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ABSTRACT ANALYTIC AND BORELIAN SETS

BY

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The aim of this note is to introduce the concept of analytic set over a given topological space Q (in the classical theory Q is the space Σ of irrational numbers) derived from a given collection $\mathfrak{M}$ of sets (in the classical theory, $\mathfrak{M}$ is the collection of all closed sets in a topological space). The main point is the general setting of the fact that, roughly speaking, closed-graph usco-compact (an abbreviation for upper-semi-continuous and compact valued) correspondences for topological spaces behave like usco-compact correspondences with separated (i. e. HAUSDORFF) range; in other words, the assumption of separatedness of the range may be replaced by closeness of the graph. It turns out that the basic properties of classical concepts extend to the present general setting; the proofs are simple and natural, and the properties of Σ needed in the descriptive theory are identified.

For the theory of analytic and Borelian spaces in topological spaces, we refer to [8], and for a survey of descriptive theory covering the last decade, we refer to [7]. In general, the notation and terminology of [1] is used throughout.

It is convenient to make use of paved spaces as introduced by P. MEYER [9]; the concept of a paved space will be explained in Section 1. In Section 2, a reflection of the concept of " closed graph " into the present more general setting will be discussed. Finally, in the concluding two sections, the concepts in the title will be introduced and investigated.

1. Paved spaces.

Following [1] by a *struct* P we mean a pair $\langle X, \alpha \rangle$, where X is a set (we will talk about the so-called comprisable structs only), designated by $|P|$, and called the underlying set of P, and α is an element designated by $\mathrm{st}(P)$ and called the structure of P. If there is no danger of misunderstanding then we write P for $|P|$, and *vice versa*; this is in accordance with commonly used conventions.

DEFINITION 1. — *A* paved space *is a struct P such that the structure of P is a non-void collection of subsets of P; the structure of P is called the pavement of P, and the elements of the pavement are called* stones. *An important assumption will be that the empty set is a stone; this is then precisely the Meyer's paved space.*

The most important example of paved spaces in this note are topological spaces. Every topological space is regarded to be a paved space; the stones are just the closed sets. This convention is the first step to explain the intuitive meaning of paved spaces in this note.

Consider a paved space P, and denote by top (P) the set $|P|$ endowed with the smallest finitely additive and completely multiplicative collection of sets containing the pavement of P. Clearly top (P) is a topological space. Thus the pavement of P is a closed sub-base for the topological space top (P).

A set X in a paved space P is called compact if, for any collection $\mathfrak{X}$ of stones such that $\mathfrak{X} \cup (X)$ have the finite intersection property, the intersection of X with the intersection of $\mathfrak{X}$ is nonvoid. It follows immediately from Alexander Lemma that X is compact in P if and only if X is compact in top (P).

Recall, *see* [11], that a correspondence f of a struct Q into a struct P is a triple $\langle \rho, Q, P \rangle$, usually written $\rho : Q \rightarrow P$, where ρ is a subset of the Cartesian product $|Q| \times |P|$, the so-called graph of f, designated by $\mathrm{gr} f$. We will follow the commonly used convention that f and $\mathrm{gr} f$ are denoted by the same symbol. In particular, if f is a correspondence then Df and Ef stand for the domain and the range of the graph of f.

The intuitive notion of a paved space in this note is deepened by definition of the morphisms we are interested in.

DEFINITION 2. — *A correspondence f of a paved space Q into a paved space P is said to be* usco *if the preimages of stones are stones, compact if the images of singletons, called values, are compact, and usco-compact if f is usco and compact.*

A correspondence f is said to be disjoint (also a fibration) if the inverse of f is single-valued. The meaning of dusco and dusco-compact seems to be obvious; d comes from disjoint.

