BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 96 (1968), p. 53-96
<http://www.numdam.org/item?id=BSMF_1968__96__53_0>

Bull. Soc. math. France,
96, 1968, p. 53 à 96.

A FOURIER SERIES METHOD FOR MEROMORPHIC AND ENTIRE FUNCTIONS

BY

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Contents.

(*) The research of the first author was partially supported by the United States Air Force Office of Scientific Research, under Grant Number AFOSR 460-63..

Introduction. — This work is a further development and application of the Fourier series method for entire functions introduced by the first author in [5]. The idea which was presented there and is exploited further here is the following : if f is a meromorphic function in the complex plane, and if

$$c_k(r, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\log |f(re^{i\theta})|)\, e^{-ik\theta} d\theta$$

is the k-th Fourier coefficient of $\log |f(re^{i\theta})|$, then the behaviour of $f(z)$ is reflected in the behaviour of the sequence $\{c_k(r, f)\}$ and *vice versa*. We prove a basic result in theorem 4.6, which characterizes the rate of growth of f in terms of the rate of growth of the $c_k(r, f)$ and the density of the poles of f, generalizing theorem 1 of [5]. We apply this theorem, as in [5], to obtain estimates for some integrals involving $|f(z)|$ and to obtain information about the distribution of the zeros of an entire function from information about its rate of growth.

By these means, we make a study of certain general classes of meromorphic and entire functions that include many of the classically studied classes as special cases. Let $\lambda(r)$ be a positive, continuous, increasing, and unbounded function defined for all positive r. We say that the meromorphic function f is of finite λ-type to mean that there exist positive constants A and B with $T(r, f) \leqq A\lambda(Br)$ for $r > 0$, where T is the Nevanlinna characteristic. An entire function f will be of finite λ-type if and only if there exist positive constants A and B such that

$$|f(z)| \leqq \exp(A\lambda(B|z|)) \quad \text{for all complex } z.$$

If we choose $\lambda(r) = r^\rho$, then the functions of finite λ-type are precisely the functions of growth not exceeding order ρ, finite exponential type. We obtain here complete answers to certain basic questions about functions of *finite λ-type*. For example, we characterize, in theorem 5.2, the zero sets of entire functions of finite λ-type. This generalizes the well-known theorem of Lindelöf that corresponds to the classical case $\lambda(r) = r^\rho$. We obtain, in theorem 5.3, a corresponding result for meromorphic functions. Then, in theorem 5.4, we give necessary and sufficient conditions on λ that each meromorphic function of finite λ-type be the quotient of two entire function of finite λ-type.

Further, we obtain, in theorem 5.7, a "generalized Hadamard product" for entire functions of finite λ-type. It serves many of the same purposes as the Hadamard canonical product, and is considerably more general. In particular, if λ satisfies some additional conditions, and if f is an entire function of finite λ-type, then there will be an unbounded set $\mathcal{R}$ of positive numbers R, and corresponding entire functions f_R of finite λ-type, such that the zeros of f_R are the zeros of f in the disc $\{z : |z| \leqq R\}$ and such that $f_R \to f$ not only uniformly on compact sets, but also in a way consistent with λ. We call the sequence $\{f_R\}$ the generalized Hadamard product. This result has been used in an essential way by the second author [6] in proving that spectral synthesis holds for mean-periodic functions in certain general spaces of entire functions.

The body of the paper is divided into five parts, the last two of which contain the main results. The first three sections are concerned with various elementary, although sometimes complicated, results on sequences of complex numbers. The first section discusses the distribution of these sequences. The " Fourier coefficients " associated with a sequence are defined in the second section and several technical propositions involving these coefficients are proved there. The third section is concerned with the property of regularity of the function λ, which is closely connected with the algebraic structure of the field of meromorphic functions of finite λ-type. The fourth section contains the generalizations of the results of [5]. Finally, in the fifth section, the results about the distribution of zeros, and the generalized Hadamard product are proved.

We urge that on a first reading, the reader read § 4 first and then § 5, referring to § 1, § 2, § 3 for the appropriate definitions and statements of necessary preliminary results. After this, the complex sequence theory of the first three chapters will seem much more natural.

We shall use, for a function $\varphi(r)$, the notation $O(\varphi(r))$ to denote a function that is bounded in modulus by $A\varphi(r)$ for some constant A, and the notation $O(\varphi(O(r)))$ to denote a function that is bounded in modulus by $A\varphi(Br)$ for some constants A and B.

It seems clear that through much of this paper, the assertions about entire functions can be replaced by corresponding assertions about subharmonic functions, and the assertions about meromorphic functions can be replaced by corresponding assertions about the differences of subharmonic functions, without requiring any real change in the proof. It is by now a standard procedure to replace the logarithm of the modulus of an entire function by a general subharmonic function, replacing the zeros of the entire function by the masses that occur in the Riesz decomposition of the subharmonic function. For numerous reasons, though, we have preferred to keep this paper in the context of entire and meromorphic functions.

1. An analysis of sequences of complex numbers.

We study here the distribution of sequences $Z = \{ z_n \}$, $n = 1, 2, 3, \ldots$, with multiplicity taken into account, of non-zero complex numbers z_n, such that $z_n \to \infty$ as $n \to \infty$. Such sequences Z are studied in relation to so-called growth functions λ. We denote by A and B generic positive constants. The actual constants so represented may vary from one occurrence to the next. In many of the results, there is an implicit uniformity in the dependence of the constants in the conclusion on the constants in the hypotheses. For a more detailed explanation of this uniformity, we refer the reader to the remak following proposition 1.11.

Let $Z = \{ Z_n \}$ be a sequence of non-zero complex numbers such that $\lim z_n = \infty$ as $n \to \infty$.

1.1. Definition. — The counting function of Z is the function

$$n(r, Z) = \sum_{|z_n| \leq r} 1.$$

1.2. Definition. — We define

$$N(r, Z) = \int_0^r \frac{n(t, Z)}{t} dt.$$

1.3. Proposition. — We have

$$N(r, Z) = \sum_{|z_n| \leq r} \log \frac{r}{|z_n|}.$$

Proof. — Note that

$$\sum_{|z_n| \leq r} \log \frac{r}{|z_n|} = \int_0^r \log\left(\frac{r}{t}\right) d[n(t, Z)].$$

The proposition follows from an integration by parts.

1.4. Proposition. — We have

$$n(r, Z) = r \frac{d}{dr} N(r, Z).$$

Proof. — Trivial.

1.5. — Definition. — We define, for $k = 1, 2, 3, \ldots$ and $r \geq 0$,

$$S(r; k : Z) = \frac{1}{k} \sum_{|z_n| \leq r} \left(\frac{1}{z_n}\right)^k.$$

1.6. Definition. — We define, for $k = 1, 2, 3, \ldots$ and $r_1, r_2 \geq 0$,

$$S(r_1, r_2; k : Z) = S(r_2; k : Z) - S(r_1; k : Z).$$

When no confusion will result, we will drop the Z from the above notation, and write $n(r)$, $S(r; k)$, etc.

1.7. Definition. — *A growth function* $\lambda(r)$ is a function, defined for $0 < r < \infty$, that is positive, non-decreasing, continuous, and unbounded.

Throughout this paper, λ will always denote a growth function.

1.8. Definition. — We say that the sequence Z has *finite λ-density* to mean that there exist constants A, B such that for all $r > 0$,

$$N(r, Z) \leq A\lambda(Br).$$

1.9. Proposition. — If Z has finite λ-density, then there are constants A, B such that

$$n(r, Z) \leq A\lambda(Br).$$

Proof. — We have

$$n(r, Z)\log 2 \leq \int_r^{2r} \frac{n(t, Z)}{t}\,dt \leq B(2r, Z).$$

1.10. Definition. — We say that the sequence Z is *λ-balanced* to mean that there exist constants A, B such that

$$(1.10.1) \qquad |S(r_1, r_2;\ k : Z)| \leq \frac{A\lambda(Br_1)}{r_1^k} + \frac{A\lambda(Br_2)}{r_2^k}$$

for all $r_1, r_2 > 0$ and $k = 1, 2, 3, \ldots$. We say that Z is *strongly λ-balanced* to mean that

$$(1.10.2) \qquad |S(r_1, r_2;\ k : Z)| \leq \frac{A\lambda(Br_1)}{kr_1^k} + \frac{A\lambda(Br_2)}{kr_2^k}$$

for all $r_1, r_2 > 0$ and $k = 1, 2, 3, \ldots$.

1.11. Proposition. — If Z has *finite λ-density* and is *λ-balanced,* then Z is *strongly λ-balanced*

Remark. — Using this result for illustrative purposes, we make explicit here the uniformity that we leave implicit in the statements of similar results. The assertion is that if Z has finite λ-density, with implied constants A, B, and is λ-balanced with implied constants A', B', then Z is strongly λ-balanced with implied constants A'', B'' that depend only on A, B, A', B' and not on Z or λ.

Proof of 1.11. — We observe first that, if $r > 0$, and if we let $r' = rk^{1/k}$ then

$$(1.11.1) \qquad |S(r, r';\ k)| \leq \frac{3n(r')}{kr^k}.$$

To prove this, we note that

$$S(r, r';\ k)| \leq \frac{1}{k}\int_r^{r'} \frac{1}{t^k}\,dn(t),$$

from which (1.11.1) follows after an integration by parts. Now, for $r_1, r_2 > 0$, let $r'_1 = r_1 k^{1/k}$ and $r'_2 = r_2 k^{1/k}$. Then

$$|S(r_1, r_2;\ k)| \leq |S(r'_1, r'_2;\ k)| + |S(r_1, r'_1;\ k)| + |S(r_2, r'_2;\ k)|.$$

On combining this inequality with (1.11.1), (1.9), and the fact that $k^{1/k} \leqq 2$, we have

$$| S(r_1, r_2; k) | \leqq | S(r'_1, r'_2; k) | + \frac{1}{kr_1^k} A\lambda(Br_1) + \frac{1}{kr_2^k} A\lambda(Br_2).$$

But, by hypothesis,

$$| S(r'_1, r'_2; k) \leqq \frac{1}{kr_1^k} A\lambda(Br_1) + \frac{1}{kr_2^k} A\lambda(Br_2)$$

for $k = 1, 2, 3, \ldots$.

1.12. Definition. — We say that the sequence Z is *λ-poised* to mean that there exists a sequence α of complex numbers $\alpha = \{ \alpha_k \}$, $k = 1, 2, 3, \ldots$ such that, for some constants A, B, we have, for $k = 1, 2, 3, \ldots$ and $r > 0$,

$$(1.12.1) \qquad | \alpha_k + S(r; k : Z) | \leqq \frac{A\lambda(Br)}{r^k}.$$

If the following stronger inequality

$$(1.12.2) \qquad | \alpha_k + S(r; k : Z) | \leqq \frac{A\lambda(Br)}{kr^k}$$

holds, we say that Z is *strongly λ-poised.*

1.13. Proposition. — If Z has *finite λ-density* and is *λ-poised*, then *Z is strongly λ-poised.*

Proof. — The proof is quite analogous to the proof of 1.11, based on the substitution $r' = rk^{1/k}$. We omit the details.

1.14. Proposition. — A sequence Z is *λ-balanced if and only if it is λ-poised,* and is *strongly λ-balanced if and only if it is strongly λ-poised.*

Proof. — We prove only the second assertion, since the proof of the first assertion is virtually the same. If it is first supposed that Z is strongly λ-poised, where $\{ \alpha_k \}$ is the relevant sequence, then we have

$$\begin{aligned} | S(r_1, r_2; k) | &= | S(r_2; k) + \alpha_k - \alpha_k - S(r_1; k) | \\ &\leqq | \alpha_k + S(r; k) | + | \alpha_k + S(r_2; k) | \end{aligned}$$

so that Z is strongly λ-balanced.

