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MAXIMAL CLASSES OF Ext-REPRODUCED ABELIAN GROUPS

BY

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1. Introduction.

In ([5], p. 131), we defined, for arbitrary abelian groups G and H, the *left iterating* Ext-*chain* $\mathfrak{L}$ inductively by

$$E_0(G, H) = G \quad \text{and} \quad E_{i+1}(G, H) = \mathrm{Ext}(E_i(G, H), H) \quad \text{for} \quad i \geq 0.$$

The group G is called *right H-periodic with respect to* $\mathfrak{L}$, if there exist integers i and j with $0 \leq i < j$ and such that

$$E_i(G, H) \cong E_j(G, H).$$

A class $\mathfrak{K}$ of groups G for which there exists a group H such that all G in $\mathfrak{K}$ are right H-periodic, is called a *right periodic class with respect to* $\mathfrak{L}$. Such a class is called *maximal* if it is not properly contained in any right periodic class. The set of all maximal right periodic classes is described in [5], Satz A.

The structure of the group of extensions of a group H by a group G is in general unknown. In an attempt to investigate the structure of $\mathrm{Ext}(G, H)$, we consider the special case $i = 0$, $j = 1$, that is

$$G \cong \mathrm{Ext}(G, H).$$

The question arises as to whether there exist non-trivial classes $\mathfrak{K}$ of groups G for which there exists a group H such that

$$G \cong \mathrm{Ext}(G, H) \quad \text{for all} \quad G \in \mathfrak{K}.$$

Such a class is called *left* Ext-*reproduced.* In Theorem 2.7 and Theorem 2.9, we show that the classes $\mathfrak{F}$ and $\mathfrak{A}$ are left Ext-reproduced, where $\mathfrak{F}$ is the class of all groups G of the form $Q^n \oplus T$, where n is a non-negative integer, and T is a finite group, and $\mathfrak{A}$ is the class defined as follows : Consider a group

$$G^{(p)} = Z(p)^{\mathfrak{m}_p} \oplus \prod_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}} \right)$$

where p is a prime, and where every $\mathfrak{m}_{p_i}$ $(i = 1, 2, \ldots)$ is a non-negative integer. Further, $\mathfrak{m}_p = \operatorname{fin} r(t\,G^{(p)})$ (*see* [2], p. 105). For each prime p, let $A(p)$ denote the direct sum of a finite number of groups $G^{(p)}$:

$$A(p) = G_1^{(p)} \oplus \ldots \oplus G_n^{(p)},$$

where the groups $G_i^{(p)}$ may, but need not be mutually isomorphic. Now $\mathfrak{A}$ denotes the class of all groups G of the form $\prod_{p \in P} A(p)$, where P denotes the set of all primes.

The classes $\mathfrak{F}$ and $\mathfrak{A}$ are also characterized in this paper. In particular, the class $\mathfrak{F}$ can be characterized by the fact that it is a maximal class of left Ext-reproduced groups which contains all groups $C(p^k)$ for all primes p and all natural numbers k, and also groups which are not reduced. The class $\mathfrak{A}$ can be characterized by the fact that it is a maximal class of left Ext-reproduced groups which contains all groups $C(p^k)$ for all primes p and all natural numbers k, and all groups in $\mathfrak{A}$ are reduced (Theorems 2.14 and 2.15). $\mathfrak{F}$ can also be characterized by the fact that it is a maximal class of left Ext-reproduced groups such that if $G \in \mathfrak{F}$ then for all pure subgroups U of G we have $U \in \mathfrak{F}$ and $G/U \in \mathfrak{F}$ (Theorem 2.14). If we assume that all torsion subgroups of a maximal class $\mathfrak{M}$ of left Ext-reproduced groups are reduced, and that the group X for which

$$G \cong \operatorname{Ext}(G, X) \quad \text{for all } G \in \mathfrak{M},$$

is torsion-free, then it follows that either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{A}$ (Theorem 2.12).

Since every class of left Ext-reproduced groups is periodic, it follows that each of the above two classes is contained in one of the maximal right periodic classes $\mathfrak{K}_p$ ([5], p. 131). Here $\mathfrak{K}_p$ denotes the class of all groups G with the property :

$\mathfrak{K}_p$: In any direct decomposition of a basic subgroup of the p-component of G, we have only a finite number of cyclic groups of a given order.

Unfortunately, we do not know whether there exist other maximal classes of left Ext-reproduced groups.

In a similar way, we defined in ([5], p. 131), for arbitrary groups G and H, the *right iterating* Ext-*chain* $\mathfrak{R}$:

$$E^0(G, H) = H,\ E^{i+1}(G, H) = \mathrm{Ext}(G, E^i(G, H)) \qquad \text{for} \quad i \geqq 0.$$

A group H is called *left G-periodic with respect to* $\mathfrak{R}$, if there exist integers i and j with $0 \leqq i < j$ such that

$$E^i(G, H) \cong E^j(G, H).$$

A class $\mathfrak{K}$ is called *left G-periodic* if there exists a group G such that all groups H in $\mathfrak{K}$ are left G-periodic. In this case, we have that the class of all abelian groups is a (maximal) left periodic class with respect to $\mathfrak{R}$. We are concerned with the special case $i = 0$, $j = 1$, and we investigate the existence of classes of groups $\mathfrak{K}$ which are maximal with respect to the property : There exists a group G such that

$$H \cong \mathrm{Ext}(G, H) \quad \text{for all groups } H \text{ in } \mathfrak{K}.$$

If we call such a class *right* Ext-*reproduced*, then it follows that the class $\mathfrak{C}$ of all reduced cotorsion groups is the only maximal class of right Ext-reproduced groups (Theorem 3.3). We show further that the isomorphism

$$H \cong \mathrm{Ext}(G, H)$$

holds for all groups H in $\mathfrak{C}$ if, and only if, $tG \cong Q/Z$ (Theorem 3.4).

NOTATION.

$A \oplus B$, $\bigoplus_{i \in I} A_i$, $A^{(\mathfrak{m})}$, direct sum;

$\prod_{i \in I} A_i$, $A^{\mathfrak{m}}$, direct product;

$\{A, B\}$,	the group generated by A and B;
$A \otimes B$,	tensor product of A and B;
tG,	maximal torsion subgroup of G;
G_p,	p-component of G;
$G[p]$,	the set of all elements of G of order p;
Z,	additive group of integers;
Q,	additive group of rational numbers;
$Z(p)$,	additive group of p-adic integers;
$C(n)$,	cyclic group of order n;
$C(p^\infty)$,	quasi-cyclic group;
$\aleph$,	the power of the continuum;
P,	the set of all prime numbers;

cotorsion group, a group X such that $\mathrm{Ext}(Q, X) = 0$.

All groups under consideration are additively written abelian groups.

Finally, we would like to emphasize the fact that if X is torsion-free and T is a torsion group, then

$$\operatorname{Ext}(T, X) \cong \operatorname{Hom}(T, X \otimes (Q/Z)).$$

In fact, it follows from

$$X \otimes Z \cong X$$

(*see* [2], p. 250), and the exact sequences

$$0 \to Z \to Q \to Q/Z \to 0$$

and

$$0 \to X \otimes Z \to X \otimes Q \to X \otimes (Q/Z) \to 0$$

that

$$(X \otimes Q)/X \cong X \otimes (Q/Z).$$

Since $X \otimes Q$ is a minimal divisible group containing X ([2], p. 256), we have ([2], p. 244)

$$\operatorname{Ext}(T, X) \cong \operatorname{Hom}(T, (X \otimes Q)/X) \cong \operatorname{Hom}(T, X \otimes (Q/Z)).$$

We shall frequently make use of this isomorphism.

2. Classes of left Ext-reproduced groups.

As a starting point for our investigations, we consider firstly an example :

EXAMPLE 2.1

(i) Let X be a subgroup of $Z(p)$ such that $Z(p)/X \cong Q$. Consider the exact sequence

$$0 \to X \to Z(p) \to Q \to 0$$

and the induced exact sequence ([1], p. 221).

$$(1) \qquad 0 \to \operatorname{Hom}(Q, Q) \to \operatorname{Ext}(Q, X) \to \operatorname{Ext}(Q, Z(p)).$$

Now, $\operatorname{Hom}(Q, Q) \cong Q$ and since $Z(p)$ is a cotorsion group ([6], p. 241) it follows that $\operatorname{Ext}(Q, Z(p)) = 0$. Hence it follows from the exact sequence (1), that $\operatorname{Ext}(Q, X) \cong Q$.

It is now clear that if n is a natural number, then we have

$$\operatorname{Ext}(Q^n, X) \cong (\operatorname{Ext}(Q, X))^n \cong Q^n.$$

(ii) Consider the group $\prod_{p \in P} C(p)$, and let X be a subgroup of $\prod_{p \in P} C(p)$

such that $\prod_{p \in P} C(p)/X \cong Q$. We have the exact sequence

$$0 \to X \to \prod_{p \in P} C(p) \to Q \to 0,$$

and the induced exact sequence ([1], p. 221)

$$(2) \qquad 0 \to \mathrm{Hom}(Q, Q) \to \mathrm{Ext}(Q, X) \to \mathrm{Ext}\left(Q, \prod_{p \in P} C(p)\right)$$

and Hom$(Q, Q) \cong Q$. Furthermore ([2], p. 80),

$$\mathrm{Ext}\left(Q, \prod_{p \in P} C(p)\right) \cong \prod_{p \in P} \mathrm{Ext}(Q, C(p)) = 0$$

and hence we deduce from the exact sequence (2) that Ext$(Q, X) \cong Q$.