It is obvious that the composite of two usco-correspondences is usco. It is not so obvious that the composite of usco-compact correspondences is usco-compact. One has to show that the image of a compact set under an usco-compact correspondence is compact, and this is done by a routine argument. It follows that if a correspondence f of a topological space Q into top (P) is usco, compact or usco-compact then so is the correspondence $f: Q \rightarrow P$. The converse is obviously true for mappings, and it does not hold for usco and usco-compact in general. Nevertheless if P is finitely multiplicative [to mean that st(P) is finitely multiplicative] and $f: Q \rightarrow P$ usco-compact, then $f: Q \rightarrow \mathrm{top}(P)$ is usco-compact.

2. S-correspondences.

An S-family in $\mathcal{M}$ over $\mathcal{B}$ is a single-valued relation M with $DM = \mathcal{B}$, $EM \subset \mathcal{M}$, such that $\mathcal{M}$ is a collection of sets and $\mathcal{B}$ is a family of sets (not necessarily one-to-one!). The associated relation to M is the set $\tilde{M}$ of all $\langle x, y \rangle$ such that $x \in \bigcup \mathcal{B}$, and y belongs to each MB_a, $B_a \in \mathcal{B}$ with $x \in B_a$. The set

$$E\tilde{M} = \bigcup \left\{ \bigcap \{ MB_a \mid x \in B_a \in \mathcal{B} \} \mid x \in \bigcup \mathcal{B} \right\}$$

is called the Souslin set of M, and designated by SM.

The intuitive meaning of these notions may be seen from the following result.

PROPOSITION 1. — *Let f be a correspondence of a topological space Q into a paved space P. If f is associated with a Souslin family in* st(P) *over an open cover of Q then f is closed-graph, that means, the graph of f is a closed set in the product topological space $Q \times \mathrm{top}(P)$. If P is topological, and if f is closed-graph then f is associated with a Souslin family in* st(P) *over an open cover of P (in fact, over any base for open sets in Q).*

Proof. — Assume that f is associated with M in st(P) where $\mathcal{B} = D\,M$ is an open cover of Q. If

$$\langle x, y \rangle \in |Q| \times |P| - f,$$

then

$$y \notin MB_a$$

for some B_a in $\mathcal{B}$ with $x \in B_a$. Clearly,

$$B_a \times (|Q| - MB_a)$$

is a neighborhood of $\langle y, y \rangle$ that does not meet f.

Conversely, assume that the graph of f is closed in $Q \times P$, and $\mathcal{B}$ is any open base for Q. Define M over $\mathcal{B}$ by setting

$$MB = cl_P\, f[B]$$

for B in $\mathcal{B}$. It is easy to verify that f is associated with M.

Definition 3. — *An S-family over a topological space Q in a paved space P is a Souslin family M in* st(P) *over a countable open cover of Q; the correspondence*

$$\tilde{M} : Q \to P$$

is called the correspondence associated with M. Finally an S-correspondence of Q into P is a correspondence associated with an S-family in P over Q.

Remark. — A closely-related notion of an S-family over a topological space is introduced in [6]; the only difference is that here the domain is a family, in [6] the domain is a collection. I decided to change the definition to get Theorem 1 below that seems to be very important, *see* Theorem 3 below that is an immediate consequence of Theorem 1. It should be remarked that we assume that the domain of an S-family is countable to get an extension of the classical theory.

The following two theorems are fundamental for development of analytic and Borelian sets.

Theorem 1. — *Let $f : R \to Q$ be an usco-compact correspondence of a topological space R into a space Q, and let P be a paved space such that the structure of P is finitely additive. If $g : Q \to P$ is an S-correspondence, then so is the composite $h = g \circ f$. More generally, if g is associated with a Souslin family in* st(P) *over an open infinite cover of Q, then h is associated with a Souslin family in* st(P) *over an open cover of R of the same cardinal as that of the cover of Q.*

Remark. — This is a generalization of Theorem 1.2 in [8] that says that the composite of an usco-compact correspondence followed by a closed-graph correspondence is closed-graph.