Suppose now that Z is strongly λ-balanced, with A, B being the relevant constants. Let

$$p(\lambda) = \inf \left\{ p = 1, 2, 3, \ldots : \liminf_{r \to \infty} \frac{\lambda(r)}{r^p} = 0 \right\}.$$

Naturally, we let $p(\lambda) = \infty$ in case $\liminf \lambda(r)/r^p > 0$ as $r^p > \infty$ for each positive integer p. For $1 \leqq k < p(\lambda)$, we have $\inf r^{-k} \lambda(Br) > 0$

for $r > 0$. Thus, there exist positive numbers r_k such that

$$\frac{\lambda(Br_k)}{r_k^k} < 2\frac{\lambda(Br)}{r^k}$$

for $r > 0$ and $1 \leq k < p(\lambda)$. For k in this range, we define

$$\alpha_k = - S(r_k; k). \tag{1.14.1}$$

For those k, if there are any, for which $k \geq p(\lambda)$, we choose a sequence $0 < \rho_1 < \rho_2 < \ldots$ with $\rho_j \to \infty$ as $j \to \infty$, such that

$$\lim_{j\to\infty} \frac{\lambda(B\rho_j)}{\rho_j^{p(\lambda)}} = 0.$$

For values of k, then, such that $k \geq p(\lambda)$, we define

$$\alpha_k = \lim_{j\to\infty} - S(\rho_j; k). \tag{1.14.2}$$

To show that the limit exists, we prove that the sequence $\{S(\rho_j; k)\}$, $j = 1, 2, \ldots$, is a Cauchy sequence. Let

$$\Delta_{j,m} = S(\rho_j; k) - S(\rho_m; k) = S(\rho_m, \rho_j; k).$$

We have

$$|\Delta_{j,m}| \leq \frac{A\lambda(B\rho_m)}{k\rho_m^k} + \frac{A\lambda(B\rho_j)}{k\rho_j^k}.$$

Since $\rho^k \geq \rho^{p(\lambda)}$ for $\rho \geq 1$, it follows from the choice of the ρ_j that $\Delta_{j,m} \to 0$ as $j, m \to \infty$. We now claim that

$$|\alpha_k + S(r; k)| \leq \frac{3A\lambda(Br)}{kr^k}.$$

For, if $1 \leq k < p(\lambda)$, then

$$|\alpha_k + S(r; k)| = |S(r_k, r; k)| \leq \frac{A\lambda(Br_k)}{kr_k^k} + \frac{A\lambda(Br_k)}{kr^k} \leq \frac{3A\lambda(Br)}{kr^k}$$

If $k \geq p(\lambda)$, then

$$|\alpha_k + S(r; k)| = \lim_{j\to\infty} |S(r, \rho_j; k)| \leq \frac{A\lambda(Br)}{kr^k} + \limsup_{j\to\infty} \frac{A\lambda(B\rho_j)}{k\rho_j^k} = \frac{A\lambda(Br)}{kr^k}.$$

1.15. Definition. — We say that the sequence Z is *λ-admissible* to mean that Z has *finite λ-density* and is *λ-balanced.*

In view of propositions 1.11 and 1.13, the following result is immediate.

1.16. Proposition. — Suppose that Z has *finite λ-density.* Then the following are equivalent :

(i) Z is *λ-balanced;*

(ii) Z is *strongly* λ*-balanced;*

(iii) Z is λ*-poised;*

(iv) Z is *strongly* λ*-poised;*

(v) Z is λ*-admissible.*

In proposition 3.3, we give a simple characterization of λ-admissible sequences in the special case $\lambda(r) = r^\rho$.

We next consider the effect that deleting from Z those finitely many terms that lie in the disk $\{ z : |z| \leqq R \}$ has on $S(r_1, r_2; k)$.

1.17. Definition. — We define $Z(R)$, for $R > 0$, to be the sequence obtained by deleting from Z those terms of modulus not exceeding R. That is,

$$Z(R) = Z \cap \{ z : |z| > R \},$$

and we call $Z(R)$ the *R-remainder* of Z.

1.18. Definition. — Let $\mathcal{R}$ be a non-empty set of positive real numbers. The *collection of remainders* $\{ Z(R) : E \in \mathcal{R} \}$ *is called complete if* $\mathcal{R}$ *is unbounded.*

1.19. Theorem. — *Each strongly* λ*-balanced sequence* Z *has a strongly* λ*-balanced supersequence* Z' *such that* $n(r, Z') = O(n(r, Z))$ *and such that* Z' *has a complete set of remainders that are uniformly strongly* λ*-balanced. In the special case in which* $\liminf_{r \to \infty} r^{-\rho} \lambda(r) = \infty$ *for each positive number* ρ, *we may take* $Z' = Z$. *In the special case in which* $\log \lambda(e^x)$ *is convex, we may take* $Z' = Z$, *and we may take the collection of remainders to be* $\{ Z(R) = R \geqq R_0 \}$ *for some number* R_0.

Before proving theorem 1.19, we derive first some elementary facts about the behaviour of the functions $r^{-k}\lambda(r)$ when λ has a particularly nice form. In the following, we will denote by $u(x)$ the function defined for $-\infty < x < \infty$ by $u(x) = \log \lambda(e^x)$. We observe that u is a non-decreasing function.

1.20. Lemma. — *Suppose that* $u(x) = \log \lambda\, e^x)$ *is convex and that* σ *is a positive number. Then the function* $r^{-\sigma}\lambda(r)$ *decreases to its infimum as r increases, and increases thereafter. If for some positive* σ *we have* $\limsup_{r \to \infty} r^{-\sigma}\lambda(r) < \infty$, *then there exists a constant* M *such that* $\lambda(2r) \leqq M\lambda(r)$. *Further, there is a positive number* R_0 *such that for every* $R \geqq R_0$, *there exists a positive number* $\sigma = \sigma(R)$ *such that*

$$\frac{\lambda(R)}{R^\sigma} = \inf_{r > 0} \frac{\lambda(r)}{r^\sigma}.$$

Proof. — Since u is convex and increasing, we may write

$$u(x) = u(0) + \int_0^x h(t)\,dt,$$

where h is a non-negative and non-decreasing function. If $x = \log r$, then

$$\frac{\lambda(r)}{r^\sigma} = \exp\left\{u(0) + \int_0^x (h(t) - \sigma)\,dt\right\},$$

from which the first assertion follows immediately. To prove the second assertion, observe that if $\limsup \frac{\lambda(r)}{r^\sigma} < \infty$, then $h(t)$ must be bounded-say $h(t) \leqq C$. Then

$$\frac{\lambda(2r)}{\lambda(r)} = \exp\{u(x + \log 2) - u(x)\} \leqq \exp\{C \log 2\}.$$

To prove the last assertion, let $x_0 = \log R$ and $\sigma = h(x_0)$. Since h is non-negative, non-decreasing, and not identically zero, it follows that σ is positive if R is sufficiently large. Then

$$(u(x) - \sigma x) - (u(x_0) - \sigma x_0) = \int_{x_0}^x (h(t) - \sigma)\,dt \geqq 0.$$

Hence,

$$\frac{\lambda(R)}{R^\sigma} = \exp\{u(x_0) - \sigma x_0\} = \inf_x \exp\{u(x) - \sigma x = \inf_{r>0} \frac{\lambda(r)}{r^\sigma}.$$

Proof of theorem 1.19. — By hypothesis, there exist constants A and B such that

$$(1.19.1) \qquad |S(r_1, r_2; k : Z)| \leqq \frac{A\lambda(Br_1)}{kr_1^k} + \frac{A\lambda(Br_2)}{kr_2^k}.$$

Let R and σ be positive numbers for which

$$\frac{\lambda(BR)}{R^\sigma} = \inf_{r>0} \frac{\lambda(Br)}{r^\sigma}.$$

We claim that then

$$(1.19.2) \qquad |S(r_1, r_2; k : Z(R))| \leqq \frac{2A\lambda(Br_1)}{kr_1^k} + \frac{2A\lambda(Br_2)}{kr_2^k}.$$

If $r_1 \leqq r_2 \leqq R$, then $S(r_1, r_2; k : Z(R)) = 0$ and there is nothing to prove. If $R \leqq r_1 \leqq r_2$, then

$$S(r_1, r_2; k : Z(R)) = S(r_1, r_2; k : Z)$$

and so (1.19.2) follows from the inequality (1.19.1). If $r_1 \leqq R \leqq r_2$, then

$$| S(r_1, r_2; k : Z(R)) | = | S(R, r_2; k : Z) |,$$

which, by (1.19.1) does not exceed

$$\frac{A\lambda(Br_2)}{kr_2^k} + \frac{A\lambda(BR)}{kR^k}.$$

However, if $k \geqq \sigma$, then

$$\frac{\lambda(BR)}{R^k} = \frac{\lambda(BR)}{R^\sigma}\frac{1}{R^{k-\sigma}} \leqq \frac{\lambda(Br_1)}{r_1^\sigma}\frac{1}{r_1^{k-\sigma}} = \frac{\lambda(Br_1)}{r_1^k}.$$

On the other hand, if $k \leqq \sigma$, then

$$\frac{\lambda(BR)}{R^k} = \frac{\lambda(BR)}{R^\sigma}\frac{1}{R^{k-\sigma}} \leqq \frac{\lambda(Br_2)}{r_2^\sigma}\frac{1}{r_2^{k-\sigma}} = \frac{\lambda(Br_2)}{r_2^k}.$$

Thus,

$$\frac{\lambda(BR)}{R^k} \leqq \max\left(\frac{\lambda(Br_1)}{r_1^k}, \frac{\lambda(Br_2)}{r_2^k}\right),$$

and (1.19.2) follows.

Let us now consider the case in which, for each $\rho > 0$, $\lim r^{-\rho}\lambda(r) = \infty$ as $r \to \infty$. We define, for $\sigma > 0$,

$$R_\sigma = \sup\left\{R : \frac{\lambda(BR)}{R^\sigma} = \inf \frac{\lambda(Br)}{r^\sigma}\right\}.$$

We have $R_\sigma > 0$ since λ is continuous. If $\sigma' > \sigma$ and $R \leqq R_\sigma$, we have

$$\frac{\lambda(BR)}{R^{\sigma'}} = \frac{\lambda(BR)}{R^\sigma}\frac{1}{R^{\sigma'-\sigma}} \geqq \frac{\lambda(BR_\sigma)}{R_\sigma^\sigma}\frac{1}{R_\sigma^{\sigma'-\sigma}} = \frac{\lambda(BR_\sigma)}{R_\sigma^{\sigma'}}.$$

It follows that R_σ is a non-decreasing function of σ. Further if $R \geqq R_\sigma$, then $\lambda(R)/R^\sigma \geqq \lambda(R_\sigma)/R_\sigma^\sigma$, so that

$$\frac{\lambda(R)}{\lambda(R_\sigma)} \geqq \left(\frac{R}{R_\sigma}\right)^\sigma,$$

and it follows that R_σ is an unbounded function of σ. If we now let

$$\mathcal{R} = \left\{R > 0 : \frac{\lambda(BR)}{R^\sigma} = \inf_{r>0}\frac{\lambda(Br)}{r^\sigma} \text{ for some } \sigma > 0\right\},$$

it follows from (1.19.2) that $\{ Z(R) : R \in \mathcal{R} \}$ is a uniformly balanced, complete set of remainders of Z. The last assertion of the theorem follows from lemma 1.20 which states that $\mathcal{R} \supseteq \{ R : R \geqq R_0 \}$ for some positive number R_0.