The following question now arises : Which are the reduced groups X which satisfy Ext$(Q, X) \cong Q$? Note that there is no loss of generality in assuming that X is reduced, for if X is not reduced then it is evident that only the reduced part of X will contribute to Ext(Q, X).

A complete answer to the above question is given in the following theorem :

THEOREM 2.2. — *A reduced group X is such that* Ext$(Q, X) \cong Q$ *if, and only if, X is isomorphic to a subgroup of a reduced cotorsion group G such that $G/X \cong Q$.*

Proof. — To prove the necessity, let X be a reduced group such that Ext$(Q, X) \cong Q$. Then the exact sequence

$$0 \to Z \to Q \to Q/Z \to 0$$

gives rise to the exact sequence

$$(3) \qquad 0 \to \mathrm{Hom}(Z, X) \to \mathrm{Ext}(Q/Z, X) \to \mathrm{Ext}(Q, X) \to 0$$

and Hom$(Z, X) \cong X$, and by assumption, Ext$(Q, X) \cong Q$. Now, since Ext$(Q/Z, X)$ is a reduced cotorsion group ([7], p. 375-376), the exact sequence (3) implies the necessity.

Conversely, let X be isomorphic to a subgroup of a reduced cotorsion group G such that $G/X \cong Q$. Then the exact sequences ([1], p. 221)

$$0 \to X \to G \to Q \to 0$$

and

$$0 \to \mathrm{Hom}(Q, Q) \to \mathrm{Ext}(Q, X) \to \mathrm{Ext}(Q, G) = 0,$$

and Hom$(Q, Q) \cong Q$ imply Ext$(Q, X) \cong Q$. This completes the proof.

EXAMPLE 2.3. — Let X be a subgroup of $\prod_{p \in P} Z(p)$ such that $\prod_{p \in P} Z(p)/X \cong Q$. Then, by Theorem 2.2, we have $\mathrm{Ext}(Q, X) \cong Q$. Further, for all primes p, we have

$$C(p) \cong \prod_{p \in P} Z(p)/p \prod_{p \in P} Z(p) \cong \left\{ X, p \prod_{p \in P} Z(p) \right\} \Big/ p \prod_{p \in P} Z(p)$$
$$\cong X/X \cap p \prod_{p \in P} Z(p) = X/pX.$$

If we consider a finite cyclic group $C(p^k)$, then ([2], p. 243)

$$\mathrm{Ext}(C(p^k), X) \cong X/p^k X \cong C(p^k),$$

and this holds for all primes p and all natural numbers k. We see therefore that if T is a finite group then ([2], p. 39)

$$T = (C(p_1^{k_1}))^{n_1} \oplus \ldots \oplus (C(p_r^{k_r}))^{n_r}$$

and hence it follows that

$$\mathrm{Ext}(T, X) \cong T.$$

Finally, if $G = Q^n \oplus T$, where n is a non-negative integer, and T is a finite group, then we have

$$\mathrm{Ext}(G, X) \cong G.$$

Let $\mathfrak{F}$ denote the class of all groups $G = Q^n \oplus T$, where n is a non-negative integer and T is a finite group. We contend that $\mathfrak{F}$ is a maximal class of left Ext-reproduced groups. However, we need three lemmas for the proof of this theorem.

LEMMA 2.4. — *A reduced group X is such that* $\mathrm{Ext}(C(p^k), X) \cong C(p^k)$ *for all primes p and all natural numbers k, i,, and only if, X is isomorphic to a pure subgroup of* $\prod_{p \in P} Z(p)$ *such that* $X/pX \cong C(p)$ *for all primes p.*

Proof. — Let X be a reduced group such that $\mathrm{Ext}(C(p^k), X) \cong C(p^k)$ for all primes p and all natural numbers k. Then ([2], p. 243)

$$C(p^k) \cong \mathrm{Ext}(C(p^k), X) \cong X/p^k X$$

implies $X/p^k X \cong C(p^k)$ for all primes p and all natural numbers k.

We assert that X is torsion-free. Indeed, if X is not torsion-free then it has a direct summand $C(p^k)$, $1 \leqq k < \infty$ ([2], p. 80), $X = C(p^k) \oplus X'$. This implies that

$$X/p^{k+1}X \cong C(p^k) \oplus X'/p^{k+1}X'$$

which is contrary to

$$X/p^{k+1}X \cong C(p^{k+1}).$$

Hence we conclude that X is torsion-free.

To recapitulate, if a reduced group X satisfies the conditions stated in the lemma then X is torsion-free and $X/pX \cong C(p)$ for all primes p. The exact sequence

$$0 \to Z \to Q \to Q/Z \to 0$$

leads to the exact sequence ([7], p. 375-376)

$$(4) \qquad 0 \to \operatorname{Hom}(Z, X) \to \operatorname{Ext}(Q/Z, X) \to \operatorname{Ext}(Q, X) \to 0$$

and

$$\operatorname{Hom}(Z, X) \cong X \quad \text{and}$$
$$\operatorname{Ext}(Q/Z, X) \cong \operatorname{Hom}(Q/Z, X \otimes (Q/Z)) \cong \prod_{p \in P} Z(p)$$

since $X \otimes (Q/Z) \cong Q/Z$ (*see* [2], p. 252 and 255). Since $\operatorname{Ext}(Q, X)$ is torsion-free and divisible ([2], p. 245), it follows from the exact sequence (4) that X is isomorphic to a pure subgroup of $\prod_{p \in P} Z(p)$.

The proof of the converse is straightforward and is omitted. This completes the proof.

LEMMA 2.5. — *Let* $U = \prod_{i=1}^{\infty} A_i$ *where* $A_i = (C(p^i))^{(\mathfrak{m}_i)}$, *and* $\mathfrak{m}_i \neq 0$ *for an infinite number of i's. Then* U/tU *has a torsion-free divisible subgroup of infinite rank.*

Proof. — Consider the exact sequence

$$0 \to tU \to U \to U/tU \to 0$$

and the exact sequence ([1], p. 221)

$$(5) \qquad 0 \to \operatorname{Hom}(Q, U/tU) \to \operatorname{Ext}(Q, tU) \to \operatorname{Ext}(Q, U).$$

It is clear that

$$\operatorname{Ext}(Q, U) \cong \prod_{i=1}^{\infty} \operatorname{Ext}(Q, A_i) = 0$$

since the A_i are cotorsion. Hence it follows from (5) that

$$\mathrm{Hom}(Q,\ U/tU) \cong \mathrm{Ext}(Q,\ tU)$$

and it is also clear that $\mathrm{Hom}(Q,\ U/tU)$ is isomorphic to the maximal divisible subgroup of U/tU.

However, by ([8], p. 606), $\mathrm{Ext}(Q, tU)$ is isomorphic to the direct sum of an infinite number of copies of Q, and hence it follows that

$$U/tU \cong Q^{(\mathfrak{m})} \oplus K,$$

where $\mathfrak{m}$ is infinite and K is reduced. This ends the proof.

LEMMA 2.6. — *Let* X *be a subgroup of* $\prod_{p \in P} Z(p)$ *such that* $X/pX \cong C(p)$ *for all primes* p.

(i) *If* $\mathrm{Hom}(Z(p), X) = 0$ *then* $t\,\mathrm{Ext}(Z(p), X) \cong C(p^\infty)$;

(*ii*) *If* $\prod_{p \in P} Z(p)/X \cong Q^{(\mathfrak{n})} \neq 0$, *then* $\mathrm{Ext}(Z(p), X)$ *is isomorphic to the direct sum of* $2^{\aleph}$ *copies of* Q *and at most one* $C(p^\infty)$.

Proof. — Firstly, $\mathrm{Ext}(Z(p), X)$ is divisible since $Z(p)$ is torsion-free, and $q\,Z(p) = Z(p)$ for all primes $q \neq p$ implies ([2], p. 245).

$$\mathrm{Ext}(Z(p), X)[q] = 0 \quad \text{for all primes } q \neq p.$$

Moreover, if $\mathrm{Hom}(Z(p), X) = 0$, then, by ([2], p. 246), we have

$$\mathrm{Ext}(Z(p), X)[p] \cong \mathrm{Hom}(Z(p), X/pX) \cong \mathrm{Hom}(Z(p)/pZ(p), C(p)) \cong C(p).$$

This proves (i).

