now the first main result of this note :

(7.1) Theorem. — *Let $\mathfrak{C}$ be the category of open subsets in linear normed spaces and let $\mathfrak{A}$ be the class of all admissible compact maps in $\mathfrak{C}$. Assume that all admissible homotopies are compact. Then, defined on $\mathfrak{A}$, the Leray-Schauder index $f \to \operatorname{Ind}(f)$ satisfies the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps f or g is compact.*

Proof. — Using the approximation theorem (4.1), lemma (4.2), properties (I)-(V) follow in a straightforward manner from the corresponding properties of the index in R^n. The proof of property (VI), which is somewhat more involved, will be given separately in section 8. It remains to establish the normalization property.

Proof of property (VII). — Given a compact map $f\colon U \to U$ let $\varepsilon > 0$ be smaller than dist $(x_f, \partial U)$, $f_\varepsilon : U \to U$ be an ε-approximation of f such that its values are in a finite dimensional subspace E^n of E and let $U_n = U \cap E^n$. Clearly every such f_ε is admissible and $f \sim f_\varepsilon$.

Consider the following commutative diagram in which all the arrows represent either the obvious inclusions or the contractions of the map f_ε :

$$\begin{array}{ccc} U_n & \xrightarrow{\ \subset\ } & U \\ {\scriptstyle f'_\varepsilon}\uparrow & \nwarrow & \uparrow{\scriptstyle f_\varepsilon} \\ U_n & \xrightarrow{\ \subset\ } & U \end{array}$$

By the definition $\operatorname{Ind}(f) = \operatorname{Ind}(f'_\varepsilon)$. By lemma (3.1) [Example (3.2)], we have $\Lambda(f') = \Lambda(f_\varepsilon)$ and, consequently, in view of theorem (7.1) (property VII), Ind $(f) = \Lambda(f_\varepsilon)$. Since f is homotopic to f_ε, this implies that Ind $(f) = \Lambda(f)$ and the proof is completed.

We remark that theorem (7.1) (the commutativity) contains the following important property of the Leray-Schauder index :

(IX) *Topological Invariance.* — Let $f: U \to E$ be an admissible compact map, and $h: E \to E'$ be a homeomorphism. Then $h \circ f \circ h^{-1}: h(U) \to E'$ is also an admissible compact map and

$$\text{Ind}\,(h \circ f \circ h^{-1}) = \text{Ind}\,(f).$$

8. Proof of the commutativity

In the proof of commutativity, we shall use the fact that the Leray-Schauder index satisfies properties (I), (IV) (V) and (VII).

Let $U \subset E$, $U' \subset E'$ be open in normed spaces E and E', respectively, $f: U' \to E$, $g: U \to E'$ be continuous and consider the composites

$$g \circ h : f^{-1}(U) \to E', \qquad f \circ g : g^{-1}(U') \to E.$$

We note that the maps $f: \varkappa(gf) \to \varkappa(fg)$ and $g: \varkappa(fg) \to \varkappa(gf)$ are inverse to each other and hence the fixed point sets $\varkappa(gf)$ and $\varkappa(fg)$ are homeomorphic; thus, if one of them is compact, then so is the other. In the proof of commutativity, we shall distinguish two cases :

Special case (both f and g are compact) : In this case, we proceed essentially as in Dold [8].

We let $\varphi : U' \times U \to E' \times E$ be given by

$$\varphi(x, y) = (g(y), f(x)). \tag{8.1.1}$$

And we define the homotopies :

$$\begin{aligned} h_t, h'_t &: U' \times U \to E' \times E, \\ H_t &: U' \times E \to E' \times E, \\ H'_t &: E' \times U \to E' \times E, \end{aligned}$$

by the following formulas :

$$\left\{ \begin{aligned} h_t(x, y) &= [tgf(x) \\ &\qquad + (1-t)g(y), f(x)], \\ h'_t(x, y) &= [g(y), tfg(y) \\ &\qquad + (1-t)f(x)], \end{aligned} \right. \quad (x, y, t) \in U' \times U \times I,$$

$$\left\{ \begin{aligned} H_t(x, y) &= [gf(x), (1-t)f(x)], \qquad (x, y, t) \in U' \times E \times I, \\ \mathrm{H}'_t(x, y) &= [(1-t)g(y), fg(y)], \qquad (x, y, t) \in U \times E' \times I. \end{aligned} \right. \tag{8.1.2}$$

We have

(8.1.3) $$\varphi = h_0 = h'_0.$$

By assumption, f and g are compact; this implies that φ and all the above homotopies are compact; since the fixed point sets $\varkappa_\varphi$ and $\varkappa_{gf}$ are homeomorphic under $x \mapsto (x, f(x))$, $(x, y) \mapsto x$, φ is admissible.