Proof of Theorem 1. — Assume that g is associated with a Souslin family in st(P) over an infinite cover $\mathcal{B} = \{ U_a \mid a \in A \}$. Let B be the set of all non-void finite subsets of A; clearly the sets A and B are of the same cardinal. Define $\{ V_b \mid b \in B \}$ and $\{ W_b \mid b \in B \}$ as follows:

$$V_b = \bigcup \{ U_a \mid a \in b \},$$
$$W_b = E \{ x \mid x \in R,\ f[(x)] \subset V_b \}.$$

Finally, define Souslin families X and Y in st(P) over $\{ V_b \}$ and $\{ W_b \}$ by setting (the sets W_b are open because f is usco)

$$XV_b = YW_b = \bigcup \{ MU_a \mid a \in b \}.$$

Clearly to show that h is associated with Y it is enough to prove that

$$f[C] = \bigcap \{ XV_b \mid b \in B,\ V_b \supset C \}$$

for each compact set C in Q. Given a y in $P - f[C]$ we can find a family $\{ax \mid x \in C\}$ in A such that $x \in U_{ax}$, and $y \notin MU_{ax}$. Since C is compact, a finite subfamily covers C, and hence $y \notin XV_b$ for some $V_b \supset C$. This concludes the proof.

THEOREM 2. — *Assume that P is a paved space, and $\mathcal{B}$ is an open cover of a topological space Q. For each B in $\mathcal{B}$ let $f_B : Q_B \to P$ be a correspondence of a topological space Q_b into P. Define a correspondance f of*

$$Q' = Q \times \Pi \{ Q_B \mid B \in \mathcal{B} \}$$

into P by setting

$$\langle\langle x, \{ x_B \mid B \in \mathcal{B} \}\rangle, y \rangle \in \operatorname{gr} f$$

if and only if

$$y \in \bigcap \{ f_B[(x_B)] \mid x \in B_a \in \mathcal{B} \}.$$

Then if $\mathcal{B}$ is countable, and if all f_B are S-correspondences then f is an S-correspondence. If, in addition, there exists a subcover $\mathcal{C}$ of $\mathcal{B}$ such that all f_B with B in $\mathcal{C}$ are usco-compact, then f is an usco-compact S-correspondence.

Remark. — Observe that Ef is the Souslin set of the Souslin family $F = \{ B \to Ef_B \mid B \in \mathcal{B} \}$, and if all f_B, and also $\tilde{F}$, are disjoint then so is f. Theorem 2 is the main result needed for the proof of idempotency of the Souslin operation, the invariance of analytic sets under Souslin operation, etc. The construction of f is given in [6], where the result concerning S-correspondences is stated. The reader is invited to state the obvious generalization suppressing the countability assumptions.

COROLLARY. — *Let $\{ f_a : R_a \to P \mid a \in A \}$ be a family of S-correspondences where R_a are topological spaces, and P be a paved space. The intersection of $\{ f_a \}$ is $f : R' \to P$, where R' is the product of $\{ R_a \}$, and $\langle x, y \rangle \in f$ if and only if $\langle x_a, y \rangle \in f_a$ for each a ; here x_a stands for the a-th coordinate of x.*

If A is countable, and if all f_a are S-correspondences, then so if f. If, in addition, at least one of f_a is usco-compact, then f is usco-compact.

Remark. — Theorem 2 is a generalization of Theorem 1.3 in [8]. It should be remarked that the intersection of two usco-compact correspondences for topological spaces need not be usco. This is to correct a statement in the concluding remark in [6] where the assumption of separatedness should be inserted.