To prove (ii), note that the proof thus far implies that $\mathrm{Ext}(Z(p), X)$ is divisible and that $\mathrm{Ext}(Z(p), X)[q] = 0$ for all primes $q \neq p$. Consider the exact sequences

$$0 \to X \to \prod_{p \in P} Z(p) \to Q^{(\mathfrak{n})} \to 0$$

and

$$(6) \quad \mathrm{Hom}(Z(p), \prod_{p \in P} Z(p)) \to \mathrm{Hom}\big(Z(p),\ Q^{(\mathfrak{n})}\big) \to \mathrm{Ext}(Z(p),\ X) \to 0.$$

By ([6], p. 239), and ([2], p. 212), we have

$$\mathrm{Hom}(Z(p), \prod_{p \in P} Z(p)) \cong \mathrm{Hom}(Z(p),\ Z(p)) \cong Z(p)$$

and the latter group is of power $\aleph$. Furthermore, it is clear that

$$\left|\operatorname{Hom}\left(Z(p), Q^{(\mathfrak{n})}\right)\right| \leqq \aleph^{\aleph} = 2^{\aleph}.$$

However $\operatorname{Hom}\left(Z(p), Q^{(\mathfrak{n})}\right)$ has a direct summand $\operatorname{Hom}(Z(p), Q)$ and, by ([2], p. 206 and 257), we have

$$\begin{aligned}\operatorname{Hom}(Z(p), Q) &\cong \operatorname{Hom}(Q, \operatorname{Hom}(Z(p), Q)) \cong \operatorname{Hom}(Q \otimes Z(p), Q) \\ &\cong (\operatorname{Hom}(Q, Q))^{\aleph}\end{aligned}$$

and the latter group is torsion-free and divisible of rank $2^{\aleph}$. Consequently,

$$\left|\operatorname{Hom}\left(Z(p), Q^{(\mathfrak{n})}\right)\right| = 2^{\aleph},$$

and hence we deduce from the exact sequence (6) that

$$|\operatorname{Ext}(Z(p), X)| = 2^{\aleph}.$$

If $\operatorname{Hom}(Z(p), X) = 0$, then (i) and the foregoing imply that

$$\operatorname{Ext}(Z(p), X) \cong Q^{(2^{\aleph})} \oplus C(p^{\infty}).$$

If $\operatorname{Hom}(Z(p), X) \neq 0$, then $\operatorname{Hom}(Z(p), X) \cong Z(p)$ ([6], p. 239), and then it follows from ([2], p. 246), that $\operatorname{Ext}(Z(p), X)[p] = 0$. In this case, the foregoing implies that

$$\operatorname{Ext}(Z(p), X) \cong Q^{(2^{\aleph})}.$$

This completes the proof.

THEOREM 2.7. — *$\mathfrak{F}$ is a maximal class of left* Ext-*reproduced groups.*

Proof. — Let $\mathfrak{D}$ denote a class of groups which contains $\mathfrak{F}$ and which is such that there exists a group X such that for all $G \in \mathfrak{D}$, we have $\operatorname{Ext}(G, X) \cong G$. We intend to show that $\mathfrak{D} = \mathfrak{F}$.

Firstly, since $C(p^k) \in \mathfrak{D}$ for all primes p and all natural numbers k, it follows from Lemma 2.4 that X is isomorphic to a pure subgroup of $\prod_{p \in P} Z(p)$ and that $X/pX \cong C(p)$ for all primes p. Moreover, since $Q \in \mathfrak{D}$, we have $\operatorname{Ext}(Q, X) \cong Q$ and hence, by Theorem 2.2 and the exact sequence (4) in the proof of Lemma 2.4, we have $\prod_{p \in P} Z(p)/X \cong Q$.

In order to show that $\mathfrak{D} = \mathfrak{F}$, we need only show that $\mathfrak{D} \subseteq \mathfrak{F}$. To this end, let $H \in \mathfrak{D}$. We assert that

(i) H contains at most a finite number of copies of Q.

Indeed, suppose that $H = Q^{(\mathfrak{m})} \oplus H'$, where $\mathfrak{m} \geqq \aleph_0$ and H' contains no torsion-free divisible subgroup. Then

$$\mathrm{H} \cong \operatorname{Ext}(H, X) \cong (\operatorname{Ext}(Q, X))^{\mathfrak{m}} \oplus \operatorname{Ext}(H', X) \cong Q^{\mathfrak{m}} \oplus \operatorname{Ext}(H', X)$$

and $Q^{\mathfrak{m}}$ is isomorphic to the direct sum of $2^{\mathfrak{m}}$ ($> \mathfrak{m}$) copies of Q. This contradiction proves (i).

We assert further that

(ii) H does not contain any quasi-cyclic group.

In fact, if $H = C(p^{\infty}) \oplus H''$, then

$$H \cong \operatorname{Ext}(C(p^{\infty}), X) \oplus \operatorname{Ext}(H'', X)$$

and

$$\operatorname{Ext}(C(p^{\infty}), X) \cong \operatorname{Hom}(C(p^{\infty}), X \otimes (Q/Z)) \cong \operatorname{Hom}(C(p^{\infty}), Q/Z) \cong Z(p).$$

Hence we have

$$\mathrm{H} \cong \operatorname{Ext}(Z(p), X) \oplus \operatorname{Ext}(\operatorname{Ext}(H'', X), X)$$

and by Lemma 2.6, $\operatorname{Ext}(Z(p), X)$ has a torsion-free divisible subgroup of infinite rank. This is however contrary to (i), and we conclude that (ii) holds.

Consider the exact sequence

$$0 \to tH \to H \to H/tH \to 0$$

and the induced exact sequence

$$0 \to \operatorname{Ext}(H/tH, X) \to \operatorname{Ext}(H, X) \cong H \to \operatorname{Ext}(tH, X) \to 0.$$

Now, $\operatorname{Ext}(H/tH, X)$ is divisible since H/tH is torsion-free ([2], p. 245), and hence we have

$$H \cong \operatorname{Ext}(H/tH, X) \oplus \operatorname{Ext}(tH, X).$$

Furthermore, $\operatorname{Ext}(tH, X)$ is a reduced cotorsion group ([5], p. 134), and hence it follows from (i) and (ii) that

$$\operatorname{Ext}(H/tH, X) \cong Q^{n},$$

where n is a non-negative integer.

We contend that

(iii) tH is finite.

Note that by (ii) tH is reduced. Firstly, we prove that the p-components $(tH)_p$ of tH are all bounded and then we deduce the finiteness of each p-component. Thereafter we show that tH has only a finite number of non-zero p-components.

We have

(7) $$\operatorname{Ext}(tH, X) \cong \operatorname{Hom}(tH, X \otimes (Q/Z)) \cong \prod_{p \in P} \operatorname{Hom}((tH)_p, Q/Z)$$

since $X \otimes (Q/Z) \cong Q/Z$ ([2], p. 252 and 255). Assume that for some prime p, $(tH)_p$ is unbounded. Let

$$B^{(p)} = B_1 \oplus \ldots \oplus B_i \oplus \ldots; \qquad B_i = (C(p^i))^{(\mathfrak{m}_{p_i})},$$

be a basic subgroup of $(tH)_p$ and let $\mathfrak{m}_p$ denote the final rank of $(tH)_p$. Then

$$\operatorname{Hom}((tH)_p, Q/Z) \cong \operatorname{Hom}((tH)_p, C(p^\infty))$$

is a direct summand of $\operatorname{Ext}(tH, X)$ and ([3], p. 137)

$$\operatorname{Hom}((tH)_p, C(p^\infty)) \cong Z(p)^{\mathfrak{m}_p} \oplus \prod_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right) = Y,$$

say. The unboundedness of $(tH)_p$ implies that $\mathfrak{m}_p \neq 0$ ([2], p. 105). Hence Y, and consequently H, has a direct summand $Z(p)$, $H \cong Z(p) \oplus H_1$, whence by Lemma 2.6

$$H \cong \operatorname{Ext}(Z(p), X) \oplus \operatorname{Ext}(H_1, X)$$

contains a torsion-free divisible subgroup of infinite rank, contrary to (i). This proves that every p-component of tH is bounded and hence is a direct sum of cyclic groups ([2], p. 44). It follows from (7) and ([5], p. 136), that $(tH)_p$ is finite for all primes p.

Suppose that tH has an infinite number of non-zero p-components. If we put $V = \operatorname{Ext}(tH, X)$ then it is clear that $tV = \bigoplus_{p \in P} \operatorname{Hom}((tH)_p, Q/Z)$ and that $V/tV \cong Q^{(\mathfrak{m})}$, where $\mathfrak{m}$ is infinite. The exact sequence

$$0 \to tV \to V \to Q^{(\mathfrak{m})} \to 0$$

leads to the exact sequence

(8) $$0 \to \operatorname{Ext}\left(Q^{(\mathfrak{m})}, X\right) \to \operatorname{Ext}(V, X).$$

Now

$$\operatorname{Ext}\left(Q^{(\mathfrak{m})}, X\right) \cong (\operatorname{Ext}(Q, X))^{\mathfrak{m}} \cong Q^{\mathfrak{m}}$$

together with the exact sequence (8) shows that $\operatorname{Ext}(V, X)$ and consequently H, has a torsion-free divisible subgroup of infinite rank, contrary to (i). Consequently, tH has at most a finite number of finite, non-zero p-components and hence tH is finite. This proves (iii).

To recapitulate, if $H \in \mathfrak{D}$ then

$$H \cong \mathrm{Ext}(H/tH, X) \oplus \mathrm{Ext}(tH, X) \cong Q^n \oplus tH,$$

where n is a non-negative integer and where tH is finite, and hence $H \in \mathfrak{F}$. Hence $\mathfrak{D} \subseteq \mathfrak{F}$ and this proves that $\mathfrak{D} = \mathfrak{F}$.

