BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 99 (1971), p. 241-245
<http://www.numdam.org/item?id=BSMF_1971__99__241_0>

Bull. Soc. math. France,
99, 1971, p. 241 à 245.

A METRIC PROPERTY OF SOME RANDOM FUNCTIONS

BY

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0. **Introduction.** — Let E_1 and E_2 be compact linear sets of Hausdorff dimensions $d_1 > 0$ and $d_2 > 0$, and suppose that $d = d_1 + d_2 < 1$. For real numbers $t > 0$, F_t is the set $E_1 + tE_2$. It is proved in [3], that $\dim F_t \geq d$ excepting a t-set of dimension at most d; in [4], the exceptional set has positive dimension. It is thus natural to search for sets " complementary " to a given set E_2; namely, sets E_1 for which the exceptional t-set is void. This particular problem is solved by some theorems in ([1], chap. XV), founded on harmonic analysis. However, the analysis of that work uses special properties of linear mappings, while the method of [3] applies without change to the following non-linear variant. Consider a continuously differentiable function $h(u, v)$ defined in the plane, whose partial derivatives satisfy inequalities $1 \leq \frac{\partial h}{\partial u}, \frac{\partial h}{\partial v} \leq C$. Setting $h_t(u, v) = h(u, tv)$ and $F_t = h_t(E_1 \times E_2)$, we see that the result of [3] holds for this definition of F_t. In the next statement, X denotes linear Brownian motion ([1], chap. XI) on $[0, \infty)$ and $d = d_1 + d_2 < 1$.

THEOREM. — *Let h_t be as defined above and $E \subseteq [0, \infty)$ a compact set of dimension $1/2\, d_1$, while $\dim E_2 = d_2$. Then it is almost sure that*

$$\dim h_t(X(E) \times E_2) \geq d \quad \textit{for all } t > 0.$$

Because it is almost sure that $\dim X(E) = d_1$ ([1], p. 143), the set $X(E)$ is " complementary " to E_2 (with respect to the transformation h).

After the proof of this theorem, the special choice $h = u + v$ is briefly considered; then it will be explained how this is indeed a simple case.

1. — Suppose that μ and λ are probability measures in E and E_2, respectively; then F_t carries a probability measure σ_t defined by the. formula

$$\int f(u)\,\sigma_t(du) = \iint f \circ h(X(x), ty)\,\mu(dx)\,\lambda(dy).$$

Let $V_k[u] = 1$ when $|u| < 2^{-k}$, $V_k[u] = 0$ otherwise $(k = 1, 2, 3, \ldots)$ Then by the method of ([2], chap. III), it is easy to see that

$$\dim F_t \geq e_3 \quad \text{provided} \quad \iint V_k[u - u']\,\sigma_t(du)\,\sigma_t(du') = O(2^{-ke_3}).$$

In order to handle all values t in an interval, $1 \leq t \leq 2$, we choose an integer k_0 so large that

$$V_k[h(a, ty) - h(b, ty_1)] \leq V_{k-k_0}[h(a, sy) - h(b, sy_1)],$$

whenever y, $y_1 \in E_2$ and $|t - s| \leq 2^{-k}$. Thus we can obtain a lower bound for $\dim F_t$, uniform over $1 \leq t \leq 2$, by majorizing

$$\max(t \in S_k) \iint V_k[h(X(x), ty) - h(X(x'), ty')]\,\mu(dx) \ldots \lambda(dy'),$$

where $S_k = \{1, 1 + 2^{-k}, \ldots, 2\}$ has $1 + 2^k$ elements. In the following paragraph, t occurs innocuously as a parameter, and so it is suppressed until the end of the proof.

2. — To each number $e_1 \in (0, 1/2\, d_1)$, there is a probability measure μ in E, fulfilling a Lipschitz condition to exponent e_1, and similarly, for each number $e_2 \in (0, d_2)$, a measure λ in E_2 (*see* [2], chap. II). For any numbers $a \geq 0$, b, we form the double integrals

$$I_k = \iint V_k[h(X(x), y) - h(X(a), b)]\,\mu(dx)\,\lambda(dy)$$
$$= \int V_k[u - h(X(a), b)]\,\sigma(du).$$

LEMMA. — *The* r*th moment of* I_k *is* $O(k\,2^{-e_1k - e_2k})^r$, *for each* $r = 1, 2, 3, \ldots$, *uniformly with respect to* a *and* b.

Proof. — The rth power of I_k is a multiple integral

$$\int \cdots \int \prod_{j=1}^{r} V_k[h(X(x_j), y_j) - h(X(a), b)]\,\mu(dx_1) \ldots \lambda(dy_r).$$

We can suppose that $x_1 \leqq x_2 \leqq \ldots \leqq x_r$, and then divide the integral into subsets depending upon the relative position of a; here we shall suppose $a \leqq x_1$. The inequality $\prod V_k \neq 0$ implies the system

$$|h(X(x_1), y_1) - h(X(a), b)| < 2^{-k},$$
$$|h(X(x_{j+1}), y_{j+1}) - h(X(x_j), y_j)| < 2^{1-k}.$$