Moreover, simple computation shows that the fixed point sets

$$\varkappa_\varphi = \varkappa(\{ h_t \}) = \varkappa(\{ h'_t \})$$

coincide and, therefore, the homotopies h_t and h'_t are admissible. By straightforward argument, one shows that H_t and H'_t are also admissible.

Therefore, by homotopy, we have, in view of (8.1.3),

(8.1.4) $$\mathrm{Ind}(\varphi) = \mathrm{Ind}(h_1) = \mathrm{Ind}(h'_1).$$

On the other hand, since

$$h_1 = H_0 \mid U' \times U, \qquad h'_1 = H'_0 \mid U' \times U,$$

we get by excision and homotopy,

(8.1.5) $$\mathrm{Ind}(\varphi) = \mathrm{Ind}(H_1) = \mathrm{Ind}(\mathrm{H}'_1).$$

Both H_1 and H'_1 are product maps

(8.1.6) $$H_1 = (gf) \times (\mathrm{Cte}), \qquad H'_1 = (\mathrm{Cte}) \times (fg).$$

By multiplicativity, in view of (8.1.5) and (8.1.6),

$$\mathrm{Ind}(gf) . \mathrm{Ind}(\mathrm{Cte}) = \mathrm{Ind}(\mathrm{Cte}) . \mathrm{Ind}(fg)$$

and hence by normalization [since Λ (Cte) $= 1$] we get Ind $(gf) =$ Ind (fg).

General case: We assume now that $f: U' \to E$ is compact and $g: U \to E'$ continuous. To show that

$$\mathrm{Ind}(gf \mid f^{-1}(U)) = \mathrm{Ind}(fg \mid g^{-1}(U')),$$

we assume that gf (and hence fg) is admissible.

Take a smaller open set $O \subset U$ such that :

(i) O isbounded;

(ii) $\partial O \subset U$;

(iii) $\varkappa_{fg} \subset O$,

and put $O' = f^{-1}(O)$. Clearly $\varkappa(gf) \subset O'$, and we may assume that $\partial O' \subset U'$ ([5]).

Now both $gf : f^{-1}(O) \to E'$ and $fg : g^{-1}(O') \to E$ are compact. By excision, it is sufficient to show that

$$\text{(8.1.7)} \qquad \operatorname{Ind}(gf \mid f^{-1}(O)) = \operatorname{Ind}(fg \mid g^{-1}(O')).$$

Let us put

$$\text{(8.1.8)} \qquad \begin{cases} \eta_1 = \inf \| x - g f(x) \| & \text{for } x \in \partial f^{-1}(O), \\ \eta_2 = \inf \| y - f g(y) \| & \text{for } y \in \partial g^{-1}(O'), \\ \eta = \min(\eta_1, \eta_2). \end{cases}$$

By lemma 4.2, the number η is positive.

Let K be a compact set containing $f(U') \subset E$. Consider the map $g : U \to E'$ at points of a compact set $K \cap \overline{O} \subset U$. Continuity of g implies that for each $y \in K \cap \overline{O}$ there is a $\delta_y > O$ such that :

(i) the open ball $V(y, \delta_y)$ with center y and radius δ_y is contained in U;

(ii) $y', y'' \in V(y, \delta_y) \Rightarrow \| (g(y') - g(y'') \| < \eta$.

From the compactness of $K \cap \overline{O}$ it follows that a finite number of balls $V(y_1, \delta_{y_1}), \ldots, V(y_k, \delta_{y_k})$ covers $K \cap \overline{O}$.