EXAMPLE 2.8. — Consider a group

$$G^{(p)} = Z(p)^{\mathfrak{m}_p} \oplus \prod_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right)$$

of the class $\mathfrak{A}$; *see* § 1. (Note that if $\mathfrak{m}_{p_i} = 0$ for $i \geq r$, where r is a natural number then $G^{(p)}$ is a finite p-group, and if $\mathfrak{m}_{p_i} \neq 0$ for an infinite number of natural numbers i, then $\mathfrak{m}_p = 2^{\aleph_0}$). Now $\bigoplus_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right)$ is a basic subgroup of $tG^{(p)}$ ([2], p. 100). Put $X = \prod_{p \in P} Z(p)$, then we have the exact sequences

$$0 \to tG^{(p)} \to G^{(p)} \to G^{(p)}/tG^{(p)} \to 0$$

and

(9) $$\mathrm{Ext}(G^{(p)}/tG^{(p)}, X) \to \mathrm{Ext}(G^{(p)}, X) \to \mathrm{Ext}(tG^{(p)}, X) \to 0$$

and since X is a cotorsion group ([7], p. 371), it follows that $\mathrm{Ext}(G^{(p)}/tG^{(p)}, X) = 0$. Hence the exact sequence (9) implies that

$$\mathrm{Ext}(G^{(p)}, X) \cong \mathrm{Ext}(tG^{(p)}, X).$$

Now, bearing in mind the fact that $\bigoplus_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right)$ is a basic subgroup of $tG^{(p)}$, it follows from ([3], p. 137) that

$$\begin{aligned}\mathrm{Ext}(tG^{(p)}, X) &\cong \mathrm{Hom}(tG^{(p)}, X \otimes (Q/Z)) \\ &\cong \mathrm{Hom}(tG^{(p)}, Q/Z) \\ &\cong \mathrm{Hom}(tG^{(p)}, C(p^\infty)) \\ &\cong Z(p)^{\mathfrak{m}_p} \oplus \prod_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right) \cong G^{(p)}.\end{aligned}$$

If we consider the group $G = \prod_{p \in P} G^{(p)}$ then it is clear that $tG = \bigoplus_{p \in P} tG^{(p)}$, and hence it follows that

$$\begin{aligned}\mathrm{Ext}(G, X) &\cong \mathrm{Ext}(tG, X) \cong \mathrm{Ext}\left(\bigoplus_{p \in P} tG^{(p)}, X\right) \\ &\cong \prod_{p \in P} \mathrm{Ext}(tG^{(p)}, X) \cong \prod_{p \in P} G^{(p)} = G.\end{aligned}$$

Note that the class $\mathfrak{A}$, which we defined in paragraph 1, contains all finite groups, all the groups $G^{(p)}$ as well as all finite direct sums of the groups $G^{(p)}$.

THEOREM 2.9. — *$\mathfrak{A}$ is a maximal class of left* Ext-*reproduced groups.*

Proof. — Let $\mathfrak{B}$ denote a class of groups which contains $\mathfrak{A}$ and which is such that there exists a group X such that

$$\mathrm{Ext}(G, X) \cong G \quad \text{for all } G \in \mathfrak{B}.$$

We intend to show that $\mathfrak{B} = \mathfrak{A}$.

Since $C(p^k) \in \mathfrak{B}$ for all primes p and all natural numbers k, it follows from Lemma 2.4 that X is isomorphic to a pure subgroup of $\prod_{p \in P} Z(p)$ and that $X/pX \cong C(p)$ for all primes p.

We now assert that $X = \prod_{p \in P} Z(p)$. To prove this, suppose on the contrary that $X \neq \prod_{p \in P} Z(p)$. We distinguish two cases. Either

(i) for some prime p, we have $X \cap Z(p) \neq Z(p)$,

or

(ii) X contains $\bigoplus_{p \in P} Z(p)$.

As far as (i) is concerned, note that since both X and $Z(p)$ are pure subgroups of $\prod_{p \in P} Z(p)$, it follows that $X \cap Z(p)$ is a proper pure subgroup of $Z(p)$, and hence $\mathrm{Hom}(Z(p), X) = 0$ ([6], p. 239). Consider the group

$$H = Z(p)^{\mathfrak{m}} \oplus \prod_{i=1}^{\infty} ((C(p^i))^{\mathfrak{m}_i}), \tag{10}$$

where each $\mathfrak{m}_i$ is finite and non-zero. This group has a direct summand $Z(p)$, i. e. $H = Z(p) \oplus H'$, and hence it follows from Lemma 2.6 that $\mathrm{Ext}(Z(p), X)$, and consequently H as well, contains a non-zero divisible subgroup, contrary to the fact that H is reduced.

If X is a proper pure subgroup of $\prod_{p \in P} Z(p)$ which contains $\bigoplus_{p \in P} Z(p)$, then $X/\bigoplus_{p \in P} Z(p)$ is a pure subgroup of $\prod_{p \in P} Z(p)/\bigoplus_{p \in P} Z(p)$. However, the latter group is torsion-free and divisible, and hence it follows that $X/\bigoplus_{p\in P} Z(p)$ is also torsion-free and divisible. Consequently,

$$\prod_{p\in P} Z(p)/X \cong \left(\prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)\right)\Big/\left(X/\bigoplus_{p\in P} Z(p)\right)$$

shows that

$$\prod_{p\in P} Z(p)/X \cong Q^{(\mathfrak{n})}.$$

For the group H in (10) above, we have $H = Z(p)\oplus H'$, and hence it follows from Lemma 2.6 that $\operatorname{Ext}(Z(p), X)$, which is a direct summand of H, is a non-trivial divisible group, contrary to the fact that H is reduced.

We conclude that $X = \prod_{p\in P} Z(p)$.

We shall now prove that $\mathfrak{B}\subseteq\mathfrak{A}$. Let $G\in\mathfrak{B}$. Since X is a cotorsion group, it follows from ([5], p. 133), that

$$G\cong \operatorname{Ext}(G, X)\cong \operatorname{Ext}(tG, X),$$

and by ([5], p. 134), G is reduced. Moreover,

$$\operatorname{Ext}(tG, X)\cong \operatorname{Hom}(tG, X\otimes(Q/Z))\cong \prod_{p\in P}\operatorname{Hom}((tG)_p, X\otimes(Q/Z))$$

and since $X\otimes(Q/Z)\cong Q/Z$ ([2], p. 252 and 255), it follows that

$$G\cong \operatorname{Ext}(tG, X)\cong \prod_{p\in P}\operatorname{Hom}((tG)_p, Q/Z)\cong \prod_{p\in P}\operatorname{Hom}((tG)_p, C((p^\infty)).$$

Let $B^{(p)} = B_1\oplus\ldots\oplus B_i\oplus\ldots;\ B_i = (C(p^i))^{(\mathfrak{m}_{p_i})}$, be a basic subgroup of $(tG)_p$, and let $\mathfrak{m}_p$ be the final rank of $(tG)_p$. We then have ([3], p. 137)

$$\operatorname{Hom}((tG)_p, C(p^\infty))\cong Z(p)^{\mathfrak{m}_p}\oplus\prod_{i=1}^{\infty}\left((C(p^i)^{\mathfrak{m}_{p_i}}\right).$$

To recapitulate, if $G\in\mathfrak{B}$, then $G\cong\prod_{p\in P} Z(p)^{\mathfrak{m}_p}\oplus\prod_{i=1}^{\infty}\left((C(p^i))^{\mathfrak{m}_{p_i}}\right)$, where the $\mathfrak{m}_p$ and the $\mathfrak{m}_{p_i}$ are defined as above. It is clear that

$$tG\cong\bigoplus_{p\in P} t\operatorname{Hom}((tG)_p, Q/Z)$$

and that

$$(tG)_p\cong t\prod_{i=1}^{\infty}\left((C(p^i))^{\mathfrak{m}_{p_i}}\right).$$

By ([5], p. 136), $\mathfrak{m}_{p_i}$ ($i = 1, 2, \ldots$) is finite for each prime p. Hence it follows from $G \in \mathfrak{B}$, that $G \in \mathfrak{A}$ and hence $\mathfrak{B} \subseteq \mathfrak{A}$ so that $\mathfrak{B} = \mathfrak{A}$.

This completes the proof.

In a certain sense, the classes $\mathfrak{F}$ and $\mathfrak{A}$ are unique. This will be demonstrated in Theorem 2.12 and Corollary 2.13. However, we need two lemmas for the proof of this theorem.

LEMMA 2.10. — *Let $\mathfrak{M}$ be a maximal class of left* Ext*-reproduced groups for which there exists a reduced torsion-free group X such that*

$$\operatorname{Ext}(G, X) \cong G \quad \textit{for all } G \in \mathfrak{M}.$$

If tG is reduced for all $G \in \mathfrak{M}$ and if, for some $G \in \mathfrak{M}$, we have $\operatorname{Ext}(G/tG, X) \neq 0$ *then $\mathfrak{M} = \mathfrak{F}$.*

Proof. — Let $G \in \mathfrak{M}$ be such that $\operatorname{Ext}(G/tG, X) \neq 0$. Consider the exact sequences

$$0 \to tG \to G \to G/tG \to 0$$

and

$$0 \to \operatorname{Ext}(G/tG, X) \to \operatorname{Ext}(G, X) \to \operatorname{Ext}(tG, X) \to 0. \tag{11}$$

The torsion-freeness of G/tG implies that $\operatorname{Ext}(G/tG, X)$ is divisible ([2], p. 245), and hence it follows from (11) that

$$G \cong \operatorname{Ext}(G, X) \cong \operatorname{Ext}(G/tG, X) \oplus \operatorname{Ext}(tG, X).$$

We maintain that

(i) $\operatorname{Ext}(Q, X) \neq 0$.