We next consider the case in which

$$\liminf r^{-p}\lambda(r) = 0 \quad \text{as } r \to \infty \text{ for some } p > 0.$$

Let $p(\lambda)$ denote the smallest positive integer for which this holds. Let $\mathcal{R}$ be any unbounded set of positive real numbers such that

$$\lim_{\substack{R\to\infty \\ R\in\mathcal{R}}} \frac{\lambda(BR)}{R^{p(\lambda)}} = 0$$

and such that for all $R \in \mathcal{R}$,

$$\frac{\lambda(BR)}{R^{p(\lambda)}} = \inf\left\{\frac{\lambda(Br)}{r^{p(\lambda)}} = r \leq R\right\}.$$

Since $\liminf r^{-p(\lambda)}\lambda(Br) = 0$ as $r \to \infty$, there is at least one such set $\mathcal{R}$. The hypothesis that $\log \lambda(e^x)$ is convex implies that we may take $\mathcal{R} = \{R : R > 0\}$, since $r^{-p(\lambda)}\lambda(Br)$ must decrease to 0 at ∞. We construct Z' as follows. Let ω be a primitive $p(\lambda)$-th root of unity, say $\omega = \exp(2\pi i/p(\lambda))$. Let $\omega^{-j}Z$ denote the sequence $\{\omega^{-j} z_n\}$, $n = 1, 2, 3, \ldots$, and then let

$$Z' = \bigcup_{j=0}^{p(\lambda)-1} \omega^{-j} Z.$$

It is clear that $n(r, Z') = p(\lambda)\, n(r, Z)$. Further, we have

$$S(r_1, r_2; k : Z') = \left(\sum_{j=0}^{p(\lambda)-1} \omega^{kj}\right) S(r_1, r_2; k : Z).$$

Consequently, for $k = 1, 2, \ldots, p(\lambda) - 1$, we have that $S(r_1, r_2; k : Z) = 0$ since the sum in parentheses is 0 for such values of k. From the above equation, it also follows that

$$| S(r_1, r_2; k : Z') | \leq p(\lambda) \, | S(r_1, r_2; k : Z) |$$

for all positive integers k. To prove, then, that $\{ Z'(R) : R \in \mathcal{R} \}$ is a complete set of uniformly λ-balanced remainders of Z', it is sufficient to prove that (1.19.2) holds for $k \geq p(\lambda)$ and for every $R \in \mathcal{R}$. As we have seen above, (1.19.2) is trivial unless $r_1 \leq R \leq r_2$, in which case

$$| S(r_1, r_2; k : Z(R)) | = | S(R, r_2; k : Z) | \leq \frac{A\lambda(BR)}{kR^k} + \frac{A\lambda(Br_2)}{kr_2^k}.$$

However, in this case $k \geq p(\lambda)$ and $r_1 \leq R$, so that

$$\frac{\lambda(BR)}{R^k} = \frac{\lambda(BR)}{R^{p(\lambda)}} \frac{1}{R^{k-p(\lambda)}} \leq \frac{\lambda(Br_1)}{r_1^{p(\lambda)}} \frac{1}{r_1^{k-p(\lambda)}} = \frac{\lambda(Br_1)}{r_1^{p(\lambda)}}.$$

Hence (1.19.2) holds, and the proof is complete.

2. The Fourier coefficients associated with a sequence.

We now present the sequence of so-called Fourier coefficients associated with a sequence Z of complex numbers, and study its properties. We will use it in § 5 to construct an entire function f whose zero set coincides with Z, and to determine some properties of entire and meromorphic functions whose growth is restricted. The reason for calling them " Fourier coefficients " will become apparent on comparing their definition with lemma 4.2 of section 4.

2.1. Definition. — We define, for $k = 1, 2, 3, \ldots,$

$$S'(r;\, k : Z) = \frac{1}{k} \sum_{|z_n| \leq r} \left(\frac{\bar{z}_n}{r}\right)^k.$$

2.2. Proposition. — We have

$$|S'(r;\, k : Z)| \leq \frac{1}{k} N(er,\, Z).$$

Proof. — It is clear that $|S'(r;\, k : Z)| \leq n(r)/k$, and we also have

$$n(r) \leq \int_r^{er} \frac{n(t)}{t}\, dt \leq N(er).$$

2.3. Definition. — Let $\alpha = \{\alpha_k\}$, $k = 1, 2, 3, \ldots,$ be a sequence of complex numbers. The sequence $\{c_k(r;\, Z : \alpha)\}$, $k = 0, \pm 1, \pm 2, \ldots,$ defined by

(2.3.1) $$c_0(r;\, Z : \alpha) = c_0(r;\, Z) = N(r,\, Z),$$

(2.3.2) $$c_k(r;\, Z : \alpha) = \frac{r^k}{2}\{\alpha_k + S(r;\, k : Z)\} - \frac{1}{2} S'(r;\, k : Z) \qquad \text{for} \quad k = 1, 2, 3, \ldots,$$

(2.3.3) $$c_{-k}(r;\, Z : \alpha) = (c_k(r;\, Z;\, \alpha))^* \qquad \text{for} \quad k = 1, 2, 3, \ldots,$$

where * denotes complex conjugation, is said to be a sequence of Fourier coefficients associated with Z.

2.4. Definition. — A sequence $\{c_k(r;\, Z : \alpha)\}$ of Fourier coefficients associated with Z is called *λ-admissible* if there exist constants A, B such that

(2.4.1) $$|c_k(r : Z;\, \alpha)| \leq \frac{A\lambda(Br)}{|k| + 1} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

2.5. Proposition. — A sequence Z is *λ-admissible* if and only if there exists a *λ-admissible* sequence of Fourier coefficients associated with Z.

Proof. — Suppose that Z is λ-admissible. Then by 1.16, Z is strongly λ-poised. Let $\alpha = (\alpha_k)$, $k = 1, 2, 3, \ldots$, be the relevant constants, and form $\{c_k(r; Z : \alpha)\}$ from them by means of (2.3.1)-(2.3.3). Now (2.4.1) holds for $k = 0$ and some constants A, B since Z has finite λ-density. For $k = \pm 1, \pm 2, \pm 3, \ldots$, we have

$$|c_k(r; Z : \alpha)| \leq \frac{r^{|k|}}{2} |\alpha_k + S(r; k)| + \frac{1}{2} |S'(r; k)|.$$

Then an inequality of the form (2.4.1) holds by proposition 2.2 since Z has finite λ-density, and because Z is strongly λ-poised with respect to the constants $\{\alpha_k\}$.

On the other hand, suppose that (2.4.1) holds. Then

$$N(r) = c_0(r) \leq A\lambda(Br),$$

so that Z has finite λ-density. Moreover,

$$\left|\frac{r^k}{2}(\alpha_k + S(r; k))\right| = \left|c_k(r; Z : \alpha) + \frac{1}{2} S'(r; k)\right|$$
$$\leq \frac{A\lambda(Br)}{|k|+1} + \frac{N(er)}{2k} \leq \frac{2A\lambda(eBr)}{k},$$

so that Z is strongly λ-poised. By proposition 1.16, it follows that Z is λ-admissible.

2.6. Proposition. — Suppose that Z and $\alpha = \{\alpha_k\}$ are such that $|c_k(r; Z : \alpha)| \leq A\lambda(Br)$. Then $\{c_k(r; Z : \alpha)\}$ is *λ-admissible.* In particular, there exist constants A', B', depending only on A, B, such that

$$|c_k(r; Z : \alpha)| \leq \frac{A'\lambda(B'r)}{|k|+1}.$$

Proof. — For $k = 1, 2, \ldots$, we have

$$|c_k(r)| \leq \frac{r^k}{2} |\alpha_k + S(r; k)| + \frac{1}{2} |S'(r; k)| \tag{2.6.1}$$

and

$$\frac{r^k}{2} |\alpha_k + S(r; k)| \leq |c_k(r)| + \frac{1}{2} |S'(r; k)|. \tag{2.6.2}$$

Since $c_0(r) = N(r) \leq A\lambda(Br)$, Z has finite λ-density. Then by (2.2), $|S'(r; k)| \leq (1/k)\, O(\lambda(O(r)))$ uniformly for $k = 1, 2, 3, \ldots$, by which we mean that there are constants A'', B'' for which $|S'(r, k)| \leq (1/k) A'' \lambda(B''r)$. From our hypothesis and (2.6.2), it then follows that

$$r^k |\alpha_k + S(r; k)| = O(\lambda(O(r))) \quad \text{uniformly for } k > 0.$$

Then by proposition 1.13, we have that

$$r^k |\alpha_k + S(r; k)| \leq \frac{1}{k} O(\lambda(O(r))) \quad \text{uniformly for } k = 1, 2, 3, \ldots.$$

Then, using (2.6.1), we have

$$c_k(r) | \leq \frac{1}{k} O(\lambda(O(r))) \quad \text{uniformly for } k = 1, 2, 3, \ldots.$$

Since $c_{-k}(r) = (c_k(r))^*$, and since Z has finite upper λ-density, the proposition follows immediately.

2.7. Definition. — The quadratic semi-norm of a sequence $\{ c_k(r; Z : \alpha) \}$ of Fourier coefficients associated with Z is given by

$$E_2(r; Z : \alpha) = \left\{ \sum_{k=-\infty}^{\infty} |c_k(r; Z : \alpha)|^2 \right\}^{1/2}.$$

2.8. Proposition. — The Fourier coefficients $\{ c_k(r; Z : \alpha) \}$ are λ-admissible if and only if $E_2(r; Z : \alpha) \leq A\lambda(Br)$ for some constants A, B.

Proof. — First, if

$$|c_k(r; Z : \alpha)| \leq \frac{A_1 \lambda(B_1 r)}{|k| + 1},$$

then $E_2(r; Z : \alpha) \leq A\lambda(Br)$, where $B = B_1$ and

$$A = A_1 \left\{ \sum_{k=-\infty}^{\infty} \left(\frac{1}{|k| + 1} \right)^2 \right\}^{1/2}.$$

On the other hand, suppose there are constants A, B for which $E_2(r; Z : \alpha) \leq A\lambda(Br)$. Then it is clear that $|c_k(r; Z : \alpha)| \leq A\lambda(Br)$ so that by proposition 2.6, $\{ c_k(r; Z : \alpha) \}$ is λ-admissible.

The next result will be used in § 5 to help develop the so-called generalized Hadamard product.

2.9. Theorem. — *Suppose that Z is λ-admissible. Then there exist a λ-admissible supersequence $Z' \supseteq Z$ and a complete, uniformly λ-balanced set of remainders of Z', $\{ Z'(R) = R \in \mathcal{R} \}$, and a family $\{ \alpha(R) \}$, $\alpha(R) = \{ \alpha_k(R) \}$, $k = 1, 2, 3, \ldots$, $R \in \mathcal{R}$, of sequences of complex numbers such that*

$$(2.9.1) \quad |c_k(r; Z'(R) : \alpha(R))| \leq \frac{A\lambda(Br)}{|k| + 1} \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

for some constants A, B, and further that

$$(2.9.2) \qquad \lim_{\substack{R \to \infty \\ R \in \mathcal{R}}} c_k(r; Z'(R) : \alpha(R)) = 0$$

for all $r > 0$ *and* $k = 0, \pm 1, \pm 2, \ldots$ *If* $\liminf r^{-k}\lambda(r) = \infty$ *for all* $k > 0$, *as* $r \to \infty$, *then we may take* $Z' = Z$ *and* $\mathcal{R} = \{R : R \geq R_0\}$ *for some positive number* R_0.

Proof. — Let Z', $Z'(R)$, and $\mathcal{R}$ be constructed as in the proof of theorem 1.19. Then for suitable constants A, B, we have that

$$n(r, Z') \leq A\lambda(Br), \qquad N(r, Z') \leq A\lambda(Br),$$
$$|S(r_1, r_2; k : Z'(R))| \leq \frac{A\lambda(Br_1)}{kr_1^k} + \frac{A\lambda(Br_2)}{kr_2^k}.$$

We now let $\alpha(R) = \{\alpha_k(R)\}$ be defined by equations (1.14.1) and (1.14.2) that occur in the proof of proposition 1.14. Then, as was proved there, we have that

$$|\alpha_k(R) + S(r; k' : Z'(R))| \leq \frac{3A\lambda(Br)}{kr^k}.$$

This inequality, together with 2.6.1, gives

$$|c_k(r; Z'(R) : \alpha(R))| \leq \frac{3A\lambda(Br)}{2k} + \frac{1}{2}|S'(r; k : Z'(R))|.$$

However, we have

$$|S'(r; k : Z'(R))| \leq \frac{1}{k}N(r; Z'(R)) \leq \frac{1}{k}N(r; Z') \leq \frac{A\lambda(Br)}{k},$$

to that (2.9.1) holds for some (possibly different) constants A, B. To prove 2.9.2, it is enough to show that $\alpha_k(R) \to 0$ as $R \to \infty$ through $\mathcal{R}$, since it is obvious that

$$\lim_{\substack{R \to \infty \\ R \in \mathcal{R}}} S(r; k : Z'(R)) = \lim_{\substack{R \to \infty \\ R \in \mathcal{R}}} S'(r; k : Z'(R)) = 0.$$

From the equations defining $\alpha_k(R)$ it is clear that $\alpha_k(R) \to 0$ as $R \to \infty$ except possibly in the case where $p(\lambda) < \infty$ and $k \geq p(\lambda)$. However, in this case, we have from (2.9.1) that, for all $r < R$,

$$\left|\frac{1}{2}\alpha_k(R)\, r^k\right| \leq \frac{A\lambda(Br)}{|k| + 1},$$

and consequently that

$$|\alpha_k(R)| \leq \frac{2A\lambda(BR)}{R^k}.$$

However, the family $\mathcal{R}$ was constructed, in the proof of theorem 1.19, in such a way that for $k \geq p(\lambda)$, we have

$$\lim_{\substack{R \to \infty \\ R \in \mathcal{R}}} \frac{\lambda(BR)}{R^k} = 0,$$

so that

$$\lim_{\substack{R \to \infty \\ R \in \mathcal{R}}} \alpha_k(R) = 0.$$

The final assertions of the theorem follow from theorem 1.19, where the family $\mathcal{R}$ was constructed.