We let

$$\text{(8.1.9)} \qquad \begin{cases} \delta = \min(\delta_{y_1}, \ldots, \delta_{y_k}), \\ V = \cup_i V(y_k, \delta_{y_i}), \qquad 1 \leqq i \leqq k, \\ \varepsilon = \min(\delta, \eta). \end{cases}$$

Clearly, from (8.1.9), it follows that

$$\text{(8.1.10)} \qquad \begin{cases} \text{if } y \in K \cap \overline{O} \text{ and } \| y' - y \| < \varepsilon, \text{ then} \\ \qquad \| g(y) - g(y') \| < \eta \\ \text{and} \\ ty + (1 - t) y' \in V \quad \text{for all } t \in [0, 1]. \end{cases}$$

Now let $f_\varepsilon : U' \to E$ be an ε-approximation of $f : U' \to E$ and $h_t : U' \to E$ be given by $h_t(x) = t f(x) + (1 - t) f_\varepsilon(x)$ clearly; h_t is an ε-homotopy joining compactly f anf f_ε. Since on $\overline{f^{-1}(O)} \subset f^{-1}(\overline{O})$ the values of h_t are in a compact subset of $V \subset U$, we may consider on $\overline{f^{-1}(O)}$ the compo-

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([5]) In view of the excision we may suppose (by taking slightly smaller open sets) that f and g are defined in fact on ∂U and ∂U′ respectively.

sition gh_t. It follows clearly from (8.1.10) that gf_t is an η-homotopy and therefore by lemma 4.2 (in view of the definition of η) it has no fixed points on $\partial f^{-1}(O)$; thus

$$gh_t : f^{-1}(O) \to E'$$

is an admissible homotopy joining gf_ε and gf on $f^{-1}(O)$.

Consequently, by homotopy, we have

$$\text{(8.1.11)} \qquad \mathrm{Ind}\,(gf \mid f^{-1}(O)) = \mathrm{Ind}\,(gf_\varepsilon \mid f^{-1}(O)).$$

Next, since f_ε is finite dimensional, $f_\varepsilon(U') \subset E^n$, we may write the following contractions

$$\text{(8.1.12)} \qquad \begin{cases} \tilde{f}_\varepsilon : & f^{-1}(O) \to E^n \cap O, \\ \tilde{g} : & E^n \cap O \to E' \end{cases}$$

of f_ε and g respectively. On $f^{-1}(O)$, we have $g \circ f_\varepsilon = \tilde{g} \circ \tilde{f}_\varepsilon$ and, therefore

$$\text{(8.1.13)} \qquad \mathrm{Ind}\,(gf_\varepsilon \mid f^{-1}(O)) = \mathrm{Ind}\,(\tilde{g}\tilde{f}_\varepsilon \mid f^{-1}(O)).$$

Further, since $\tilde{g} \subset g \mid E^n \cap \overline{O}$ and O is bounded, we conclude that $\tilde{g}$ is compact. Thus, both $\tilde{f}_\varepsilon$ and $\tilde{g}$ being compact, we may apply the special case of commutativity. We have

$$\text{(8.1.14)} \qquad \mathrm{Ind}\,(\tilde{g}\tilde{f}_\varepsilon \mid f^{-1}(O)) = \mathrm{Ind}\,(\tilde{f}_\varepsilon \circ \tilde{g} \mid \tilde{g}^{-1} f^{-1}(O))$$

and finally, since $O' = f^{-1}(O)$, we obtain from (8.1.11), (8.1.13) and (8.1.14),

$$\text{(8.1.15)} \qquad \mathrm{Ind}\,(gf \mid f^{-1}(O)) = \mathrm{Ind}\,(f_\varepsilon \circ \tilde{g} \mid \tilde{g}^{-1}(O')).$$

On the other hand, consider the composition $h_t g$ on $\overline{g^{-1}(O')} \subset g^{-1}(\overline{O'})$. Clearly $h_t g$ is a compact ε-homotopy joining fg and $f_\varepsilon \circ g$; since $\varepsilon < \eta$, it is an η-homotopy and, hence, by lemma 4.2, it is fixed point free on $\partial g^{-1}(O')$. In other words, $h_t g : g^{-1}(O') \to E$ is an admissible compact homotopy joining $f_\varepsilon \circ g$ and fg on $g^{-1}(O')$, and consequently (by homotopy) we have

$$\text{(8.1.16)} \qquad \mathrm{Ind}\,(fg \mid g^{-1}(O')) = \mathrm{Ind}\,(f_\varepsilon \circ g \mid g^{-1}(O')).$$