Indeed, if $\operatorname{Ext}(Q, X) = 0$, then X is a torsion-free cotorsion group, and it follows that $\operatorname{Ext}(G/tG, X) = 0$, contrary to the assumption that $\operatorname{Ext}(G/tG, X) \neq 0$. We conclude that $\operatorname{Ext}(Q, X) \neq 0$.

(ii) G contains at most a finite number of copies of Q.

As a matter of fact, if $G = Q^{(\mathfrak{m})} \oplus G'$, where $\mathfrak{m} \geq \aleph_0$ and where G' contains no torsion-free divisible subgroup, then

$$G \cong (\operatorname{Ext}(Q, X))^{\mathfrak{m}} \oplus \operatorname{Ext}(G', X)$$

and $(\operatorname{Ext}(Q, X))^{\mathfrak{m}}$ is a torsion-free divisible group of rank $\geq 2^{\mathfrak{m}} > \mathfrak{m}$. This contradiction proves (ii).

(iii) If $G_p \neq 0$, then $X/pX \neq 0$.

Assume on the contrary that $X = pX$, then by ([2], p. 245), we have $pG = G$ since $G \cong \operatorname{Ext}(G, X)$. However, $pG = G$ implies the divisibility of the non-zero p-component G_p and this is contrary to the fact that tG is reduced. We conclude that (iii) holds. We assert that

(iv) $\operatorname{Ext}(tG, X)$ is bounded.

Suppose that $\operatorname{Ext}(tG, X)$ is unbounded. Then it follows from

$$\operatorname{Ext}(tG,\ X) \cong \operatorname{Hom}(tG,\ X \otimes (Q/Z)) \cong \prod_{p \in P} \operatorname{Hom}((tG)_p,\ X \otimes (Q/Z))$$

that either $\operatorname{Hom}(tG)_p,\ X \otimes (Q/Z))$ is unbounded for some prime p, or $\operatorname{Hom}((tG)_p,\ X \otimes (Q/Z))$ is non-zero for an infinite number of primes p.

(a) Let $\operatorname{Hom}((tG)_p, X \otimes (Q/Z))$ be unbounded for some prime p. Then, by ([2], p. 255), and (iii), we have

$$\begin{aligned} \operatorname{Hom}((tG)_p,\ X \otimes (Q/Z)) &\cong \operatorname{Hom}((tG)_p,\ X \otimes C(p^\infty)) \\ &\cong \operatorname{Hom}\big((tG)_p,\ (C(p^\infty))^{(\mathfrak{n}_p)}\big), \end{aligned}$$

where $\mathfrak{n}_p = r(X/pX) \neq 0$. Now, $\operatorname{Hom}\big((tG)_p,\ (C(p^\infty))^{(\mathfrak{n}_p)}\big)$ can be unbounded only if $(tG)_p$ is unbounded. Let

$$B^{(p)} = B_1 \oplus \ldots \oplus B_i \oplus \ldots; \qquad B_i = (C(p^i))^{(\mathfrak{m}_{p_i})},$$

be a basic subgroup of $(tG)_p$, and let $\mathfrak{r}_p = \operatorname{fin} r((tG)_p)$. Then $B^{(p)}$ is unbounded, and we have ([3], p. 137)

$$\begin{aligned} &\operatorname{Hom}\big((tG)_p,\ (C(p^\infty))^{(\mathfrak{n}_p)}\big) \\ &\quad \cong \operatorname{Hom}((tG)_p,\ C(p^\infty) \oplus T), \quad (C(p^\infty))^{(\mathfrak{n}_p)} = C(p^\infty) \oplus T \\ &\quad \cong \operatorname{Hom}((tG)_p,\ C(p^\infty)) \oplus \operatorname{Hom}((tG)_p,\ T) \\ &\quad \cong Z(p)^{\mathfrak{r}_p} \oplus \prod_{i=1}^{\infty} \big((C(p^i))^{\mathfrak{m}_{p_i}}\big) \oplus \operatorname{Hom}((tG)_p,\ T). \end{aligned}$$

Now, $\prod_{i=1}^{\infty} \big((C(p^i))^{\mathfrak{m}_{p_i}}\big) = U$ is a direct summand of $\operatorname{Hom}\big((tG)_p, (C(p^\infty))^{(\mathfrak{n}_p)}\big)$ and the latter group is in turn a direct summand of $\operatorname{Ext}(tG, X)$. Let $\operatorname{Ext}(tG,\ X) = U \oplus H$, then $\operatorname{Ext}(U, X)$ is a direct summand of $G \cong \operatorname{Ext}(G, X)$.

By Lemma 2.5, we have the exact sequence

$$0 \to tU \to U \to Q^{(\mathfrak{m})} \oplus K \to 0,$$

where $\mathfrak{m}$ is infinite. This exact sequence leads to the exact sequence

$$(12) \qquad 0 \to \operatorname{Ext}\big(Q^{(\mathfrak{m})} \oplus K,\ X\big) \to \operatorname{Ext}(U,\ X).$$

Furthermore, by (i), $\operatorname{Ext}(Q, X) \neq 0$, and hence it follows that

$$\operatorname{Ext}\big(Q^{(\mathfrak{m})},\ X\big) \cong (\operatorname{Ext}(Q,\ X))^{\mathfrak{m}}$$

is torsion-free and divisible of infinite rank. Consequently, we deduce from (12) that $\operatorname{Ext}(U, X)$, which is a direct summand of G, has a torsion-free divisible subgroup of infinite rank, contrary to (ii). We conclude that $(tG)_p$ is bounded for all primes p.

(*b*) If $\mathrm{Hom}((tG)_p, X \otimes (Q/Z)) \neq 0$ for an infinite number of primes p, then $(tG)_p \neq 0$ and by (iii), $X/pX \neq 0$ for the relevant primes. Moreover, each $(tG)_p$ is a direct sum of cyclic groups since it is bounded ([2], p. 44). Hence we have

$$\mathrm{Ext}(tG, X) \cong \prod_{p \in P} \mathrm{Hom}((tG)_p, X \otimes (Q/Z)) = W \text{ say,}$$

and it is clear that

$$tW \cong \bigoplus_{p \in P} \mathrm{Hom}((tG)_p, X \otimes (Q/Z))$$

and that $W/tW \cong Q^{(\mathfrak{n})}$, where $\mathfrak{n}$ is infinite. The exact sequence

$$0 \to tW \to W \to Q^{(\mathfrak{n})} \to 0$$

yields the exact sequence

$$0 \to \mathrm{Ext}(Q^{(\mathfrak{n})}, X) \to \mathrm{Ext}(W, X)$$

and

$$\mathrm{Ext}(Q^{(\mathfrak{n})}, X) \cong (\mathrm{Ext}(Q, X))^{\mathfrak{n}}$$

is a torsion-free divisible group of infinite rank, since, by (i), $\mathrm{Ext}(Q,X) \neq 0$. Hence $\mathrm{Ext}(W, X)$, which is a direct summand of G, has a torsion-free divisible subgroup of infinite rank, contrary to (ii).

Hence $(tG)_p \neq 0$ for only a finite number of primes p and by ([5], p. 136), each $(tG)_p$ is finite. Hence we deduce that tG is finite, and clearly this holds for all $G \in \mathfrak{M}$. This proves (iv).

We return to

$$G \cong \mathrm{Ext}(G/tG, X) \oplus \mathrm{Ext}(tG, X).$$

The reducedness of tG implies that $t\,\mathrm{Ext}(G/tG, X) = 0$ and hence, on account of (ii),

$$\mathrm{Ext}(G/tG, X) \cong Q^m,$$

where m is a non-zero positive integer. Moreover, since $\mathrm{Ext}(Q, X)$ is a direct sum of copies of Q ([2], p. 245), it follows that we must necessarily have $\mathrm{Ext}(Q, X) \cong Q$.

Now $\mathfrak{M}$, being maximal, cannot contain only torsion-free groups, for if this were the case and $H \in \mathfrak{M}$ then $H \cong \mathrm{Ext}(H, X)$ implies that H is divisible, and, by (ii), $H \cong Q^n$. Hence Example 2.3 implies that $\mathfrak{M}$ is not a maximal class. Therefore $\mathfrak{M}$ contains a group H such that $tH \neq 0$, and it follows, from the proof of (iv), that tH is finite.

Hence H has a direct summand $C(p^k)$, where k is a non-zero positive integer, $H = C(p^k) \oplus H'$. Furthermore, by (iii), $X/pX \neq 0$. Now

$$H \cong \operatorname{Ext}(C(p^k), X) \oplus \operatorname{Ext}(H', X) \cong X/p^k X \oplus \operatorname{Ext}(H', X)$$

and hence $X/p^k X \cong C(p^k)$ since tH is finite and X is torsion-free.

We maintain that

(v) $C(p^n) \in \mathfrak{M}$ for all natural numbers n.