The reader is invited to prove a particular case of Corollary to Theorem 2 that every S-correspondence into a compact paved space is usco. The point of the proof is used in

Proof of Theorem 2. — Write $\mathcal{B} = \{ B_a \mid a \in A \}$, f_a instead of f_{B_a}. Assume that each f_a is associated with M_a in P over $\mathcal{L}_a$. For each b

in A, and for each member L of $\mathcal{L}_b$ denote by L'_b the set of all $\xi = \langle x, \{x_a\}\rangle \in Q'$ such that $x \in B_b$, and $x_b \in L$. We get a cover $\mathcal{B}' = \{L'_a \mid a \in A,\ L \in \mathcal{L}_a\}$. Define

$$KL'_a = M_a L.$$

It is easy to verify that f is associated with K. Assume, now, that P is finitely multiplicative, and that for some cover $\{B_a \mid a \in o\}$, $o \subset A$, all f_a with a in o are usco-compact. Choose any $\xi = \langle x, \{x_a\}\rangle$ in Q', and let F be a stone disjoint to $f[(\xi)]$; we have to find a neighborhood U of ξ such that $f[U] \cap F = \emptyset$. By definition,

$$f[(\xi)] = \bigcap \{f_a[(x_a)] \mid x \in B_a\},$$

the values of $f'_a s$ are intersections of stones, and at least one of them is compact in P, say $f_b[(x_b)]$; hence there exists a finite set $\alpha \subset A$ such that $x \in B_a$ for a in α, $b \in \alpha$, and the set

$$C = \bigcap \{f_a[(x_a)] \mid a \in \alpha\}$$

is disjoint to F. Since f_a is associated with M_a we have

$$\bigcap \{M_a L \mid a \in \alpha,\ x_a \in L \in \mathcal{L}_a\} = C,$$

and as the set

$$C_b = F \cap f_b[(x_b)]$$

is compact and disjoint to C, some finite intersection X should be disjoint to C_b. For the proof of Theorem 2, we need just to know that there exists a neighborhood V of ξ, and a stone X such that

$$f[V] \subset X \qquad \text{and} \qquad X \cap C_b = \emptyset.$$

Now consider the stone $X \cap F$. Since f_b is usco, and the stone $X \cap F$ is disjoint to $f_b[(x_b)]$, there exists a neighborhood W of x_b in Q such that

$$f_b[W] \cap X \cap F = \emptyset,$$

and hence

$$f[W'] \cap X \cap F = \emptyset,$$

where W' is the cylinder in Q' over W. Put $U = V \cap W'$. Obviously,

$$F \cap f[U] \subset F \cap f[V] \cap f[W'[\subset F \cap X \cap f[W'] = \emptyset.$$

This concludes the proof.

3. **Souslin and analytic sets over Q.**

Denote by Σ the set of all infinite sequences in the set N of natural numbers, endowed with the topology of point-wise convergence. The space Σ is known to be homeomorphic with the space of all irrational numbers. All results of this section apply to $Q = \Sigma$.

DEFINITION 4. — Let Q be a topological space, and let P be a paved space. A Souslin set in P over Q is the image of Q under an S-correspondence of Q into P. An analytic set in P over Q is the image of Q under an usco-compact S-correspondence of Q into P. We denote by $S_Q(P)$ or $A_Q(P)$ the set $|P|$ endowed with the pavement consisting of all Souslin or analytic sets in P over Q. If $\mathfrak{M}$ is a collection of sets then Souslin-$\mathfrak{M}$ or analytic-$\mathfrak{M}$ sets over Q are defined to be the Souslin or analytic, respectively, sets in P over Q, where P is the union of $\mathfrak{M}$ endowed with the pavement $\mathfrak{M}$. The symbols $S_Q(\mathfrak{M})$ and $A_Q(\mathfrak{M})$ have the obvious meaning.

Convention. — If $Q = \Sigma$ then we speak just about Souslin or analytic sets in P, and also the subscript Σ in symbols is omitted.

For further references we note an obvious proposition.