Since the values of $f_\varepsilon \circ g$ are in $E^n \cap O$, we have, by the definition of the Leray-Schauder index,

$$\text{(8.1.17)} \qquad \mathrm{Ind}\,(f_\varepsilon \circ g \mid g^{-1}(O')) = \mathrm{Ind}\,(f_\varepsilon \circ g \mid g^{-1}(O') \cap O \cap E^n),$$

and hence, because $g^{-1}(O') \cap O \cap E^n = \tilde{g}^{-1}(O')$, we get

$$\text{Ind}(fg \mid g^{-1}(O')) = \text{Ind}(f_\varepsilon \circ \tilde{g} \mid \tilde{g}^{-1}(O')). \tag{8.1.18}$$

By comparing formulas (8.1.18) and (8.1.15), we obtain the desired conclusion (8.1.7), and thus, the proof of commutativity is completed.

9. Compact maps of the ANR-spaces

We propose now as the first consequence of the Leray-Schauder index a general fixed point theorem which on the one hand contains the classical Lefschetz theorem for compact ANR-s, and on the other hand contains various fixed point theorems of the non-linear functional analysis.

We denote by ANR (resp. AR) the class of metrizable absolute neighbourhood retracts (resp. absolute retracts). We recall (*cf.* [3]) that a metrizable space Y is an ANR (resp. AR) provided for any metrizable pair (X, A), with A closed in X and any continuous $f_0 : A \to Y$, there exists an extension $f : U \to Y$ of f_0 over a neighbourhood U of A in X (resp. an extension $f : X \to Y$ of f_0 over X).

(9.1) *Example.* — The following are some typical and important properties of the ANR spaces :

(i) If X is r-dominated by Y, then $Y \in \text{ANR}$ implies $X \in \text{ANR}$;

(ii) If U is open in X, and $X \in \text{ANR}$, then $U \in \text{ANR}$;

(iii) If X is convex subset of a normed (or locally convex metrizable) linear space, then $X \in \text{AR}$ (J. Dugundji [9]).

(iv) A metrizable space which is locally ANR is an ANR; in particular manifolds, Banach manifolds are ANR-s.

In what follows, we shall use essentially the following fact from general topology :

(9.2) (Arens-Eells [1]) : *Every metrizable space can be embedded as a closed subset of a linear normed space.*

The above Arens-Eells embedding theorem permits to give the following simple characterization of the ANR-s :

(9.3) *In order that* $Y \in \text{ANR}$ (*resp.* $Y \in \text{AR}$, *it is necessary and sufficient that* Y *be r-dominated by an open set of a normed space* (*resp. by a normed space*).

Proof. — Let $Y \in \text{ANR}$. By theorem (9.2), there exists an embedding $\mathfrak{I} : Y \to E$ of Y into a normed space E such that $\mathfrak{I}(Y)$ is closed in E. Take a retraction $r : U \to \mathfrak{I}(Y)$ of an open set $U \supset \mathfrak{I}(Y)$. Then

$\vartheta^{-1} r : U \to Y$ is clearly an r-map. The converse follows from the general properties of the ANR-s [*cf.* Example (9.1)]. The proof of the second part is similar.

(9.4) THEOREM (*cf.* [12]). — *Let X be an* ANR *and* $f : X \to X$ *be a compact map. Assume further that U is open in a normed space E and* $s : X \to U$, $r : U \to X$ *be an arbitrary pair of maps with* $rs = 1_X$. *Then f is a Lefschetz map and the Lefschetz number of f is equal to the Leray-Schauder index of the map sfr,* $\Lambda(f) = \text{Ind}(sfr) = \Lambda(sfr)$.

Proof. — Theorem (9.4) clearly follows from theorem (7.1) and lemma (3.1) [Example (3.3)].

As a consequence of theorems (9.4), (7.1), we get the following generalization of the Lefschetz fixed point theorem, established by the author in [11].