Firstly, if $k > 1$, then $C(p) \in \mathfrak{M}$. In fact, this follows from the exact sequences

$$0 \to C(p) \to C(p^k) \to C(p^{k-1}) \to 0$$

and

$$\operatorname{Ext}(C(p^k), X) \cong C(p^k) \to \operatorname{Ext}(C(p), X) \cong X/pX \to 0$$

and the fact that $X/pX \neq 0$. Hence $X/pX \cong C(p) \in \mathfrak{M}$. By induction, assume that $C(p^k) \in \mathfrak{M}$ for all $k < n$ and consider $C(p^n)$. Then the exact sequences

$$0 \to C(p^{n-1}) \to C(p^n) \to C(p) \to 0$$

and

$$0 \to \operatorname{Ext}(C(p), X) \to \operatorname{Ext} C(p^n), X) \to \operatorname{Ext}(C(p^{n-1}), X) \to 0$$

yield

$$\operatorname{Ext}(C(p^n), X) \cong X/p^n X \cong C(p^n),$$

for the induction hypothesis implies that

$$\operatorname{Ext}(C(p), X) \cong C(p), \qquad \operatorname{Ext}(C(p^{n-1}), X) \cong C(p^{n-1})$$

and in addition we have

$$\operatorname{Ext}(C(p^n), X) \cong X/p^n X \cong (C(p^n))^{(n)}$$

since X is torsion-free. This proves (v).

Consider the set of all primes p for which there exists a group $G \in \mathfrak{M}$ with $G_p \neq 0$ (Recall that for all $G \in \mathfrak{M}$, tG is finite). This set must be the set of all primes, for if there exists a prime p such that $G_p = 0$ for all $G \in \mathfrak{M}$ then Example 2.3 shows that $\mathfrak{M}$ is not maximal. Hence it follows from (v) that $C(p^k) \in \mathfrak{M}$ for all primes p and all natural numbers k, consequently, by Theorem 2.2 and Lemma 2.4, we have $\prod_{p \in P} Z(p)/X \cong Q$. Finally, the maximality of $\mathfrak{M}$, Example 2.3 and Theorem 2.7, imply that $\mathfrak{M} = \mathfrak{F}$. The proof is complete.

LEMMA 2.11. — *Let $\mathfrak{M}$ be a maximal class of left* Ext-*reproduced groups for which there exists a reduced torsion-free group X such that*

$$\operatorname{Ext}(G, X) \cong G \quad \textit{for all } G \in \mathfrak{M}.$$

If $\operatorname{Ext}(G/tG, X) = 0$ *for all* $G \in \mathfrak{M}$, *then* $\mathfrak{M} = \mathfrak{A}$.

Proof. — To begin with, we give a brief outline of the proof. First we prove that X is a torsion-free cotorsion group and then we show that $X \cong \prod_{p \in P} Z(p)$ and that $\mathfrak{M} = \mathfrak{A}$.

The exact sequence (11) in the proof of Lemma 2.10 yields in this case for $G \in \mathfrak{M}$

$$G \cong \operatorname{Ext}(G, X) \cong \operatorname{Ext}(tG, X).$$

Hence, by virtue of ([5], p. 134), all groups G in $\mathfrak{M}$ are reduced.

Now $\mathfrak{M}$, being a maximal class, cannot contain only finite groups. This follows from the fact that $\mathfrak{F}$ and $\mathfrak{A}$ are maximal classes of left Ext-reproduced groups, which contain all finite groups. It is also clear that $\mathfrak{M}$ cannot contain only bounded groups, for if this were the case, let B be a bounded group. Then B is a direct sum of cyclic groups and hence ([5], p. 136) implies that B is finite.

Hence we conclude that $\mathfrak{M}$ contains at least one group G for which tG is unbounded. The fact that tG is unbounded implies that tG has either an unbounded p-component or an infinite number of non-zero p-components.

Assume that tG has an unbounded p-component $(tG)_p$. Then it follows from statement (iii) in the proof of the previous Lemma that $X/pX \neq 0$. Let $B^{(p)} = B_1 \oplus \ldots \oplus B_i \oplus \ldots$; $B_i = (C(p^i))^{(\mathfrak{m}_{p_i})}$ be a basic subgroup of $(tG)_p$ and let $\mathfrak{m}_p$ be the final rank of $(tG)_p$. By repeating the proof of statement (iv), (a) in the proof of the previous Lemma, we find that $\operatorname{Hom}((tG)_p, X \otimes C(p^\infty))$, and consequently $\operatorname{Ext}(tG, X) \cong G$ as well, contains a direct summand $V = \prod_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right)$. Hence $\operatorname{Ext}(V, X)$ is a direct summand of G.

Now, it follows from Lemma 2.5 that

$$V/tV \cong Q^{(\mathfrak{n})} \oplus K,$$

where $\mathfrak{n}$ is infinite. The exact sequence

$$0 \to tV \to V \to Q^{(\mathfrak{n})} \oplus K \to 0$$

gives rise to the exact sequence

$$0 \to \operatorname{Ext}\left(Q^{(\mathfrak{n})} \oplus K, X\right) \to \operatorname{Ext}(V, X) \to \operatorname{Ext}(tV, X) \to 0$$

and since G is reduced, it follows that

$$\operatorname{Ext}\left(Q^{(\mathfrak{n})} \oplus K, X\right) \cong (\operatorname{Ext}(Q, X))^{\mathfrak{n}} \oplus \operatorname{Ext}(K, X) = 0$$

whence we deduce that X is a torsion-free cotorsion group.

If $(tG)_p \neq 0$ for an infinite number of primes p, then $X/pX \neq 0$ for the relevant primes. Let us consider

$$G \cong \operatorname{Ext}(tG, X) \cong \prod_{p \in P} \operatorname{Hom}((tG)_p, X \otimes (Q/Z)).$$

Notice that if $(tG)_p \neq 0$ then

$$\operatorname{Hom}((tG)_p, X \otimes (Q/Z)) \cong \operatorname{Hom}((tG)_p, X \otimes C(p^\infty)) \neq 0.$$

Further, it is evident thet

$$tG \cong \bigoplus_{p \in P} t \operatorname{Hom}((tG)_p, X \otimes (Q/Z))$$

and that G/tG contains a divisible subgroup of infinite rank $\mathfrak{r}$,

$$G/tG = Q^{(\mathfrak{r})} \oplus L.$$

Consider the exact sequence

$$0 \to tG \to G \to Q^{(\mathfrak{r})} \oplus L \to 0$$

and the induced exact sequence

$$0 \to \operatorname{Ext}(Q^{(\mathfrak{r})} \oplus L, X) \to \operatorname{Ext}(G, X) \to \operatorname{Ext}(tG, X) \to 0.$$

Now, since G is reduced, we have

$$\operatorname{Ext}(Q^{(\mathfrak{r})} \oplus L, X,) \cong (\operatorname{Ext}(Q, X))^{\mathfrak{r}} \oplus \operatorname{Ext}(L, X) = 0$$

and hence X is a torsion-free cotorsion group.

In any event, we see that X is a torsion-free cotorsion group. It therefore follows that if $G \in \mathfrak{M}$ then G is never torsion-free.

Let $G \in \mathfrak{M}$ and consider $tG = \bigoplus_{p \in P} (tG)_p$. Let $B^{(p)}$ be a basic subgroup of $(tG)_p$, then $B^{(p)} = B_1 \oplus \ldots \oplus B_i \oplus \ldots$, where $B_i = (C(p^i))^{(\mathfrak{m}_{p_i})}$, and it follows from ([5], p. 136), that $\mathfrak{m}_{p_i}$ is finite for each i. Let $\mathfrak{r}_p$ denote the final rank of $(tG)_p$, then we have

$$G \cong \operatorname{Ext}(tG, X) \cong \prod_{p \in P} \operatorname{Hom}((tG)_p, X \otimes (Q/Z)).$$

It is clear that only

$$\operatorname{Hom}((tG)_p, X \otimes (Q/Z)) \cong \operatorname{Hom}((tG)_p, X \otimes C(p^\infty))$$

will contribute to $(tG)_p$ and it is further evident that

$$tG \cong \bigoplus_{p \in P} t \operatorname{Hom}((tG)_p, X \otimes C(p^\infty))$$

and $(tG)_p \neq 0$ implies $X/pX \neq 0$.

For those primes p for which $(tG)_p \neq 0$, we must necessarily have $X/pX \cong C(p)$, for if $X/pX \cong (C(p))^{(\mathfrak{n})}$, where $\mathfrak{n} \geq 2$, then ([3], p. 137) $\operatorname{Hom}((tG)_p, X \otimes C(p^\infty)) \cong \operatorname{Hom}((tG)_p, (C(p^\infty))^{(\mathfrak{n})})$

$$= \operatorname{Hom}((tG)_p, C(p^\infty) \oplus T), (C(p^\infty))^{(\mathfrak{n})} = C(p^\infty) \oplus T$$

$$\cong Z(p)^{\mathfrak{r}_p} \oplus \prod_{i=1}^{\infty} \left((C(p^i))^{\mathfrak{m}_{p_i}}\right) \oplus \operatorname{Hom}((tG)_p, T)$$

and the finiteness of the $\mathfrak{m}_{p_i}$ implies that a basic subgroup of the latter group will never be isomorphic to $B^{(p)}$. Since $(tG)_p$ is contained in $\operatorname{Hom}((tG)_p, X \otimes C(p^\infty))$ and since all basic subgroups of a p-group are isomorphic ([2], p. 104), we conclude that $X/pX \cong C(p)$.