To these, we can adjoin the inequalities

$$|X(x_1) - X(a)| < k|x - a|^{\frac{1}{2}}, \qquad |X(x_{j+1}) - X(x_j)| < k|x_{j+1} - x_j|^{\frac{1}{2}}$$

for the set on which one of these fails has probability $< e^{-\frac{1}{2}k^2}$. These systems of inequalities yield

$$|y_1 - b| \leqq 2^{-k} + Ck|x - a|^{\frac{1}{2}}, \qquad |y_{j+1} - y_j| \leqq 2^{1-k} + Ck|x_{j+1} - x_j|^{\frac{1}{2}},$$

where C depends also on t, but is bounded for $1 \leqq t \leqq 2$. Because $\frac{\partial h}{\partial u} \geqq 1$, the probability of the first system of inequalities is

$$\leqq \inf\left(1, 2^{-k}|x_1 - a|^{-\frac{1}{2}}\right) \prod_{1}^{r-1} \inf\left(1, 2^{1-k}|x_{j+1} - x_j|^{-\frac{1}{2}}\right).$$

To each determination of $x_1, \ldots, x_r$ a domain of values $(y_1, \ldots, y_r)$ is defined, of measure

$$O\left(2^{-k} + k|x - a|^{\frac{1}{2}}\right)^{e_2} \prod_{1}^{r-1} \left(2^{-k} + k|x_{j+1} - x_j|^{\frac{1}{2}}\right)^{e_2}.$$

To majorize the rth moment, we perform iterated integration; we require a bound for integrals of the type

$$\int \inf\left(1, 2^{-k}|x - c|^{-\frac{1}{2}}\right)\left(2^{-k} + k|x - c|^{\frac{1}{2}}\right)^{e_2} \mu(dx).$$

For the intervals $|x - c| < 4^{-k}$, a contribution $O(4^{-e_1 k} . k^{e_2} 2^{-e_2 k})$ is obtained. For the intervals $4^{-n} \leqq x| - c| < 4^{1-n}$, the magnitude does not exceed

$$2^{n-k} . 4^{-ne_1} . k^{e_2} 2^{-e_2 n} = 2^{-k} k^{e_2} 2^{n(1 - 2e_1 - e_2)}.$$

As $2e_1 + e_2 \leqq d_1 + d_2 < 1$, the sum, for $k \geqq n$, of all partial integrals, is of magnitude $2^{-e_1 k - e_2 k} k^{e_2}$, and from this the required estimate follows.

Now, setting

$$J_k = \int \cdots \int V_k[h(X(x), y) - h(X(x'), y']\, \mu(dx) \ldots \lambda(dy'),$$

we find by Jensen's inequality the same estimate for the rth moment of J_k.

Let now $e_3 < 2e_1 + e_2$. Then

$$P\{J_k \geq 2^{-ke_3}\} = O(2^{ke_3 - 2ke_1 - ke_2})^r k^r.$$

At this point, we restore the parameter t, and find

$$P\{\max J_k(t) \geq 2^{-ke_3},\ t \in S_k\} = O(2^{ke_3 - 2ke_1 - ke_2})^r 2^k k^r.$$

Choosing r so large that $r(e_3 - 2e_1 - e_2) + 1 < 0$, we find $\max J_k(t) = o(2^{-ke_3})$ almost surely. Since e_3 is arbitrarily close to $d = d_1 + d_2$, we have finally $\dim F_t \geq d$ for $1 \leq t \leq 2$, almost surely.

In the proof, we allowed t to operate on the coordinate v, since E_2 is bounded. But $X(E)$ is almost surely bounded, and so t might also operate on u. In fact, the proof is valid for functions $h(u, v, t)$ continuously differentiable in u, v, t provided $\frac{\partial h}{\partial u} > 0$, $\frac{\partial h}{\partial v} > 0$ everywhere.

3. — In the special case $h = u + v$, we use harmonic analysis of Fourier-Stieltjes transforms. Let μ and λ be the measures introduced in the beginning of paragraph 2, and ν the transform of μ by the trajectory X:

$$\hat{\nu}(s) \equiv \int e^{-isX(x)} \mu(dx) \equiv \int e^{-isx'} \nu(dx').$$

By Theorem 1 of ([1], chap. XV),

$$\hat{\nu}(s) = o(|s|^{-e}) \quad \text{for every } e < \frac{1}{2} e_1.$$

Also,

$$\int_{|s|>1} |\hat{\lambda}(s)|^2 |s|^f\, ds < \infty \quad \text{for every } f < e_2 - 1.$$

Hence

$$\int_{|s|>1} |\hat{\lambda}(s)|^2 |\hat{\nu}(s)|^2 |s|^f\, ds < \infty \quad \text{for every } g < e_2 + e_1 - 1.$$

Now the measure $\nu \star \lambda$ has Fourier transform $\hat{\nu}\hat{\lambda}$, and is supported by $X(E) + E_2$, so that $X(E) + E_2$ has dimension $\geq e_1 + e_2$. (For this paragraph, *see* [2], chap. III.) The set $X(E)$ is therefore complementary to every set E_2, but the proof of this fact seems to have little relation to the non-linear problem. Observe that when $e_1 + e_2 > 1$, the set $X(E) + E_2$ has positive measure, by the Plancherel formula; it would be extremely interesting to obtain a theorem of this type for non-linear mappings, valid for all $t > 0$ uniformly; it would also be interesting to find properties of h_t dependent upon the higher derivatives of h when these exist.

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(Texte reçu le 5 août 1970.)

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