2.10. Corollary. — *If the sequences* $\{c_k(r; Z'(R) : \alpha(R))\}$ *of Fourier coefficients are as described in theorem* 2.9, *then for each* $r > 0$,

$$\lim_{\substack{R \to \infty \\ R \in \mathcal{R}}} E_2(r; Z'(R) : \alpha(R)) = 0.$$

Proof. — For each $r > 0$ and $n = 1, 2, 3, \ldots$, we have

$$\limsup_{\substack{R \to \infty \\ R \in \mathcal{R}}} [E_2(r; Z'(R) : \alpha(R))]^2 \leq \limsup_{\substack{R \to \infty \\ R \in \mathcal{R}}} \sum_{|k| > n} |c_k(r; Z'(R) : \alpha(R))|^2$$
$$\leq A^2(\lambda(Br))^2 \sum_{|k| > n} \left(\frac{1}{|k| + 1}\right)^2.$$

The result follows on letting n tend to ∞.

3. Sequences that are λ-balanceable.

In this section, were are concerned with the process of enlarging a sequence Z so that it becomes λ-balanced. Growth functions λ for which this is always possible are called regular and give rise to associated fields of meromorphic functions with special properties. For example, see theorem 5.4. The principal results of this section are propositions 3.5 and 3.6 which give simple conditions that λ be regular. In addition, we give in proposition 3.3 a simple characterization of λ-admissible sequences for the case $\lambda(r) = r^\rho$.

3.1. Definition. — The sequence Z is λ-*balanceable* if there exists a λ-admissible supersequence Z' of Z.

3.2. Definition. — The growth function λ is *regular* if every sequence Z that has finite λ-density is λ-balanceable.

3.3. Proposition. — Suppose that $\lambda(r) = r^\rho$ where $\rho > 0$. Then

(i) the sequence Z is of finite λ-density if and only if $\limsup r^{-\rho} n(r, Z) < \infty$ as $r \to \infty$;

(ii) if ρ is not an integer, then every sequence of finite λ-density is λ-admissible;

(iii) if ρ is an integer, then Z is λ-admissible if and only if Z is of finite λ-density and $S(r; \rho : Z)$ is a bounded function of r;

(iv) the function $\lambda(r) = r^\rho$ is regular.

Proof. — To prove (i), we have that $n(r) = O(r^\rho)$ whenever Z has finite λ-density. On the other hand, if $\limsup r^{-\rho} n(r) < \infty$, then $n(r) \leq A r^\rho$ for some positive constant A so that

$$N(r) = \int_0^r t^{-1} n(t)\, dt \leq A \rho^{-1} r^\rho.$$

To prove (ii), suppose that $N(t) \leq A t^\rho$. Then so long as $k \neq \rho$, we have

$$\int_{r_1}^{r_2} \frac{1}{t^k}\, dn(t) \leq \left(A + \frac{A}{|\rho - k|}\right)\left(\frac{\lambda(r_1)}{r_1^k} + \frac{\lambda(r_2)}{r_2^k}\right). \tag{3.3.1}$$

For, on integrating by parts, we have that the integral is equal to

$$\frac{n(r_2)}{r_2^k} - \frac{n(r_1)}{r_1^k} + k\int_{r_1}^{r_2} \frac{n(t)}{t^{k+1}}\, dt.$$

But

$$\frac{n(r_2)}{r_2^k} \leq A \frac{r_2^\rho}{r_2^k} = \frac{A\lambda(r_2)}{r_2^k},$$

and similarly

$$\frac{n(r_1)}{r_1^k} \leq \frac{A\lambda(r_1)}{r_1^k}.$$

Moreover,

$$\int_{r_1}^{r_2} \frac{n(t)}{t^{k+1}}\, dt \leq A\int_{r_1}^{r_2} \frac{t^\rho}{t^{k+1}}\, dt \leq \frac{A}{|\rho - k|}\left(\frac{r_2^\rho}{r_2^k} + \frac{r_1^\rho}{r_1^k}\right),$$

and the inequality (3.3.1) follows. Hence, so long as ρ is not an integer, every sequence Z of finite r^ρ-density is r^ρ-balanced.

To prove (iii), suppose that Z has finite r^ρ-density and that ρ is an integer. Then by (3.3.1), we see that all the conditions that Z be λ-balanced are satisfied except for $k = \rho$. For this case, the condition that $S(r_1, r_2; \rho)$ be bounded by $r_1^{-\rho} A\lambda(Br_1) + r_2^{-\rho} A\lambda(Br_2)$ for some A, B is precisely the condition that $S(r; \rho)$ be bounded, as is quite easy to see.

To prove (iv), we observe first that if ρ is not an integer then $\lambda(r) = r^\rho$ is trivially regular by (ii). If ρ is an integer, and Z has finite r^ρ-density, let Z' be the sequence obtained by adding to Z all numbers of the form $\omega^{-1} Z_n$, where $\omega^\rho = -1$. Then Z' has finite r^ρ-density and $S(r; \rho : Z') = 0$ for all $r > 0$. Hence by (iii), Z' is r^ρ-admissible, and it follows that $\lambda(r) = r^\rho$ is regular.

The next two results give simple conditions, both satisfied in case $\lambda(r) = r^\rho$, that imply that λ is regular. We do not know whether there exists a growth function λ that is not regular.

3.4. Definition. — We say that the growth function λ is *slowly increasing* to mean that $\lambda(2r) \leqq M\lambda(r)$ for some constant M.

If λ is slowly increasing, it is easy to show that for some positive number ρ, $\lambda(r) = O(r^\rho)$ as $r \to \infty$.

3.5. Proposition. — If λ is *slowly increasing*, then λ is *regular*.

3.6. Proposition. — If $\log \lambda(e^x)$ is *convex*, then λ is *regular*.

The proofs of these results use the next lemma.

3.7. Lemma. — *The growth function λ is slowly increasing if and only if there exist an integer p_0 and constants A, B such that*

$$(3.7.1) \qquad \int_r^\infty \frac{\lambda(t)}{t^{p+1}}\,dt \leqq \frac{A\lambda(Br)}{pr^p} \qquad \textit{for} \quad p \geqq p_0.$$

If $\lambda(r) = r^\rho$, then we may choose $p_0 = [\rho] + 1$.

Proof. — Suppose first that (3.7.1) holds. We may clearly suppose that $B \geqq 1$. Then

$$\frac{A\lambda(Br)}{pr^p} \geqq \int_r^\infty \frac{\lambda(t)}{t^{p+1}}\,dt \geqq \int_{2Br}^\infty \frac{\lambda(t)}{t^{p+1}}\,dt \geqq \frac{\lambda(2Br)}{p(2B)^p r^p}$$

whenever $p \geqq p_0$. Taking $p = p_0$, we have $\lambda(2Br) \leqq M\lambda(Br)$, where

$$M = A(2B)^{p_0},$$

so that $\lambda(r)$ is slowly increasing. Suppose next that $\lambda(r)$ is slowly increasing, say $\lambda(2r) \leqq M\lambda(r)$. Then

$$\int_r^\infty \frac{\lambda(t)}{t^{p+1}}\,dt = \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \frac{\lambda(t)}{t^{p+1}}\,dt \leqq \sum_{k=0}^\infty \frac{\lambda(2^{k+1}r)}{pr^p(2^k)^p} \leqq M\frac{\lambda(r)}{pr^p}\sum_{k=0}^\infty \left(\frac{M}{2p}\right)^k.$$

Hence, if p_0 is taken so large that $2^{p_0} > M$, we have an inequality of the form (3.7.1). In case $\lambda(r) = r^\rho$, we have $M = 2^\rho$, and the final assertion follows.

Proof of proposition 3.5. — Let λ be slowly increasing, and let Z be a sequence of finite λ-density. Choose p_0 as in the last lemma so that, for $p \geqq p_0$,

$$\int_r^\infty \frac{\lambda(t)}{t^{p+1}}\,dt \leqq \frac{A\lambda(Br)}{pr^p}.$$

Define

$$Z' = \bigcup_{k=0}^{p_0} \omega^{-k} Z,$$

where $n_0 = p_0 + 1$, $\omega = \exp(2\pi i/n_0)$, and $w^{-k}Z = \{\omega^{-k} z_n\}$, $n = 1, 2, 3, \ldots$. Then we have $S(r_1, r_2; k : Z') = 0$ for $k = 1, 2, \ldots, p_0$, since

$$1 + \omega^k + \omega^{2k} + \ldots + \omega^{p_0 k} = 0$$

so long as $\omega^k \neq 1$, and this is true for $k = 1, 2, \ldots, p_0$. Hence, to prove that Z' is λ-balanced, we need consider only $k > p_0$. For such k, with $r < r'$, we have

$$| S(r, r'; k : Z) | = \frac{1}{k} \left| \sum_{r < |z_n| < r'} \left(\frac{1}{z_n}\right)^k \right| \leq \frac{1}{k} \int_r^{r'} \frac{1}{t^k} dn(t, Z').$$

On integrating by parts, we have

$$\left| \int_r^{r'} \frac{1}{t^k} dn(t, Z') \right| \leq \frac{n(r', Z')}{(r')^k} + \frac{n(r, Z')}{r^k} + k \int_r^{r'} \frac{n(t, Z')}{t^{k+1}} dt.$$

Since Z is of finite λ-density and $n(r, Z') = (p_0 + 1)\, n(r, Z)$, we have $n(r, Z') \leq A_1 \lambda(B_1 r)$ for some constants A_1, B_1 by proposition 1.9.
Since λ is slowly increasing, we have $\lambda(B_1 r) \leq M\lambda(r)$ for some positive constant $A_2 > 0$. To complete the proof of the theorem, we have only to prove that

$$\int_r^{r'} \frac{n(t, Z')}{t^{k+1}} dt \leq \frac{A' \lambda(B' r)}{kr^k}$$

for some constants A', B'. However,

$$\int_r^{r'} \frac{n(t, Z')}{t^{k+1}} dt \leq A_2 \int_r^{\infty} \frac{\lambda(t)}{t^{k+1}} dt \leq \frac{AA_2 \lambda(Br)}{kr^k}$$

since $k > p_0$.

Proof of proposition 3.6. — It is no loss of generality to suppose that $r^{-p}\lambda(r) \to \infty$ as $r \to \infty$ for each $p > 0$, since otherwise λ is slowly increasing by lemma 1.20, and then proposition 3.5 applies. Now for $p = 1, 2, 3, \ldots$, let R_p be the largest number such that

$$\frac{\lambda(R_p)}{R_p^p} = \inf_{r > 0} \frac{\lambda(r)}{r^p},$$

and let $R_0 = 0$. Then, as was shown in the proof of theorem 1.19 with the numbers R_σ, we have that $R_0 \leq R_1 \leq R_2 \leq \ldots$, and that $R_p \to \infty$ as $p \to \infty$. Further, by lemma 1.20, $r^{-p}\lambda(r)$ decreases for $r \leq R_p$ and increases for $r \geq R_p$. We also have the inequality

$$(3.6.1) \qquad 2^p \lambda(r) \leq 2\lambda(2r) \quad \text{if} \quad r \geq R_{p-1},$$

since by the above remark,

$$\frac{\lambda(r)}{r^{p-1}} \leqq \frac{\lambda(2r)}{(2r)^{p-1}}.$$

Now let Z be of finite λ-density. For convenience of notation, we suppose that $N(r) \leqq \lambda(r)$ and $n(r) \leqq \lambda(r)$, since we could otherwise replace the function $\lambda(r)$ by the function $A\lambda(Br)$ for suitable constants A, B. We then claim that

$$\int_r^{r'} \frac{1}{t^k}\, dn(t) \leqq \frac{4\lambda(r)}{r^k} \tag{3.6.2}$$

if $k \geqq 2^p$ and $r \leqq r' \leqq R_p$.