PROPOSITION 2. — *If X is a stone in a finitely multiplicative P, and if Y is Souslin or analytic in P over Q, then so is $X \cap Y$.*

THEOREM 3. — *Assume that f is an S-correspondence of a topological space Q into a finitely additive paved space P, and let A be the image of a topological space R under an usco-compact correspondence of R into Q (the latter assumption is fulfilled when A is analytic in P over R). Then $f[A]$ is Souslin in P over R, and if, in addition, f is usco-compact, then $f[A]$ is analytic in P over R.*

Proof. — Let $g : R \to Q$ be usco-compact, and let $A = f[R]$. The correspondence $h = f \circ g$ is an S-correspondence by Theorem 1, and h is usco-compact if g is usco-compact by the concluding part of section 1.

Remark. — The set A in Theorem 3 need not be Souslin in Q over R, even if $Q = \Sigma$ and the values of g are closed in Q; *see* [8].

THEOREM 4. — *Let P be a finitely multiplicative paved space, and let Q be a topological space that maps continuously onto $Q^{\aleph_0}$. Then*

$$S_Q(S_Q(P)) = S_Q(P), \qquad S_Q(A_Q(P)) = A_Q(P).$$

Remark. — If Q is a space then $Q^{\aleph_0}$ stands for the topological product of a countably infinite number of copies of Q, say for $\Pi\{Q \mid n \in N\}$. Since $\Sigma = N^{\aleph_0}$, $\Sigma^{\aleph_0}$ is homeomorphic to Σ.

The proof of Theorem 4 follows from Theorem 2 and the following lemma.

LEMMA 1. — *If there exists a continuous mapping of Q_1 onto Q then*

$$S_{Q_1}(P) \supset S_Q(P) \qquad \text{and} \qquad A_{Q_1}(P) \supset A_Q(P)$$

for any paved space P.

Proof. — Let f be a contiuous mapping of Q_1 into Q. If $g : Q \to P$ is an S-correspondence, then $g \circ f$ is an S-correspondence without any assumption on P, *see* the subsequent remark to Theorem 1.

Remark. — The relations in Lemma 1 hold if Q is a closed subspace of Q_1 and empty set is a stone.

Proof of Theorem 4. — Assume that g is an S-correspondence of Q into $S_Q(P)$ or $A_Q(P)$, respectively, and say that f is associated with a Souslin family M over a countable open cover $\mathcal{B}$ of Q. For each B in $\mathcal{B}$ there exists and S-correspondence or an usco-compact S-correspondence f_B et Q into P such that $MB = Ef_B (= f_B[Q])$. The correspondence f in Theorem 2 is an S-correspondence or an usco-compact correspondence. Thus

$$Eg = Ef$$

is Souslin or analytic in P over $Q \times Q^{\mathcal{B}}$. By Lemma 1, Eg is Souslin or analytic in P over Q.

COROLLARY (*to Theorem* 4). — *Assume that P is a finitely multiplicative paved space that maps continuously onto $Q^{\aleph_0}$. Then a set X is analytic in P over Q if and only if X is Souslin in P over Q, and X is contained in an analytic set in P over Q. A Souslin set over Q in a Souslin set in P over Q is Souslin in P over Q.*

Proof. — A subspace of a paved space is defined in obvious way : the stones in the subspace are just the traces of stones in the space. Thus by Proposition 2, the relative stones in Souslin or analytic sets in P have the respective property in P. It remains to apply Theorem 4.

PROPOSITION 3. — *Assume that Q is a topological space and P is a paved space. Then*

$$\mathrm{st}\, S_Q(P) \supset (\mathrm{st}\, P)_\delta;$$

if Q is regular and contains an infinite closed discrete set, and if the empty set is a stone then also

$$\mathrm{st}\ S_Q(P) \supset (\mathrm{st}\, P)_\sigma.$$

Proof. — Let $\{X_n\}$ be a sequence of stones in P. To verify the first inclusion, take the constant cover $\{U_n\}$ of Q and define $MU_n = X_n$. Clearly $SM = \bigcap \{X_n\}$. To verify the second inclusion, consider a one-to-one sequence $\{x_n\}$ of points in Q such that the set F of all x_n is closed and discrete; choose a disjoint sequence $\{U_n\}$ of open sets such that $x_n \in U_n$, and put $U = Q - F$. Define M by setting $MU_n = X_n$, $MU = \emptyset$. Clearly M is a Souslin family in P over the cover consisting of all U_n and U. Clearly $SM = \bigcup\{X_n\}$.