(9.5) THEOREM. — *Let X be an* ANR *and* $f : X \to X$ *be a compact map. Then :*

(i) *f is a Lefschetz map;*

(ii) $\Lambda(f) \neq 0$ *implies that f has a fixed point.*

As an illustration, we list a number of well-known consequences of theorem (9.5) :

COROLLARY 1 (*Lefschetz fixed point theorem for compact* ANR-s). — *Let X be a compact* ANR *and* $f : X \to X$ *be continuous. Then* $\lambda(f) \neq 0$ *implies that f has a fixed point.*

COROLLARY 2. — *Let X be an acyclic* ANR *or, in particular, an* AR. *Then any compact map* $f : X \to X$ *has a fixed point.*

COROLLARY 3 (*Schauder fixed point theorem* [20]). — *Let X be a convex (not necessarily closed) subset of a normed (or locally convex metrizable) linear space. Then any compact* $f : X \to X$ *has a fixed point.*

Proof. — X is an AR [*cf.* Example (9.1)] and hence the assertion follows from corollary 3.

COROLLARY 4 (*Birkhoff-Kellog theorem* [2]). — *Let* $S = \{ x \in E; \| x \| = 1 \}$ *be the unit sphere in an infinite dimensional normed space E and* $f : S \to E$ *be a compact map satisfying*

(9.5.1) $$\| f(x) \| \geqq \alpha > 0 \quad \text{for all } x \in S.$$

Then there exists an invariant direction for f, i. e., for some $x_0 \in S$ *and* $\mu > 0$, *we have* $f(x_0) = \mu x_0$.

Proof. — Let us put for each $x \in S$,

$$\varphi(x) = f(x)/\| f(x) \|.$$

Then (9.5.1) implies that the map $\varphi : S \to S$ is compact. Since S is clearly an acyclic ANR (and even an AR, *cf.* [9]), φ has a fixed point, i. e.,

$$\varphi(x_0) = f(x_0)/\| f(x_0) \| = x_0.$$

for some x_0 and the proof of our assertion is competed.

Corollary 5 (Browder-Eells [6]). — *Let X be a Banach (or more generally a Fréchet) manifold and $f : X \to X$ a compact map. Then $\Lambda(f)$ is defined, and $\Lambda(f) \neq 0$ implies that f has a fixed point.*

10. Fixed point index theory for arbitrary ANR-s

Now we turn to the main application of the Leray-Schauder index by establishing the existence of the fixed point index theory for compact maps of arbitrary ANR-s.

Definition (*cf.* [7] and [8]). — Let X be an ANR, and $f : U \to X$ an admissible compact map. To define Ind (f), take an open set V in a normed space E which r-dominates X. Let $s : X \to V$, $r : V \to X$ be a pair of maps with $rs = 1$. Since the composite map

$$r^{-1}(U) \xrightarrow{r} U \xrightarrow{f} X \xrightarrow{s} V$$

is compact and admissible [because $\varkappa(f) = \varkappa(sfr)$] its Leray-Schauder index is defined by theorem (7.1), and we define

$$\text{(10.1.1)} \qquad \operatorname{Ind}(f) =_{\text{df}} \operatorname{Ind}(sfr).$$

Let $V' \subset E'$ be another open set in a normed space E', which r-dominates X, with $s' : X \to V'$, $r' : V' \to X$, $r's' = 1$. Then, since the second of the maps $sr' : V' \to V$, $s'fr : r^{-1}(U) \to V'$ is compact we may apply the commutativity property for the Leray-Schauder index and hence

$$\operatorname{Ind}((s'fr) \circ (sr') \mid (sr')^{-1}(r^{-1}(U)) = \operatorname{Ind}((sr') \circ (s'fr) \mid r^{-1}(U)).$$

Since $(s'fr) \circ (sr') = s'fr'$, $(sr') \circ (s'fr) = sfr$ and because

$$(sr')^{-1}(r^{-1}(U)) = r'^{-1}(U),$$

we get

$$\operatorname{Ind}(s'fr' \mid r'^{-1}(U)) = \operatorname{Ind}(sfr \mid r^{-1}(U)),$$

which proves that our definition is independent of the choices involved.