To prove (3.6.2), we first integrate by parts, replacing the integral by

$$\frac{n(r')}{(r')^k} - \frac{n(r)}{r^k} + k\int_r^{r'} \frac{n(t)}{t^{k+1}}\, dt.$$

Now

$$\frac{n(r')}{(r')^k} \leqq \frac{\lambda(r')}{(r')^k} \leqq \frac{\lambda(r)}{r^k}$$

since $r \leqq r'$ and $r^{-k}\lambda(r)$ is decreasing for $r \leqq R_p \leqq R_k$. Also,

$$\int_r^{r'} \frac{n(t)}{t^{k+1}}\, dt \leqq \int_r^{r'} \frac{\lambda(t)}{t^p} \frac{1}{t^{k+1-p}}\, dt \leqq \frac{\lambda(r)}{r^p} \frac{1}{k-p} \frac{1}{r^{k-p}},$$

since $t^{-p}\lambda(t) \leqq r^{-p}\lambda(r)$ for $r \leqq t \leqq r' \leqq R_p$. Thus

$$\int_r^{r'} \frac{1}{t^k}\, dn(t) \leqq \frac{\lambda(r)}{r^k} + \frac{\lambda(r)}{r^k} + \frac{k}{k-p} \frac{\lambda(r)}{r^k}.$$

We have $k/(k-p) \leqq 2$ since $k \geqq 2^p$, and (3.6.2) follows.

We now define Z' as follows. For each $z_n \in Z$ with $R_{p-1} < |z_n| \leqq R_p$, we introduce into Z' the numbers

$$z_n, \quad \frac{1}{\omega} z_n, \quad \ldots, \quad \frac{1}{\omega^{m-1}} z_n$$

where $m = m(p) = 2^p$ and $\omega = \omega(m) = \exp(2\pi i/m)$. We make the following assertions :

$$n(r, Z') - n(R_{p-1}, Z') = 2^p(n(r) - n(R_{p-1})) \quad \text{if} \quad R_{p-1} \leqq r \leqq R_p, \tag{3.6.3}$$

$$n(r, Z') \leqq 2^p n(r) \quad \text{if} \quad r \leqq R_p, \tag{3.6.4}$$

(3.6.5) $$N(r, Z') \leqq 2^p \lambda(r) \quad \text{if} \quad r \leqq R_p,$$

(3.6.6) $$\int_r^{r'} \frac{1}{t^k} dn(t, Z') \leqq k \int_r^{r'} \frac{1}{t^k} dn(t) \quad \text{if} \quad k \geqq 2^p \quad \text{and} \quad r \leqq r' \leqq R_p.$$

(3.6.7) $$S(r, r'; k : Z') = 0 \quad \text{if} \quad r, r' \geqq R_p \text{ and } k \text{ is not a multiple of } 2^p.$$

The assertions (3.6.3) and (3.6.4) follow immediately from the definition of Z', while (3.6.5) follows from (3.6.4), and (3.6.6) follows easily from (3.6.3). To prove (3.6.7), it is enough to prove that $S(r, r'; k : Z') = 0$ if $R_{j-1} \leqq r \leqq r' \leqq R_j$, $j \geqq p$, and k is not a multiple of 2^p. But, in this case, we have

$$S(r, r'; k : Z') = \gamma S(r, r'; k : Z),$$

where

$$\gamma = 1 + \omega^k + \omega^{2k} + \ldots + \omega^{(m-1)k},$$

where $m = m(j) = 2^j$ and $\omega = \omega(m) = \exp(2\pi i/m)$. Since k is not a multiple of 2^p, k is therefore certainly not a multiple of 2^j, so that $\omega^k \neq 1$. We then have

$$\gamma = \frac{1 - \omega^{km}}{1 - \omega^k} = 0,$$

and our assertion is proved.

We now prove that Z' is λ-admissible. To see that Z' has finite λ-density, let $r > 0$ and let p be such that $R_{p-1} \leqq r \leqq R_p$. Then by (3.6.5) and (3.6.1), we have that $N(r, Z') \leqq 2^p \lambda(r) \leqq 2\lambda(2r)$. To see that Z' is λ-balanced, let k be a positive integer and suppose that $0 < r \leqq r'$. Write k in the form $2^p q$, where q is odd. Then, by (3.6.7), $S(r, r'; k : Z') = 0$ if $R_p \leqq r < r'$. Suppose $r \leqq R_p$. Then $S(r, r'; k : Z') = S(r, r''; k : Z')$, where $r'' = \min(r', R_p)$, by (3.6.7). However,

$$|S(r, r''; k : Z')| \leqq \frac{1}{k} \int_r^{r''} \frac{1}{t^k} dn(t, Z').$$

By (3.6.6), this last term does not exceed

$$\int_r^{r''} \frac{1}{t^k} dn(t),$$

and this, in turn, does not exceed $4r^{-k}\lambda(r)$, by (3.6.2). Consequently, we always have $|S(r, r'; k : Z)| \leqq 4r^{-k}\lambda(r)$, so that Z' is λ balanced, and the proof is complete.

4. The Fourier coefficients associated with a meromorphic function.

4.1. In this section, we associate a Fourier series with a meromorphic function, and use it to study properties of the function. As we mentioned in the introduction, the results of this section are generalized versions of the results of the earlier paper [5], and the proofs are essentially the same. Our notation follows the notation of [5] and the usual notation from the theory of meromorphic and entire functions. We first recall the results from the theory of meromorphic functions that will be needed.

For a non-constant meromorphic function f, we denote by $Z(f)$ [respectively $W(f)$] the sequence of zeros (respectively poles) of f, each occurring the number of times indicated by its multiplicity. We suppose throughout this paper that $f(0) \neq 0, \infty$. It requires only minor modifications to treat the case where $f(0) = 0$ or $f(0) = \infty$. By $n(r, f)$ we denote the number of poles of f in the disc $\{z : |z| \leq r\}$. By $N(r, f)$ we denote the function,

$$N(r, f) = \int_0^r \frac{n(t, f)}{t} dt,$$

and by $m(r, f)$ the function

$$m(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta,$$

where $\log^+ x = \max(\log x, 0)$. We have, of course, that $n(r, f) = n(r, W(f))$ and $N(r, f) = N(r, W(f))$. The Nevanlinna characteristic, that measures the growth of f, is the function

$$T(r, f) = m(r, f) + N(r, f).$$

Three fundamental facts about $T(r, f)$ are that

$$T(r, f) = T\left(r, \frac{1}{f}\right) + \log |f(0)|, \tag{4.1.2}$$

$$T(r, fg) \leq T(r, f) + T(r, g), \tag{4.1.3}$$

$$T(r, f+g) \leq T(r, f) + T(r, g) + \log 2. \tag{4.1.4}$$

Proofs of these facts may be found in [2], pages 4 and 5. An easy consequence of (4.1.2) is that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \log |f(re^{i\theta})| \right| d\theta \leq 2T(r, f) + \log |f(0)|. \tag{4.1.5}$$

This follows from (4.1.2) by observing that the first term is equal to $m(r, f) + m(r, 1/f)$, which is dominated by $T(r, f) + T(r, 1/f)$.

For the entire functions f, we use the notation

$$M(r, f) = \sup \{ |f(z)| : z = r \}.$$

The following inequality relates these two measures of the growth of f in case f is entire :

$$T(r, f) \leqq \log^+ M(r, f) \leqq \frac{R+r}{R-r} T(R, f) \tag{4.1.6}$$

for $0 \leqq r \leqq R$. For a proof of this, *see* [2], page 18. We will use (4.1.6) mostly in the form

$$T(r, f) \leqq \log^+ M(r, f) \leqq 3T(2r, f), \tag{4.1.7}$$

which results from setting $R = 2r$ in (4.1.6).

The following lemma, which is fundamental in our method, was proved in [1] and [5]. For completeness, we reproduce the proof of [5] here.

4.2. Lemma. — *If $f(z)$ is meromorphic in $|z| \leqq R$, with $f(0) \neq 0, \infty$, and $Z(f) = \{z_n\}$, $W(f) = \{w_n\}$, and if $\log(f(z)) = \sum_{k=0}^{\infty} \alpha_k z^k$ near $z = 0$, then for $0 < r \leqq R$, we have*

$$\log |f(re^{i\theta})| = \sum_{k=-\infty}^{\infty} c_k(r, f) e^{ik\theta}, \tag{4.2.1}$$

where the $c_k(r, f)$ are given by

$$c_0(r, f) = \log |f(0)| + \sum_{|z_n| \leqq r} \log \frac{r}{|z_n|} - \sum_{|w_n| \leqq r} \log \frac{r}{|w_k|} \tag{4.2.2}$$

$$= \log |f(0)| + N\left(r, \frac{1}{f}\right) - N(r, f).$$

(4.2.3) *For $k = 1, 2, 3, \ldots,$*

$$c_k(r, f) = \frac{1}{2} \alpha_k r^k + \frac{1}{2k} \sum_{|z_n| \leqq r} \left[\left(\frac{r}{z_n}\right)^k - \left(\frac{\bar{z}_n}{r}\right)^k \right]$$

$$- \frac{1}{2k} \sum_{|w_n| \leqq r} \left[\left(\frac{r}{w_n}\right)^k - \left(\frac{\bar{w}_n}{r}\right)^k \right].$$

(4.2.4) *For $k = 1, 2, 3, \ldots,$*

$$c_k(r, f) = c_k(r, f))^*,$$

*where * denotes complex conjugation.*

There are appropriate modifications if $f(0) = 0$ or $f(0) = \infty$.

Remark. — Observe that in the notation of § 1 and § 2, formula (4.2.3) becomes

$$(4.2.5)\quad c_k(r, f) = \frac{1}{2}\alpha_k r^k + \frac{r^k}{2}\{S(r; k : Z(f)) - S(r; k : W(f))\}$$
$$- \frac{1}{2}\{S'(r; k : Z(f)) - S'(r; k : W(f)).$$

Proof. — We may suppose that f is holomorphic, since the result for meromorphic functions will then follow by writing f as the quotient of two holomorphic functions. We may further suppose that f has no zeros on $\{z : |z| = r\}$, since the general case follows from the continuity of both sides of (4.2.2) and (4.2.3) as functions of r. Formula (4.2.2) is of course, Jensen's Theorem, and (4.2.4) is trivial since $\log|f|$ is real. To prove (4.2.3), write

$$I_k(r) = \frac{1}{\pi}\int_{-\pi}^{\pi}[\log f(re^{i\theta})]\cos(k\theta)\,d\theta$$

for some determination of the logarithm, and $k = 1, 2, 3, \ldots$. Then, by integrating by parts, we have

$$I_k(r) = -\frac{1}{\pi k}\int_{-\pi}^{\pi}\frac{f'(re^{i\theta})}{f(re^{i\theta})}\sin(k\theta)\,ire^{i\theta}\,d\theta.$$

This may be rewritten as

$$I_k(r) = \frac{1}{2\pi ki}\int_{|z|=r}\frac{f'(z)}{f(z)}\left\{\frac{r^p}{z^p} - \frac{z^p}{r^p}\right\}dz.$$

This last integral may be evaluated as a sum of residues, and on taking real parts, we get the k-th cosine coefficient of $\log|f|$. Similarly, considering the integral

$$J_k(r) = \frac{1}{k}\int[\log(f(re^{i\theta}))]\sin(k\theta)\,d\theta,$$

where the integration is now between $\pi/2k$ and $2\pi + \pi/2k$, we get the k-th sine coefficient. On combining these, we get (4.2.3).