Remark. — If $Q = \Sigma$, then we can choose U_n such that $\{U_n\}$ is a disjoint cover, and hence we need not assume that the empty set is a stone in P. Next, the slight distinction of the definition of S-family in this note and in [6] influences the first relation in Proposition 3; if we use the definition in [6] then the first relation does not hold. Notice that the cover in the proof was the constant cover $\{Q \mid n \in N\}$. If Q is a one-point space then $\mathrm{st}(P) = \mathrm{st}\, S_Q(P)$ for any P if we use the definition in [6]. *See* also the remark following Definition 3. On the other hand,

COROLLARY. — *For any non-void finitely multiplicative collection of sets, we have*

$$\mathcal{B}(S(\mathfrak{M})) = S(\mathfrak{M}) \supset \mathcal{B}(\mathfrak{M}) \supset \mathfrak{M},$$
$$\mathcal{B}(A(\mathfrak{M})) = A(\mathfrak{M}) \supset \mathfrak{M},$$

where $\mathcal{B}(\mathfrak{N})$ *is the smallest collection of sets* $\mathcal{R} \supset \mathfrak{N}$ *with* $\mathcal{R}_\sigma \cup \mathcal{R}_\delta = \mathcal{R}$.

4. **Souslin and analytic sets over Σ.**

For idempotency of S_Q, we needed just the fact that Q continuously maps onto $Q^{\aleph_0}$. In this section, we are interested in consequences of more special properties of Σ. We shall need the following notation.

Let S_n, $n = 1, 2, \ldots$ be the set of all sequences $\{i_k \mid k < n\}$ of natural numbers, and let S be the union of all S_n. Let $f < g$ mean that f is a restriction of g. For each s in S, put

$$\Sigma s = E\{\sigma \mid \sigma \in \Sigma,\ s < \sigma\}.$$

Clearly $\{\Sigma s\}$ is an open base for Σ. We shall need the following simple lemma.

LEMMA 2. — *If* $\{B_a\}$ *is an open cover of* Σ *then there exists a homeomorphism* k *of* Σ *onto* Σ *such that* $\{k[\Sigma s]\}$ *refines* $\{B_a\}$ *(or equivalently,* $\{\Sigma s\}$ *refines* $\{k^{-1}[B_a]\}$*).*

Proof. — Let S' be the set of all minimal elements of the set of all $s \in S$ with $\Sigma s \subset B_a$ for some a. Pick a one-to-one mapping of N onto S', and let k be the mapping of Σ into Σ that assigns to each $\{i_n \mid n \in N\}$ the point $\sigma = \{i_n\}$ such that $\varphi i_0 < \sigma$, say $\varphi i_0 = \{j_n \mid n < k\}$, and $j_{k+n+1} = i_{n+1}$. Clearly k is a homeomorphism onto, and $\{\Sigma s\}$ refines $\{k^{-1}[B_a]\}$.

PROPOSITION 4. — *If f is an S-correspondence of Σ into a finitely multiplicative paved space P, then there exists a homeomorphism k of Σ onto Σ such that $g = f \circ k$ is associated with a Souslin family in P over $\{ \Sigma s \}$.*

Proof. — Let f be associated with an open cover $\{ U_n \}$. First assume that $\{ \Sigma s \}$ refines $\{ U_n \}$. Set

$$K \Sigma s = \bigcap \{ M U_n \mid n \in Ns \},$$

where Ns is defined as follows : let k be the length of s, and consider the set $N's$ of all n such that $U_n \supset \Sigma s$. Now $Ns = N's$ if the cardinal of $N's$ is less than k, and Ns consists of the first k elements of $N's$, otherwise. Clearly f is associated with K.