To summarize, X is a reduced torsion-free cotorsion group such that $X/pX \cong C(p)$ for all primes p for which there exists a group $G \in \mathfrak{M}$ with $G_p \neq 0$. Moreover ([7], p. 372),

$$X \cong \operatorname{Ext}(Q/Z, X) \cong \prod_{p \in P} \operatorname{Hom}(C(p^\infty), X \otimes C(p^\infty)).$$

Now consider the set of all primes p for which there exists a group $G \in \mathfrak{M}$ with $G_p \neq 0$. We assert that this set is the set of all primes. Indeed, if there exists a prime q such that $G_q = 0$ for all $G \in \mathfrak{M}$, then consider the reduced torsion-free cotorsion group

$$Y = \prod_{\in P} \operatorname{Hom}\left(C(p^\infty), (C(p^\infty))^{(\mathfrak{n}_p)}\right),$$

where $\mathfrak{n}_p = r(X/pX)$ for all primes $p \neq q$ and where $\mathfrak{n}_q = 1$. For this group Y, we have $G \cong \operatorname{Ext}(G, Y)$ for all $G \in \mathfrak{M}$, and furthermore $C(q^k) \cong \operatorname{Ext}(C(q^k), Y)$ for all natural numbers k, contrary to the maximality of $\mathfrak{M}$. We conclude that X is a reduced torsion-free cotorsion group such that $X/pX \cong C(p)$ for all primes p. Consequently,

$$X \cong \operatorname{Ext}(Q/Z, X) = \prod_{p \in P} \operatorname{Hom}(C(p^\infty), X \otimes C(p^\infty)) \cong \prod_{p \in P} Z(p).$$

It now follows from Theorem 2.9 and Example 2.8 that $\mathfrak{M} = \mathfrak{A}$. This completes the proof of the Lemma.

We are now in a position to prove the following theorem :

THEOREM 2.12. — *Let $\mathfrak{M}$ be a maximal class of left* Ext-*reproduced groups for which there exists a reduced torsion-free group X such that*

$$\operatorname{Ext}(G, X) \cong G \quad \textit{for all } G \in \mathfrak{M}.$$

If tG is reduced for all $G \in \mathfrak{M}$ then either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{A}$.

Proof. — There are two alternatives viz. either

(i) $\mathfrak{M}$ contains a group G which is not reduced

or

(ii) all groups G in $\mathfrak{M}$ are reduced.

As far as (i) is concerned, note that $G/tG \neq 0$, i. e. G is not a torsion group, for if G is a torsion group then

$$G \cong \mathrm{Ext}(G, X) \cong \mathrm{Hom}(G, X \otimes (Q/Z)),$$

and the latter group is reduced ([4], p. 20), contrary to (i). Consider the exact sequences

$$0 \to tG \to G \to G/tG \to 0$$

and

$$0 \to \mathrm{Ext}(G/tG, X) \to \mathrm{Ext}(G, X) \cong G \to \mathrm{Ext}(tG, X) \to 0. \tag{13}$$

The above remark implies that $\mathrm{Ext}(G/tG, X) \neq 0$, and hence it follows, from Lemma 2.10, that $\mathfrak{M} = \mathfrak{F}$.

As regards (ii), we deduce from the exact sequence (13) that $\mathrm{Ext}(G/tG, X) = 0$ for all $G \in \mathfrak{M}$, whence, by Lemma 2.11, $\mathfrak{M} = \mathfrak{A}$. This ends the proof.

The following corollary is a direct consequence of the foregoing.

COROLLARY 2.13. — *The groups X which are such that $\prod_{p \in P} Z(p)/X \cong Q$, and $X \cong \prod_{p \in P} Z(p)$, are the only torsion-free and reduced groups for which there exist maximal classes $\mathfrak{M}$ of left* Ext-*reproduced groups such that, for all $G \in \mathfrak{M}$, tG is reduced, and* $\mathrm{Ext}(G, X) \cong G$.

In the following two theorems, we characterize the classes $\mathfrak{F}$ and $\mathfrak{A}$. We draw attention to the fact that X will denote a group such that

$$\mathrm{Ext}(G, X) \cong G \quad \text{for all } G \in \mathfrak{M}.$$

THEOREM 2.14. — *Let $\mathfrak{M}$ be a class of left* Ext-*reproduced groups. Then the following properties of $\mathfrak{M}$ are equivalent :*

1° *$\mathfrak{M}$ is the class of all groups $Q^n \oplus T$, where T is a finite group and n is a non-negative integer.*

2° *$\mathfrak{M}$ is a maximal class such that :*

(a) $C(p^k) \in \mathfrak{M}$ for all primes p and all natural numbers k;

(b) $\mathfrak{M}$ contains groups which are not reduced.

3° *$\mathfrak{M}$ is a maximal class which contains Q, and*

(*a*) *X is torsion-free;*

(*b*) *tG is reduced for all $G \in \mathfrak{M}$.*

4° *$\mathfrak{M}$ is a maximal class such that :*

(*a*) *X is torsion-free;*

(*b*) *for all $G \in \mathfrak{M}$, tG is reduced, and $G/pG \in \mathfrak{M}$ for all primes p.*

5° *$\mathfrak{M}$ is a maximal class such that if $G \in \mathfrak{M}$ then, for all pure subgroups U of G, we have $U \in \mathfrak{M}$ and $G/U \in \mathfrak{M}$.*

6° *$\mathfrak{M}$ is a maximal class for which X satisfies $\prod_{p \in P} Z(p)/X \cong Q$.*

Proof. — We shall prove that the following properties are equivalent : 1° and 2°, 1° and 3°, 1° and 4°, 1° and 5°, 1° and 6°, and this will complete the proof.

The equivalence of 1° and 6° is an immediate consequence of Theorem 2.2, Example 2.3 and Theorem 2.7. The equivalence of 1° and 3° follows from Theorem 2.7 and Lemma 2.10. It is also clear, in view of Theorem 2.7, that 1° implies 2°, 4° and 5°.

We now prove that 2° implies 1°. Assume that 2° holds. Then, by Lemma 2.4, X is isomorphic to a pure subgroup of $\prod_{p \in P} Z(p)$ such that $X/pX \cong C(p)$ for all primes p. Moreover, X is not isomorphic to $\prod_{p \in P} Z(p)$ for if this were the case then, for all $G \in \mathfrak{M}$, we would have

$$G \cong \operatorname{Ext}(tG, X) \cong \operatorname{Hom}(tG, X \otimes (Q/Z))$$

and this would imply by ([4], p. 20), that all groups in $\mathfrak{M}$ were reduced, contrary to our assumption. Hence it follows, from the exact sequence (4) in the proof of Lemma 2.4, that $\prod_{p \in P} Z(p)/X \cong Q^{(\mathrm{m})}$.

Let $G \in \mathfrak{M}$ be a group which is not reduced. Then it is clear that G contains at most a finite number of copies of Q. We assert that

(14) G contains no subgroup $C(p^\infty)$.

In order to prove this, suppose that $G = C(p^\infty) \oplus G'$. Then

$$\begin{aligned} G &\cong \operatorname{Ext}(C(p^\infty), X) \oplus \operatorname{Ext}(G', X) \\ &\cong \operatorname{Hom}(C(p^\infty), X \otimes C(p^\infty)) \oplus \operatorname{Ext}(G', X) \end{aligned}$$

and since $X \otimes C(p^\infty) \cong C(p^\infty)$ ([2], p. 255), it follows that

$$\operatorname{Hom}(C(p^\infty), C(p^\infty)) \cong Z(p)$$

is a direct summand of G. Consequently, $\mathrm{Ext}(Z(p), X)$ is a direct summand of G. However, by Lemma 2.6, $\mathrm{Ext}(Z(p), X)$, and hence G as well, contains a torsion-free divisible subgroup of infinite rank, contrary to the fact that G has at most a finite number of copies of Q. Therefore (14) holds.

We conclude that if $G \in \mathfrak{M}$ is not reduced then G has a subgroup Q and since $\mathrm{Ext}(Q, X)$ is torsion-free and divisible ([2], p. 245), we necessarily must have $\mathrm{Ext}(Q, X) \cong Q$. Now, Theorem 2.2 implies that $\prod_{p \in P} Z(p)/X \cong Q$ and hence it follows, from Example 2.3 and Theorem 2.7, that $\mathfrak{M} = \mathfrak{F}$. This proves that 2° implies 1°.

Next we show that 4° implies 1°. To this end, assume that 4° holds. Then, by Theorem 2.12, either $\mathfrak{M} = \mathfrak{F}$ or $\mathfrak{M} = \mathfrak{A}$. We contend that $\mathfrak{M} = \mathfrak{F}$. Indeed, if $\mathfrak{M} = \mathfrak{A}$ consider the group

$$G = Z(p)^{2^{\aleph_0}} \oplus \prod_{i=1} ((C(p^i))^{\mathfrak{m}_i}),$$

where each $\mathfrak{m}_i$ is a non-zero positive integer. For this group, we clearly have that $G/pG \cong (C(p))^{(\mathfrak{n})}$ is of infinite rank, whence by ([5], p. 136), $G/pG \notin \mathfrak{M}$. This is however contrary to the assumption that $G/pG \notin \mathfrak{M}$ and hence we conclude that $\mathfrak{M} = \mathfrak{F}$. This proves that 4° implies 1°.

Finally, we show that 5° implies 1°. Let $G \in \mathfrak{M}$. Then since tG is pure in G our hypothesis implies that $tG \in \mathfrak{M}$ and that $G/tG \in \mathfrak{M}$. Note that tG is reduced, for if G contains a subgroup $C(p^\infty)$, then $C(p^\infty) \in \mathfrak{M}$, i. e.

$$C(p^\infty) \cong \mathrm{Ext}(C(p^\infty), X).$$

However, $\mathrm{Ext}(C(p^\infty), X)$ is a reduced cotorsion group ([5], p. 134), and this contradiction shows that tG is reduced.