We now define the classes of functions that we shall study.

4.3. Definition. — Let λ be a growth function We say that $f(z)$ is of *finite λ-type*, and write $f \in \Lambda$, to mean that f is meromorphic and that $T(r, f) \leqq A\lambda(Br)$ for some constants A, B and all positive r.

4.4. Definition. — We denote by Λ_E the class of *all entire functions of finite λ-type.*

4.5. Proposition. — Let f be entire. Then f is of *finite λ-type if and only if* $\log M(r, f) \leqq A\lambda(Br)$ *for some constants A, B and all positive r.*

Proof. — This follows immediately from (4.1.7).

Note in particular that if $\lambda(r) = r^\rho$ then $f \in \Lambda$ if and only if f is of growth at most order ρ, exponential type. Note also that the definition given above coincides with the definition given in [5], since only slowly increasing functions (*see* definition 3.4) were considered there. A disadvantage of the earlier definition was that the classes Λ contained only functions of finite order. A possible disadvantage of the present definition is that for functions λ of very rapid growth [e. g. $\lambda(r) = \exp(\exp(r))$], $\lambda(2r)$ is much larger than any constant multiple of $\lambda(r)$.

We also note that by inequalities (4.1.3) and (4.1.5), Λ is a field and Λ_E is an integral domain under the usual operations.

The main theorem of this paper is the following.

4.6. THEOREM. — *Let f be a meromorphic function. If f is of finite λ-type, then $Z(f)$ and $W(f)$ have finite λ-density, and there exist constants A, B such that*

$$(4.6.1) \qquad |c_k(r, f)| \leq \frac{A\lambda(Br)}{|k|+1} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

In order that f should be of finite λ-type, it is sufficient that $Z(f)$ [or $W(f)$] have finite λ-density, and that the weaker inequality

$$(4.6.2) \qquad |c_k(r, f)| \leq A\lambda(Br)$$

hold for some (possibly different) *constants A, B. Thus, in order that f should be of finite λ-type, it is necessary and sufficient that $Z(f)$ have finite λ-density and that* (4.6.1) *should hold. It is also necessary and sufficient that $Z(f)$ have finite λ-density and that* (4.6.2) *should hold.*

We will first give a proof based on the Fourier coefficients and later give an alternate proof.

Proof. — The order of the steps in the proof will be as follows. We first show that if f satisfies an inequality of the form (4.6.2), and if either $Z(f)$ or $W(f)$ has finite λ-density, then f must satisfy an inequality of the form (4.6.1). We then show that if f is of finite λ-type, then $Z(f)$ and $W(f)$ are of finite λ-density, and f satisfies an inequality of the form (4.6.2). Finally, we prove that if $Z(f)$ [or $W(f)$] has finite λ-density and if f satisfies an inequality of the form (4.6.1), then f must be of finite λ-type.

We shall suppose that $f(0) = 1$. The case $f(0) = 0$ or $f(0) = \infty$ causes no difficulty since we may multiply f by an appropriate power of z, and the resulting function will still be of finite λ-type. This is because if $\liminf(\lambda(r)/\log r) = 0$ as $r \to \infty$, then by a well known result, essentially Liouville's theorem, the class Λ contains only the constants.

Let us suppose that either $Z(f)$ or $W(f)$ is of finite λ-density and that $|c_k(r, f)| = O(\lambda(O(r)))$ uniformly for $k = 0, \pm 1, \pm 2, \ldots$. On considering the case $k = 0$, we see that both $Z(f)$ and $W(f)$ have finite λ-density. It is enough to prove that f satisfies an inequality of the form (4.6.1) for $k = 1, 2, 3, \ldots$, since c_{-k} is the complex conjugate of c_k. We prove this exactly as proposition 2.6 was proved. From (4.2.5), we have

$$(4.6.3)\quad |c_k(r, f)| \leq \frac{r^k}{2}|\alpha_k + S(r; k : Z(f)) - S(r; k : W(f))| + \frac{1}{2}|S'(r; k : Z(f))| + \frac{1}{2}|S'(r; k : W(f))|,$$

and

$$(4.6.4)\quad \frac{r^k}{2}|\alpha_k + S(r; k : Z(f)) - S(r; k : W(f)) \leq |c_k(r, f)| + \frac{1}{2}|S'(r; k : Z(f))| + \frac{1}{2}|S'(r; k : W(f))|.$$

Then, by proposition 2.2, for $Z = Z(f)$ or $Z = W(f)$, we have

$$|S'(r; k : Z)| = k^{-1}O(\lambda(O(r))) \quad \text{uniformly for } k > 0.$$

By (4.6.3), it is therefore enough to prove that

$$|\alpha_k + S(r; k : Z(f)) - S(r; k : W(f))| = k^{-1}r^{-k}O(\lambda(O(r)))$$
$$\text{uniformly for } k = 1, 2, 3, \ldots.$$

But, we already have from (4.6.4) that

$$|\alpha_k + S(r; k : Z(f)) - S(r; k : W(f))| = r^{-k}O(\lambda(O(r))) \quad \text{uniformly for such } k.$$

Replacing r by $r' = k^{1/k}r$, and observing that $r' \leq 2r$, we have that

$$|\alpha_k + S(r'; k : Z(f)) - S(r'; k : W(f)) = k^{-1}r^{-k}O(\lambda(O(r))).$$

Thus, the assertion will be proved if we can show that, for $Z = Z(f)$ and $Z = W(f)$, we have

$$|S(r, r'; k : Z)| = k^{-1}r^{-k}O(\lambda(O(r))).$$

This was proved in proposition 1.11 (*see* 1.11.1).

Now suppose that f has finite λ-type. Then

$$N(r, W(f)) = N(r, f) \leq T(r, f),$$

so that $W(f)$ has finite λ-density. By (4.1.2), the function $1/f$ also has finite λ-type. Hence $Z(f) = W(1/f)$ also has finite λ-density.

To see that an inequality of the form (4.6.2) holds, note that

$$|c_k(r, f)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\log|f(re^{i\theta})|\} e^{-ik\theta}\, d\theta \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\log|f(re^{i\theta})|\right| d\theta \leq 2T(r, f) + \log|f(0)|$$

by (4.1.5).

Finally, suppose that $W(f)$ has finite λ-density and that (4.6.1) holds. If $Z(f)$ has finite λ-density, we apply the argument below to the function $1/f$. Then $N(r, f) = O(\lambda(O(r)))$. It remains to prove that $m(r, f) = O(\lambda(O(r)))$. However,

$$m(r, f) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\log|f(re^{i\theta})|\right| d\theta,$$

which by the Schwarz inequality does not exceed

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\log|f(re^{i\theta})|\right|^2 d\theta \right)^{1/2}.$$

By Parseval's Theorem, we have, for suitable constants A, B,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\log|f(re^{i\theta})|\right|^2 d\theta = \sum_{k=-\infty}^{\infty} |c_k(r, f)|^2 \leq A^2(\lambda(Br))^2 \sum_{k=-\infty}^{\infty} \left(\frac{1}{|k|+1}\right)^2.$$

Hence $m(r, f) = O(\lambda(O(r)))$, which completes the proof of the theorem.

Specializing theorem 4.6 to entire functions, we have the next result.

4.7. Theorem. — *Let f be an entire function. If f is of finite λ-type then there exist constants A, B such that*

$$(4.7.1) \qquad |c_k(r, f)| \leq \frac{A\lambda(Br)}{|k|+1} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

It is sufficient, in order that f be of finite λ-type, that there exist (possibly different) *constants A, B such that*

$$(4.7.2) \qquad |c_k(r, f)| \leq A\lambda(Br) \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Thus, in order that f should be of finite λ-type, it is necessary and sufficient that (4.7.1) should hold, and it is also necessary and sufficient that (4.7.2) should hold).

Proof. — This result is an immediate corollary of theorem 4.6 since $W(f)$ is empty in case f is entire.

We now give a direct proof of theorems 4.6 and 4.7, that does not depend on the formula for the Fourier series coefficients given in lemma 4.2, and that is independent of proposition 1.11. We wish to thank P. MALLIAVIN for suggesting that there might be a proof of this kind. It has also the advantage that it is immediately applicable to subharmonic functions. The steps are the same as in the above proof except that we must prove directly that if f is of finite λ-type, then $|c_k(r, f)| \leq (|k|+1)^{-1} A \lambda(Br)$ for suitable constants A, B.

Suppose that f is of finite λ-type. Then, by the Poisson-Jensen formula,

$$\log |f(re^{i\theta})| = F(\theta) + G(\theta) - H(\theta),$$

where, choosing $\rho > r$, and writing $z = re^{i\theta}$,

$$G(\theta) = \sum_{|z_n| \leq \rho} \log \left| \frac{\dfrac{z}{\rho} - \dfrac{z_n}{\rho}}{1 - \dfrac{\bar{z}_n z}{\rho^2}} \right|,$$

$$H(\theta) = \sum_{|w_n| < \rho} \log \left| \frac{\dfrac{z}{\rho} - \dfrac{w_n}{\rho}}{1 - \dfrac{\bar{w}_n z}{\rho^2}} \right|,$$

and

$$F(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\rho, r, \varphi, \theta) \log |f(\rho e^{i\varphi})| \, d\varphi.$$

Here,

$$P(\rho, r, \varphi, \theta) = \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\theta - \varphi) + r^2}$$

is the Poisson kernel. We choose $\rho = 2r$. Then each term in the expressions for $G(\theta)$ and $H(\theta)$ is of the form

$$\log |w - a| - \log |1 - \bar{a}w|,$$

where $w = \frac{1}{2} e^{i\theta}$ and $|a| \leq 1$. Using 4.2, we see that the Fourier coefficients of such as function are, for $k > 0$, either

$$-\frac{1}{2k}(2\bar{a})^k - \frac{(\bar{a})^k}{2k}$$

or

$$-\frac{1}{2k}\left(\frac{1}{2a}\right)^k - \frac{(\bar{a})^k}{2k},$$

according as $0 < |a| \leq \frac{1}{2}$ or $\frac{1}{2} < |a| \leq 1$. In either case, we see that the k-th Fourier coefficient of $G(\theta) - H(\theta)$ cannot exceed

$$\frac{1}{k}\left\{n(2r, f) + n\left(2r, \frac{1}{f}\right)\right\}.$$

To estimate the k-th Fourier coefficient of $F(\theta)$, we use the estimate [7]:

$$|c_k| \leq \frac{1}{2}\omega_1\left(\frac{\pi}{|k|}, F\right),$$

where

$$\omega_1(\alpha, F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(x+\alpha) - F(x)|\, dx.$$

A simple estimate shows that

$$\omega_1(\alpha, F) \leq \omega^*(\alpha)\frac{1}{2\pi}\int_{\pi}^{\pi}\big|\log|f(2re^{i\theta})|\big|\, d\theta,$$

where

$$\omega^*(\alpha) = \sup_{\theta,\varphi} |P(2r, r, \theta+\alpha, \varphi) - P(2r, r, \theta, \varphi)|.$$

But it is easy to see that

$$\omega^*(\alpha) \leq 12 \sup_{\theta,\varphi} |\cos(\theta + \alpha - \varphi) - \cos(\theta - \varphi)| \leq c\alpha,$$

for a suitable constant c. Combining these estimates with the fact that f is of finite λ-type and proposition 1.9, we obtain the desired estimate

$$|c_k(r, f) \leq \frac{A\lambda(Br)}{|k| + 1}.$$

The next results are proved from theorems 4.6 and 4.7 in much the same way that the corresponding results of [5] were proved from theorem 1 of that paper. For the sake of completeness, we include the proofs.

4.8. Definition. — For a meromorphic function f, we define

$$E_q(r, f) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\big|\log|f(re^{i\theta})|\big|^q d\theta\right\}^{1/q}.$$

Notice that if f is entire with $f(0) = 1$, and if $\alpha = \{\alpha_k\}$ is such that $c_k(r, f) = c_k(r; Z(f) : \alpha)$, $k = 0, \pm 1, \pm 2, \ldots$, then $E_2(r, f) = E_2(r; Z(f); \alpha)$, where this last quantity is the one defined in definition 2.7.