The general case is reduced to the particular case that has just been proved by Lemma 2.

Remark. — It follows from Proposition 4 that if $\mathcal{M}$ is a finitely multiplicative collection of sets, then $S(\mathcal{M})$ is the collection of all Souslin-$\mathcal{M}$ sets in the sense of usual definition, *see* [6]. One can prove the equality is also true if the empty set is a stone, *see* [6]. We do not need it, and therefore we will not discuss it in more details.

Now we formulate the first separation principale for analytic sets.

LEMMA 3. — *Assume that P is a finitely multiplicative paved space, and $\mathcal{C}$ is a collection of sets in P such that $\mathcal{B}(\mathcal{C}) = \mathcal{C}$.*

(a) Assume that if X is a compact set in $(\mathrm{st}\,P)_\delta$, and Y is a stone disjoint to X, then there exists a neighborhood $U \in \mathcal{C}$ of Y with $U \cap Y = \emptyset$. Then if X is analytic in P, and if Y is Souslin in P disjoint to X, then $X \subset C \subset P\text{-}Y$ for some C in $\mathcal{C}$.

(b) Assume that for each disjoint compact X and Y in $(\mathrm{st}\,P)_\delta$ there exist disjoint C and D in $\mathcal{C}$ such that C is a neighborhood of X, and D is a neighborhood of Y. Then if X and Y are disjoint analytic sets in P then there exist disjoint C and D in $\mathcal{C}$ with $X \subset C$, and $Y \subset D$.

Proef. — Follows the pattern of the proof of Theorem 1 in [5] and Theorem 5 in [3].

Remark. — The « separation » assumption in Lemma 3 is satisfied if $\mathrm{top}(P)$ is separated and locally belongs to $\mathcal{C}$. We refer to [7] for further development.

5. Borelian and *d*-Souslin sets.

In general setting the role of absolute Borel sets and of Borel sets in the classical theory in separable metric spaces is sometimes played by Borelian and *d*-Souslin sets. Our technique developed for analytic and Souslin sets applies, and therefore we give the definition and just formulate the basic theorems.

DEFINITION 5. — Assume that Q is a topological space, and P is a paved space. A set X is d-Souslin in P over Q if X is the image of Q under a disjoint S-correspondence of Q into P. A set X is Borelian in P over Q if X is the image of Q under a dusco-compact S-correspondence. In an obvious way, we use the notation $S^d_Q(P)$, $A^d_Q(P)$, $S^d_Q(\mathfrak{M})$, $A^d_Q(\mathfrak{M})$, $S^d(P)$, ... Borelian sets in completely regular spaces were introduced in [3].

THEOREM 5. — *Assume that f is a disjoint S-correspondence of a topological space Q into a finitely additive paved space P, and let A be the image of a space R under a dusco-compact correspondence of R into Q. Then $f[A]$ is d-Souslin in P over R, and if, in addition, f is usco-compact then $f[A]$ is Borelian in P over Q.*

THEOREM 6. — *If Q continuously 1 — 1 maps onto $Q\aleph_0$ then*

$$S^d_Q(S^d_Q(P)) = S^d_Q(P), \qquad S^d_Q(A^d_Q(P)) = A^d_Q(P)$$

for each finitely multiplicative paved space P.

For the theory of d-Souslin sets and Borelian set in a topological space, we refer to 18]. Here we want to discuss another important example.

For a topological space P denote by exact (P) the set P endowed with all zero-sets in P (called in [1] exact closed). A Baire set in a topological space P is an element of $\mathcal{B}(\mathrm{st}(\mathrm{exact}(P)))$. It is easy to see that X is Borelian in exact (P) if and only if X is a Borelian Baire set in P; for the theory of these sets, we refer to [4], [5] and [10]. Similarly, for analytic sets in exact (P). The present theory gives new proofs. As concerns d-Souslin sets in exact (P) perhaps the second separation principle technique should be applied, *see* [11].

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