Now we may state our second main result :

(10.1) THEOREM. — *Let* $\mathfrak{C}$ *be a category of metric* ANR-s *and* $\mathfrak{A}$ *be a class of all admissible compact maps in* $\mathfrak{C}$. *Assume further that all admissible homotopies are compact. Then the fixed point index function* $f \to \mathrm{Ind}(f)$ *defined by formula* (10.1.1) *satisfies all the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps f or g is compact, which implies that the fixed point index is a topological invariant.*

Proof. — The normalization property was already established in the previous section. All the remaining properties follow easily from the corresponding properties of the Leray-Schauder index. Let us prove for instance property (VI). The proofs of other properties, being similar, are omitted.

Proof of property (VI). — Let $X, X' \in \mathrm{ANR}$, $f : U' \to X$, $g : U \to X'$ be admissible maps and assume that f is compact. Let V (resp. V') be an open set in a normed space E (resp. E') which r-dominates X (resp. X'); denote by $X \xrightarrow{s} V \xrightarrow{r} X$, $X' \xrightarrow{s'} V' \xrightarrow{r'} X'$ two pairs of maps satisfying $rs = 1_X$, $r' s' = 1_{X'}$.

Consider the following maps :

$$\begin{aligned} s\,fr' &: \; r'^{-1}(U') \to V, \\ s'\,gr &: \; r^{-1}(U) \to V', \end{aligned}$$

and note that the first of them is compact. It follows by commutativity of the Leray-Schauder index (applied to the above maps) that

$$\mathrm{Ind}\,((sfr')\,(s'\,gr) \mid (s'\,gr)^{-1}\,r'^{-1}(U')) = \mathrm{Ind}\,((s'\,gr)\,(sfr') \mid (sfr')^{-1}\,r^{-1}(U))$$

and hence, in view of

$$\begin{aligned} (s'\,gr)^{-1}\,r'^{-1}(U') &= r^{-1}\,g^{-1}(U'), \\ (sfr')^{-1}\,r^{-1}(U) &= r'^{-1}\,f^{-1}(U); \end{aligned}$$

we get

$$\mathrm{Ind}\,(sfgr \mid r^{-1}\,g^{-1}(U')) = \mathrm{Ind}\,(s'\,gfr' \mid r'^{-1}\,f^{-1}(U)).$$

From this (in view of the definition of the fixed point index), we get

$$\mathrm{Ind}\,(fg \mid g^{-1}(U')) = \mathrm{Ind}\,(gf \mid f^{-1}(U)),$$

and the proof is completed.

11. Remarks on the non-metrizable case

First, we note that the approximation theorem (4.1) extends (with appropriate modifications) to the case when U is open in locally convex topological space E.

This fact permits to extend the Leray-Schauder index to the case of locally convex spaces and to state theorem (7.1) in the following more general form :

(11.1) THEOREM. — *Let* $\mathfrak{C}$ *be the category of open subsets of locally convex topological spaces. Let* $\mathfrak{A} = \mathfrak{A}(\mathfrak{C})$ *be the class of all admissible compact maps and assume that all admissible homotopies are compact. Then, there exists a functions* Ind : $\mathfrak{A} \to Z$ *(the Leray-Schauder index) which satisfies the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps* f *or* g *is compact.*

Now, by proceeding as in the metrizable case, one gets from theorem (11.1) the following generalization of theorem (10.1) :

(11.2) THEOREM. — *Let* $\mathfrak{C}$ *be the category of spaces which are r-dominated by open sets in linear locally convex topological spaces. Let* $\mathfrak{A} = \mathfrak{A}(\mathfrak{E})$ *be the class of all admissible compact maps and assume that all admissible homotopies are compact. Then, there is on* $\mathfrak{A}$ *an integer valued function* $f \to \text{Ind}(f)$, *which satisfies all the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps* f *or* g *is compact; in particular,* Ind (f) *is topologically invariant.*

Let X be a compact ANR for normal spaces and $h : X \to E'$ be an embedding of X into a locally convex space E'. It can be shown that the linear span E of the compact set $h(X)$ in E' is normal. It follows that X is r-dominated by a set open in a locally convex space. We obtain, therefore, as a special case of theorem (11.2) the following :

COROLLARY (*Fixed point index for compact non-metrizable* ANR-s). — *Let* $\mathfrak{C}$ *be the category of compact* ANR-s *for normal spaces and* $\mathfrak{A}$ *be the class of all continuous admissible maps in* $\mathfrak{C}$. *Then there is on* $\mathfrak{A}$ *the fixed point index which satisfies all the properties* (I)-(VII).