4.9. THEOREM. — *Let f be an entire function. If f is of finite λ-type and $1 \leq q < \infty$, then*

$$E_q(r, f) \leq A\lambda(Br) \tag{4.9.1}$$

for suitable constants A, B and all $r > 0$.

Conversely, if (4.9.1) holds for some $q \geq 1$, then f is of finite λ-type.

Proof. — If f is of finite λ-type, then by the Hausdorff-Young Theorem ([7], p. 190), the L^q norm of $\log|f(re^{i\theta})|$, as a function of θ, is bounded by the l^p norm of the sequence $\{c_k\}$, where $(1/p)+(1/q)=1$. By theorem 4.7, this l^p norm, is dominated by an expression of the form $A\lambda(Br)$. Conversely, using Hölder's inequality,

$$|c_k(r, f)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \bigl|\log|f(re^{i\theta})|\bigr| d\theta$$
$$\leq \frac{1}{2\pi}\left\{\int_{-\pi}^{\pi} \bigl|\log|f(re^{i\theta})|\bigr|^q d\theta\right\}^{1/q} \leq A\lambda(Br),$$

for suitable constants A, B, and it follows from theorem 4.7 that f has finite λ-type.

4.10. THEOREM. — *Let f be a meromorphic function of finite λ-type, with $f(0) \neq 0, \infty$. Then for each positive number ε there exist positive constants α, β such that, for all $r > 0$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(\frac{\alpha}{\lambda(\beta r)}\bigl|\log|f(re^{i\theta})|\bigr|\right) d\theta \leq 1 + \varepsilon. \tag{4.10.1}$$

Remark. — We have as a consequence that, for all $r > 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|f(re^{i\theta})|^{\alpha/\lambda(\beta r)}} d\theta \leq 1 + \varepsilon,$$

which is somewhat surprising, even in case f is entire, since it is by no means evident that the integral is even finite.

Proof. — There is a number $\beta > 0$ such that

$$\frac{|c_k(r, f)|}{\lambda(\beta r)} \leq \frac{M}{|k|+1} \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Let

$$F(\theta) = F(\theta, r) = \frac{1}{\lambda(\beta r)} \log|f(re^{i\theta})|.$$

Then

$$F(\theta) = \sum \gamma_k e^{ik\theta}, \qquad \text{where } \gamma_k = \frac{c_k(r, f)}{\lambda(\beta r)}.$$

We may also suppose that the constant M satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(\theta)|\, d\theta \leq M,$$

by theorem 4.9. By a slight modification of [7] (p. 234, example 4), we know that for any such F there exists a constant $\alpha > 0$, where α depends only on M and ε, such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(\alpha\, |F(\theta)|\, d\theta \leq 1 + \varepsilon,$$

from which (4.10.1) follows.

5. Applications to entire functions.

We present, in theorem 5.2, a simple necessary and sufficient condition on a sequence Z of complex numbers that it be the precise sequence of zeros of some entire function of finite λ-type. The condition is that Z should be λ-admissible in the sense of definition 1.15. This generalizes a well-known theorem of Lindelöf (*see* the remarks following the proof of theorem 5.2). Our proof depends on theorem 4.7 and a method, presented in theorem 5.1, for constructing an entire function with certain properties from an appropriate sequence of Fourier coefficients associated with a sequence of complex numbers. In an appendix, we give an alternate proof of theorem 5.2, due to H. Delange. We also prove, in theorem 5.4, that λ has the property that each meromorphic function of finite λ-type is the quotient of two entire functions of finite λ-type if and only if λ is regular in the sense of definition 3.2. Accordingly, propositions 3.5 and 3.6 give a large class of growth functions λ for which this is the case, including the classical case $\lambda(r) = r^{\rho}$. Even this case seems not to be known.

Finally, we develop the so-called generalized canonical product, a detailed discussion of which is given in the remarks preceding the proof of lemma 5.5.

We turn now to our first task, the construction of an entire function f from a sequence Z and a sequence $\{c_k(r; Z : \alpha)\}$ of Fourier coefficients associated with Z. We recall that we have assumed that $Z = \{z_n\}$ is a sequence of non-zero complex numbers such that $z_n \to \infty$ as $n \to \infty$.

5.1. Theorem. — *Suppose that* $\{c_k(r)\} = \{c_k(r; Z : \alpha)\}$, $k = 0, \pm 1, \pm 2, \ldots$, *is a sequence of Fourier coefficients associated with* Z *such that for each* $r > 0$, $\sum |c_k(r)|^2 < \infty$. *Then there exists a unique entire function* f *with* $Z(f) = Z$, $f(0) = 1$, *and* $c_k(r, f) = c_k(r)$ *for* $k = 0, \pm 1, \pm 2, \ldots$.

Proof. — We define

$$\Phi(\rho e^{i\varphi}) = \sum_{k=-\infty}^{\infty} c_k(\rho) e^{ik\varphi}.$$

Since $\sum |c_k(\rho)|^2 < \infty$, this defines $\Phi(\rho e^{i\varphi})$ as an element of $L^2[-\pi, \pi]$ for each $\rho > 0$, by the Riesz-Fischer Theorem. For $\rho > 0$, we define the following functions :

(5.1.1) $$B_\rho(z; z_n) = \frac{\bar{z}_n}{|z_n|} \frac{\rho(z_n - z)}{\rho^2 - \bar{z}_n z},$$

(5.1.2) $$P_\rho(z) = \prod_{|z_n| \leq \rho} B_\rho(z; z_n),$$

(5.1.3) $$K(w; z) = \frac{w + z}{w - z},$$

(5.1.4) $$Q_\rho(z) = \exp\left\{ \frac{1}{2\pi i} \int_{|w|=\rho} K(w, z) \Phi(w) \frac{dw}{w} \right\},$$

(5.1.5) $$f_\rho(z) = P_\rho(z) Q_\rho(z).$$

We make the following assertions :

(5.1.6) The function f_ρ is holomorphic in the disc $\{z : |z| < \rho\}$ and its zeros there are those z_n in Z that lie in this disc.

(5.1.7) $$f_\rho(0) = 1.$$

(5.1.8) If $r < \rho$, then $c_k(r, f) = c_k(r)$.

Now (5.1.6) is clear from the definition of f_ρ. Also,

$$f_\rho(0) = P_\rho(0) Q_\rho(0) = Q_\rho(0) \prod_{|z_n| \leq \rho} \frac{|z_n|}{\rho}.$$

However,

$$Q_\rho(0) = \exp\left\{ \frac{1}{2\pi i} \int_{|w|=\rho} \Phi(w) \frac{dw}{w} \right\} = \exp\{c_0(\rho)\} = \prod_{|z_n| \leq \rho} \frac{\rho}{|z_n|},$$

and it follows that $f_\rho(0) = 1$.

To prove (5.1.8), we see by lemma 4.2 that it is enough to show that, near $z = 0$,

$$\log f_\rho(z) = \sum_{k=1}^{\infty} \alpha_k z^k,$$

where the α_k are such that $\alpha = \{\alpha_k\}$ and $c_k(r) = c_k(r; Z : \alpha)$. That is, near $z = 0$,

$$\frac{f'_\rho(z)}{f_\rho(z)} = \sum_{k=1}^{\infty} k\alpha_k z^{k-1}. \tag{5.1.9}$$

We now make this computation. First, we have that

$$\frac{B'_\rho(z; z_n)}{B_\rho(z; z_n)} = \frac{|z_n|^2 - \rho^2}{(z_n - z)(\rho^2 - \bar{z}_n z)} = \frac{\bar{z}_n}{\rho^2 - \bar{z}_n z} - \frac{1}{z_n - z}$$

$$= \sum_{k=1}^{\infty} \left(\frac{\bar{z}_n}{\rho^2}\right)^k z^{k-1} - \sum_{k=1}^{\infty} \left(\frac{1}{z_n}\right)^k z^{k-1}.$$

Thus,

$$\frac{P'_\rho(z)}{P_\rho(z)} = \sum_{k=1}^{\infty} U_{k,\rho} z^{k-1} \qquad \text{near} \quad z = 0,$$

where

$$U_{k,\rho} = \sum_{|z_n| \leq \rho} \left(\frac{\bar{z}_n}{\rho^2}\right)^k - \sum_{|z_n| \leq \rho} \left(\frac{1}{z_n}\right)^k.$$

For $k = 0, 1, 2, \ldots$, we write $w = \rho e^{i\varphi}$ and $c_k(\rho) e^{ik\varphi} = \Omega_k w^k$. Then by the definition of $c_k(\rho)$, we see that

$$\Omega_0 = N(\rho, Z)$$

and

$$\Omega_k = \frac{1}{2}\alpha_k + \frac{1}{2k} \sum_{|z_n| \leq \rho} \left\{ \left(\frac{1}{z_n}\right)^k - \left(\frac{\bar{z}_n}{\rho^2}\right)^k \right\} \qquad (k = 1, 2, 3, \ldots).$$

Then

$$\Phi(w) = N(\rho, Z) + \sum_{k=1}^{\infty} \{ \Omega_k w^k + \bar{\Omega}_k \bar{w}^k \}$$

$$= N(\rho, Z) + \sum_{k=1}^{\infty} \left\{ \Omega_k w^k + \bar{\Omega}_k \rho^{2k} \left(\frac{1}{w}\right)^k \right\},$$

so that

$$\frac{1}{2\pi i} \int_{|w|=\rho} \Phi(w) K_z(w, z) \frac{dw}{w} = \frac{2}{2\pi i} \int_{|w|=\rho} \frac{\Phi(w)}{(w - z)^2} dw,$$

where

$$K_z(w, z) = \frac{\partial}{\partial z} K(w, z) = \frac{2w}{(w - z)^2}.$$

But

$$\frac{1}{2\pi i} \int_{|w|=\rho} \frac{w^k}{(w - z)^2} dw = kz^{k-1} \qquad \text{for} \quad k = 0, 1, 2, \ldots,$$

and

$$\frac{1}{2\pi i}\int_{|w|=\rho}\left(\frac{1}{w}\right)^k \frac{1}{(w-z)^2}\,dw = 0 \qquad \text{for} \quad k = 1, 2, 3, \ldots.$$

Hence

$$\frac{Q'_\rho(z)}{Q_\rho(z)} = \sum_{k=1}^{\infty} V_{k,\rho} z^{k-1},$$

where

$$V_{k,\rho} = 2\,\Omega_k = \alpha_k + \frac{1}{k}\sum_{|z_n|\leq\rho}\left\{\left(\frac{1}{z_n}\right)^k - \left(\frac{\bar{z}_n}{\rho^2}\right)^k\right\}.$$

Hence, near $z = 0$, we have

$$\frac{f'_\rho(z)}{f_\rho(z)} = \frac{P'_\rho(z)}{P_\rho(z)} + \frac{Q'_\rho(z)}{Q_\rho(z)} = \sum_{k=1}^{\infty} k\,\alpha_k z^{k-1},$$

and (5.1.9) is proved.

It next follows from (5.1.6)-(5.1.8) that

(5.1.10) if $\rho' > \rho$, then $f_{\rho'}$ is an analytic continuation of f_ρ.

For if we define, for $|z| < \rho$,

$$F(z) = \frac{f_{\rho'}(z)}{f_\rho(z)},$$

then

$$c_k(r, F) = c_k(r, f_{\rho'}) - c_k(r, f_\rho) = c_k(r) - c_k(r) = 0,$$

for $0 \leq r < \rho$, and therefore $|F(z)| = 1$. On the other hand, $F(0) = 1$ and it follows that F is the constant function 1.

We now define the function f of theorem 5.1 by setting $f(z) = f_\rho(z)$ if $\rho > |z|$. It is clear that f is entire, and, by (5.1.6), that $Z(f) = Z$. Also $f(0) = 1$, and $c_k(r, f) = c_k(r, f_\rho)$ for $\rho > r$, so that $c_k(r, f) = c_k(r)$. An argument analogous to that used in proving (5.1.10) proves that f is unique, and the proof of the theorem is complete.