If $G/tG = 0$ then G is a torsion group, and then we have $G = \bigoplus_{p \in P} G_p$. By assumption, $G_p \in \mathfrak{M}$ for all primes p and hence

$$G \cong \mathrm{Ext}(G, X) \cong \prod_{p \in P} \mathrm{Ext}(G_p, X) \cong \prod_{p \in P} G_p,$$

consequently, since G is torsion, it follows that $G_p \neq 0$ for only a finite number of primes, i. e. $G \cong \bigoplus_{i=1}^{n} G_{p_i}$. Moreover, every G_{p_i} is bounded, for if G_{p_i} is unbounded let $B^{(p_i)}$ denote a basic subgoup of G_{p_i}. Then $B^{(p_i)}$ is unbounded and since it is pure in G_{p_i}, we have $B^{(p_i)} \in \mathfrak{M}$. But then $B^{(p_i)}$ must be isomorphic to the direct product of its cyclic direct summands since every direct summand belongs to $\mathfrak{M}$, and this is evidently impossible. Hence each G_{p_i} is bounded and therefore a direct sum of cyclic groups. By ([5], p. 136), each G_{p_i} is finite and hence G is finite.

Now $\mathfrak{M}$, being a maximal class, cannot contain only finite groups and hence it must necessarily contain infinite groups. Let $G \in \mathfrak{M}$ be an infinite group. Then either G contains elements of infinite order or G is an infinite torsion group. From what we have proved above it follows that G cannot be an infinite torsion group since tG is finite for all $G \in \mathfrak{M}$ and hence G must necessarily contain elements of infinite order. Consequently, G/tG is torsion-free and hence

$$G/tG \cong \mathrm{Ext}(G/tG, X)$$

is torsion-free and divisible. Hence Q, being a direct summand of G/tG, belongs to $\mathfrak{M}$, i. e.

$$Q \cong \mathrm{Ext}(Q, X)$$

and hence for all natural numbers n, we have $Q^n \in \mathfrak{M}$.

To summarize, if 5° holds then $\mathfrak{M}$ contains groups Q^n for all natural numbers n as well as finite groups. Moreover, tG is reduced and finite for all $G \in \mathfrak{M}$. The maximality of $\mathfrak{M}$ implies that $\mathfrak{M}$ contains all finite groups and hence $\mathfrak{M}$ contains all groups $Q^n \oplus T$, where n is a non-negative integer and T is a finite group. This proves that 5° implies 1° and the proof of the theorem is complete.

THEOREM 2.15. — *Let $\mathfrak{M}$ be a class of left* Ext-*reproduced groups. Then the following properties of $\mathfrak{M}$ are equivalent :*

(α) *$\mathfrak{M}$ is the class of all groups $\prod_{p \in P} A(p)$, where $A(p) = G_1^{(p)} \oplus \ldots \oplus G_n^{(p)}$, and where the $G_i^{(p)}$ are defined as in* Example 2.8.

(β) *$\mathfrak{M}$ is a maximal class such that :*

(*a*) *$C(p^k) \in \mathfrak{M}$ for all primes p and all natural numbers k;*

(*b*) *$\mathfrak{M}$ contains only reduced groups.*

(γ) *$\mathfrak{M}$ is a maximal class such that :*

(*a*) *X is torsion-free;*

(*b*) *for all $G \in \mathfrak{M}$, we have $G \cong \mathrm{Ext}(tG, X)$.*

(δ) *$\mathfrak{M}$ is a maximal class for which $X \cong \prod_{p \in P} Z(p)$.*

Proof. — W shall prove that the following properties are equivalent : (α) and (β), (α) and (γ), (α) and (δ), and this will furnish the proof.

The equivalence of (α) and (β) follows from Lemma 2.4, Example 2.8, Theorem 2.9, and Lemma 2.11. The equivalence of (α) and (γ) is a consequence of Theorem 2.9 and Lemma 2.11, and finally, Example 2.7 and Theorem 2.8 show that (α) and (δ) are equivalent. This completes the proof.

3. Classes of right Ext-reproduced groups.

LEMMA 3.1. — *The class $\mathfrak{C}$ of all reduced cotorsion groups is a class of right* Ext-*reproduced groups.*

Proof. — Let $C \in \mathfrak{C}$, and consider the exact sequences

$$0 \to Z \to Q \to Q/Z \to 0$$

and

$$\text{(15)} \qquad \operatorname{Hom}(Q, C) \to \operatorname{Hom}(Z, C) \to \operatorname{Ext}(Q/Z, C) \to \operatorname{Ext}(Q, C).$$

Since C is reduced, we have $\operatorname{Hom}(Q, C) = 0$. Furthermore, $\operatorname{Hom}(Z, C) \simeq C$, and since C is cotorsion, $\operatorname{Ext}(Q, C) = 0$. Hence it follows, from the exact sequence (15), that

$$\operatorname{Ext}(Q/Z, C) \cong C$$

and this completes the proof of the lemma.

LEMMA 3.2. — *Every class $\mathfrak{R}$ of right* Ext-*reproduced groups is contained in $\mathfrak{C}$.*

Proof. — Let A be a group such that

$$G \cong \operatorname{Ext}(A, G) \quad \text{for all } G \in \mathfrak{R}.$$

Then G is a cotorsion group ([5], p. 134), and by ([5], p. 133), we have

$$\operatorname{Ext}(A, G) \cong \operatorname{Ext}(tA, G).$$

Now, since tA is torsion, $\operatorname{Ext}(tA, G)$ is reduced ([5], p. 134), and hence G is also reduced. Consequently $G \in \mathfrak{R}$ implies $G \in \mathfrak{C}$ and this proves that $\mathfrak{R} \subseteq \mathfrak{C}$.

The following theorem is an immediate consequence of Lemma 3.1 and Lemma 3.2.

THEOREM 3.3. — *$\mathfrak{C}$ is the only maximal class of right* Ext-*reproduced groups.*

We now investigate the existence of the groups A which satisfy

$$C \cong \operatorname{Ext}(A, C) \quad \text{for all } C \in \mathfrak{C}.$$

A complete answer is given in the following theorem :

THEOREM 3.4. — *A group A satisfies*

$$C \cong \operatorname{Ext}(A, C) \text{ for all } C \in \mathfrak{C}$$

if, and only if, tA is isomorphic to Q/Z.

Proof. — Suppose that $tA \cong Q/Z$, and let $C \in \mathfrak{C}$, then, by Lemma 3.1, we have

$$C \cong \operatorname{Ext}(tA, C).$$

Since C is cotorsion, it follows from ([5], p. 133), that

$$C \cong \operatorname{Ext}(tA, C) \cong \operatorname{Ext}(A, C).$$

Conversely, let A be a group such that

$$C \cong \operatorname{Ext}(A, C) \quad \text{for all } C \in \mathfrak{C}.$$

For all primes p and all natural numbers i, we have $C(p^i) \in \mathfrak{C}$ and hence

$$C(p^i) \cong \operatorname{Ext}(A, C(p^i)) \cong \operatorname{Ext}(tA, C(p^i)).$$

However, since

$$\operatorname{Ext}(P, P') = 0$$

for a p-group P and a q-group P' with $q \neq p$, it follows that

$$\operatorname{Ext}(A, C(p^i)) \cong \operatorname{Ext}(tA, C(p^i)) \cong \operatorname{Ext}((tA)_p, C(p^i)) \cong C(p^i).$$

Hence $(tA)_p \neq 0$ and consequently, by ([2], p. 80), there exists an integer $1 \leq k \leq \infty$ such that

$$(tA)_p = C(p^k) \oplus X.$$

We maintain that $X = 0$. Assume on the contrary that $X \neq 0$. Then X has a direct summand of the form $C(p^l)$, $1 \leq l \leq \infty$. Hence $\operatorname{Ext}((tA)_p, C(p))$, and therefore $\operatorname{Ext}(A, C(p))$, contains a direct summand of the form $C(p) \oplus C(p)$, for firstly, we have for every natural number k

$$\operatorname{Ext}(C(p^k), C(p)) \cong C(p),$$

and secondly

$$\operatorname{Ext}(C(p^\infty), C(p)) \cong C(p).$$

Hence we have

$$\operatorname{Ext}(C(p^k) \oplus C(p^l), C(p)) \cong C(p) \oplus C(p),$$

contrary to

$$\operatorname{Ext}(A, C(p)) \cong C(p).$$

We conclude that $X = 0$ and hence, for every prime p, $(tA)_p$ is of the form $C(p^k)$ with $1 \leq k \leq \infty$. If there exists a finite p-component of tA, viz.

$$(tA)_p = C(p^k),$$

then we have

$$\operatorname{Ext}(A, C(p^{k+1})) \cong \operatorname{Ext}(C(p^k), C(p^{k+1})) \cong C(p^k),$$

contrary to

$$\operatorname{Ext}(A, C) \cong C \quad \text{for all } C \in \mathfrak{C}.$$

Hence

$$(tA)_p \cong C(p^\infty) \quad \text{for all primes } p,$$

and we have

$$tA \cong \bigoplus_{p \in P} C(p^\infty) \cong Q/Z.$$

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