We remark that the fixed point index for compact (non metrizable) ANR-s was established previously by combinatorial means (and in a different form) by several authors (*cf.* J. LERAY [15], A. DELEANU [7], D. BOURGIN [4], F. BROWDER [5]).

12. Other generalizations

In the definition of the fixed point index of f, only of importance is the behaviour of f in the neighbourhood of the fixed point set $\varkappa_f$. This general remark indicates how to enlarge the class of maps for which the fixed point index is defined.

Definition. — Let X be an ANR and $f : U \to X$ an admissible map satisfying the following condition :

(12.1.1) for some neighbourhood V of the fixed point set $\varkappa_f$, the restriction $f \mid V$ is compact.

For such f we define the fixed point index of f by putting

$$\text{Ind}\,(f) = \text{Ind}\,(f \mid V). \tag{12.1.2}$$

[Example : every admissible map which is locally compact satisfies condition (12.1.1).]

With the above definition, we have the following generalization of theorem (10.1) :

(12.1) Theorem. — *Let* $\mathfrak{C}$ *be the category of metric* ANR-s *and* $\mathfrak{A}$ *a class of all admissible maps satisfying condition* (12.1.1). *Assume further that, given an admissible homotopy* h_t *there is a neighbourhood* W *of* $\varkappa(\{h_t\})$ *such that* h_t *is compact on* W. *Then the function* $f \to \text{Ind}\,(f)$ *defined by* (12.1.2) *satisfies properties* (I)-(VII). *In* (VI), *it is assumed that* f *is compact in some neighbourhood of* $\varkappa(gf)$ *and in* (VII) *it is assumed that* f *is compact.*

13. The uniqueness of the fixed point index

Let $\mathfrak{C}_0$ be the category of open sets in finite dimensional normed space and $\mathfrak{A}_0$ the class of admissible maps. It can be proved that the Dold index Ind : $\mathfrak{A}_0 \to Z$, defined in section 6, is determined *uniquely* by properties (I)-(VII). We indicate now how the uniqueness of Dold's index implies that of the other fixed point indices discussed in this paper.

Let $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$) be the category of open subsets of normed (resp. ANR) spaces, $\mathfrak{A}_1$ (resp. $\mathfrak{A}_2$) the class of admissible compact maps and assume that all admissible homotopies are compact. Let ind : $\mathfrak{A}_1 \to Z$ be an integer valued function satisfying properties (I)-(VII). The excision and commutativity imply that ind satisfies also the contraction property (similar to that in section 6). Since every compact map is compactly homotopic to a finite dimensional map, it follows by homotopy, excision and contraction, that the function ind is completely determined by its values on maps in $\mathfrak{A}_0$. Consequently, in view of the uniqueness of Dold's index on $\mathfrak{A}_0$, ind must coincide with the Leray-Schauder index Ind.

Next, let ind : $\mathfrak{A}_2 \to Z$ be defined on $\mathfrak{A}_2$ and assume that it satisfies properties (I)-(VII). Let $f : U \to X$ be a map in $\mathfrak{A}_2$, V be an open set in as normed space which r-dominates X with $s : X \to V$, $r : V \to X$, $rs = 1$. By commutativity applied to maps

$$s : \ X \to V, \qquad fr : \ r^{-1}(U) \to X$$

we get

$$\text{ind}\,(frs \mid s^{-1}\,r^{-1}\,(U)) = \text{ind}\,(f) = \text{ind}\,(sfr \mid r^{-1}\,(U)).$$

Thus if the function ind satisfies commutativity, then it is completely determined by its values on maps in $\mathfrak{A}_1$. Consequently, if it satisfies also properties (I)-(VI), it must necessarily be the unique extension of the Leray Schauder index from $\mathfrak{A}_1$ over $\mathfrak{A}_2$. Thus we get the uniqueness of the index constructed in section 7.

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