We now characterize the zero sets of entire functions of finite λ-type.

5.2. Theorem. — *A necessary and sufficient condition that the sequence Z be the precise sequence of zeros of an entire function f of finite λ-type is that Z be λ-admissible in the sense of definition* 1.15, *that is, that Z have finite λ-density and be λ-balanced.*

Proof. — If $Z = Z(f)$ for some $f \in \Lambda_E$, then by theorem 4.7, the sequence $\{c_k(r, f)\}$, is a λ-admissible sequence of Fourier coefficients associated with Z, and thus Z is λ-admissible by proposition 2.5. Conversely, suppose that Z is λ-admissible. Then by proposition 2.5 there exists a λ-admissible sequence $\{c_k(r)\}$ associated with Z. Then by theorem 5.1, there exists an entire function f with $Z = Z(f)$ and $\{c_k(r, f)\} = \{c_k(r)\}$. Then by theorem 4.7 and the fact that $\{c_k(r)\}$ is λ-admissible, it follows that $f \in \Lambda_E$, and the proof is done.

Remark. — This theorem generalizes a well-known result of Lindelöf [3], which may be stated as follows.

THEOREM. — *Let Z be a sequence of complex numbers, and let $\rho > 0$ be given. If ρ is not an integer, then in order that there exist an entire function of growth at most order ρ, finite type, it is necessary and sufficient that there exist a constant A such that $n(r, Z) \leq A^{\rho}$. If ρ is an integer, it is necessary and sufficient that both this and the following condition be satisfied for some constant B :*

$$\left| \sum_{|z_n| \leq r} \left(\frac{1}{z_n} \right)^{\rho} \right| \leq B.$$

This result follows immediately from theorem 5.2 and the characterization of r^{ρ}-admissible sequences given in proposition 3.3. Our result shows that, in general, the angular distribution of the sequence of zeros of a function, and not only its density, is involved in an essential way in determining the rate of growth of the function.

We turn now to the second problem of this section, that of determining when Λ is the field of quotients of the ring Λ_E. We first prove the following result.

5.3. THEOREM. — *In order that a sequence Z of complex numbers be the precise sequence of zeros of a meromorphic function of finite λ-type, it is necessary and sufficient that Z have finite λ-density.*

Proof. — The necessity follows immediately from the fact that if f is a meromorphic function, then $N(r, f) \leq T(r, f)$. For the sufficiency, we remark first that the method used in proving theorem 5.1 can be used to construct suitable meromorphic functions. Indeed, suppose that we are given two disjoint sequences Z, W of non-zero complex numbers with no finite limit point, and constants γ_k, $k = 1, 2, 3, \ldots$, such that the coefficients defined by

$$c_0(r) = N(r, Z) - N(r, W),$$

$$c_k(r) = \frac{r^k}{2} \{ \gamma_k + S(r; k : Z) - S(r; k : W) \}$$

$$- \frac{1}{2} \{ S'(r; k : Z) - S'(r; k : W) \qquad (k = 1, 2, 3, \ldots),$$

$$c_{-k}(r) = (c_k(r))^* \qquad (k = 1, 2, 3, \ldots),$$

satisfy $\sum |c_k(r)|^2 < \infty$ for every $r > 0$. Then by defining

$$D_\rho(z; w_n) \prod_{|w_n| \leq \rho} B_\rho(z; w_n)$$

and

$$f_\rho(z) = \frac{P_\rho(z)\, Q_\rho(z)}{D_\rho(z)},$$

one can show, as in theorem 5.1, that the meromorphic function defined by $f(z) = f_\rho(z)$ for $\rho > |z|$ has zero sequence Z, pole sequence W, and Fourier coefficients $\{c_k(r)\}$. It is therefore enough to prove that given a sequence Z of finite λ-density, there exists a disjoint sequence W of finite λ-density, and constants γ_k, $k = 1, 2, 3, \ldots$, such that the $c_k(r)$ satisfy $|c_k(r)| \leq A\lambda(Br)$ for some constants A, B and all $r > 0$. For then, by the first part of the proof of theorem 4.6, the $c_k(r)$ must satisfy the stronger inequality

$$|c_k(r)| \leq \frac{A'\lambda(B'r)}{|k| + 1} \qquad (r > 0),$$

for some constants A', B', so that the function f synthesized from the $c_k(r)$ must be of finite λ-type by theorem 4.6.

Supposing now that $Z = \{z_n\}$ has finite λ-density, we define $W = \{w_n\}$ by $w_n = z_n + \varepsilon_n$, $n = 1, 2, 3, \ldots$, where the ε_n are small complex numbers so chosen that $|w_n| = |z_n|$, $n = 1, 2, 3, \ldots$, all the numbers w_n and z_k are different, and such that

$$\sum \frac{|\varepsilon_n|}{|z_n|} < \lambda(0).$$

Then $N(r, W) = N(r, Z)$ so that W has finite λ-density. Hence

$$|S'(r; k : Z)| = k^{-1} O(\lambda(O(r)))$$

and

$$|S'(r; k : W)| = k^{-1} O(\lambda(O(r))) \qquad \text{for} \quad k = 1, 2, 3, \ldots.$$

We define

$$\gamma_k = -\frac{1}{k} \sum \left\{ \left(\frac{1}{z_n}\right)^k - \left(\frac{1}{w_n}\right)^k \right\}.$$

It remains to prove that

$$\frac{r^k}{2} |\gamma_k + S(r; k : Z) - S(r; k : W)| = O(\lambda(O(r)))$$

uniformly for $k = 1, 2, 3, \ldots$. Now

$$\frac{r^k}{2}\,|\,\gamma_k + S(r;\, k : Z) - S(r;\, k : W)\,|$$

$$= \frac{r^k}{2}\left|\frac{1}{k}\sum_{|z_n|>r}\left\{\left(\frac{1}{z_n}\right)^k - \left(\frac{1}{w_n}\right)^k\right\}\right|$$

$$= \frac{r^k}{2}\left|\frac{1}{k}\sum_{|z_n|>r}\frac{(w_n)^k - (z_n)^k}{(w_n z_n)^k}\right| \leqq \frac{r^k}{2}\,\frac{1}{k}\sum_{|z_n|>r}\frac{|(w_n)^k - (z_n)^k|}{|z_n|^{2k}}.$$

However, $|(w_n)^k - (z_n)^k| \leqq k\varepsilon_n |z_n|^{k-1}$, so that we have

$$\frac{r^k}{2}\,|\,\gamma^k + S(r;\, k : Z) - S(r;\, k : W)\,|$$

$$\leqq \frac{r^k}{2}\sum_{|z_n|>r}\frac{|\varepsilon_n|}{|z_n|^{k+1}} \leqq \frac{1}{2}\sum_{|z_n|>r}\frac{|\varepsilon_n|}{|z_n|} \leqq \frac{1}{2}\lambda(0) \leqq \lambda(r).$$

5.4. Theorem. — *The field Λ of all meromorphic functions of finite λ-type is the field of quotients of the rings Λ_E of all entire functions of finite λ-type if and anly if λ is regular in the sense of definition* 3.2, *that is, if and only if every sequence of finite λ-density is λ-balanceable.*

Proof. — First, suppose that λ is regular and that $f \in \Lambda$. Then $Z(f)$ has finite λ-density by theorem 5.13. There then exists a sequence $Z' \supseteq Z(f)$ such that Z' is λ-admissible. [We may suppose, by the remarks preceding the proof of theorem 4.6, that $f(0) \neq 0, \infty$]. Then by theorem 5.2, there exists a function $g \in \Lambda_E$ such that $Z(g) = Z'$. Since we have then that $Z(g) \subseteq Z(f)$, the function $h = g/f$ is entire. However,

$$T(r, h) \leqq T(r, g) + T\left(r, \frac{1}{f}\right) = T(r, g) + T(r, f) - \log|f(0)|$$

by (4.1.2) and (4.1.3), so that $h \in \Lambda_E$, and $f = g/h$ is the desired representation.

Conversely, suppose that $\Lambda = \Lambda_E/\Lambda_E$. Let Z have finite λ-density. Then by theorem 5.3, there exists a function $f \in \Lambda$ with $Z(f) = Z$. We write $f = g/h$ with $g, h \in \Lambda_E$. Then $Z(g)$ is λ-admissible, and $Z(g) \supseteq Z(f) = Z$, and we have proved that λ is regular.

We turn now to the so-called " generalized Hadamard product ", which is somewhat analogous to the canonical product of Hadamard, which represents an entire function of finite order [this is the case of the growth function $\lambda(r) = r^\rho$] as the limit of a sequence of functions, namely the partial products, each having as its zeros those zeros of f that lie in discs with center at the origin. The sequence converges to f not only uniformly on compact sets, but also in a way consistent with the growth of f. The fact that this sequence is usually written as a product is not essential, at least for our purposes. In the case of more general growth functions λ, there still exists a sequence or family of functions with the desired properties, and we call this sequence the generalized Hadamard product. For completely general growth functions λ, it seems to be necessary first to multiply f by a suitable entire function.

In the case $\lambda(r)=r^{\rho}$, once the function f and the growth function λ are given, there is a canonical or natural choice of the product. If $\liminf r^{-k}\lambda(r)=0$ for each $k=1, 2, 3, \ldots$, then there is still a canonical choice. In general, there is some leeway in the choice, that comes from the leeway in the choice of the constants α_k that appear in the Fourier coefficients associated with the partial products.

Before presenting these facts in the form of theorem 5.7, we require some lemmas.

5.5. LEMMA. — *Suppose that the function f is holomorphic in the disct $\{z : |z|<\rho\}$ and that $r<\rho/2$. Then*

$$\log M(r, f) \leqq 3E_2(2r, f),$$

where E_2 is the quantity defined in definition 4.8.

Proof. — First since f is holomorphic, we have

$$T(r, f)=m(r, f)\leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\log|f(re^{i\theta})|\right|d\theta \leqq E_2(r, f)$$

from Schwarz's inequality. However, since f is holomorphic for $|z|<\rho$, we have by (4.1.6) that $\log M(r, f)\leqq 3T(2r, f)$.

5.6. LEMMA. — *Let $\mathcal{R}$ be an unbounded set of positive numbers. Let $\{g_R : R\in\mathcal{R}\}$ be a family of entire functions such that $g_R(0)=1$, g_R has no zeros in the disc $\{z : |z|<R\}$, and such that*

$$(5.6.1)\qquad |c_k(r, g_R)|\leqq \frac{A\lambda(Br)}{|k|+1} \qquad (k=0, \pm 1, \pm 2, \ldots, R\in\mathcal{R}),$$

$$(5.6.2)\qquad \lim_{\substack{R\to\infty\\ R\in\mathcal{R}}} c_k(r, g_R)=0,$$

for each $k=0, \pm 1, \pm 2, \ldots$ and each positive r. Then

$$\lim_{\substack{L\to\infty\\ R\in\mathcal{R}}} g_R(z)=1$$

uniformly on compacts sets.

Proof. — Since $g_R(0)=1$, it is enough to prove that $\log|g_R(z)|\to 0$ uniformly on compact sets as $R\to\infty$ through $\mathcal{R}$. Now

$$\log|g_R(e^{i\theta})|\leqq \log M(r, g_R)\leqq \frac{\rho+r}{\rho-r}T(\rho, g_R)$$

for any $\rho > r$, by (4.1.5). However, as was shown in the proof of lemma 5.5, $T(r, f) \leq E_2(r, f)$ if f is holomorphic in $\{z : |z| \leq r\}$. Hence

$$\log |g_R(re^{i\theta})| \leq \frac{\rho + r}{\rho - r} E_2(\rho, g_R).$$

Moreover, if $R > \rho$, we may apply the same argument to $1/g_R$. Doing this, and observing that $E_2(r, f) = E_2(r, 1/f)$, we have

$$\big| |\log |g_R(re^{i\theta})| \big| \leq \frac{\rho + r}{\rho - r} E_2(\rho, g_R).$$

However, from (5.6.1) and (5.6.2), we have that

$$E_2(r, g_R) = \left(\sum_{k=-\infty}^{\infty} |c^k(\rho, g_R)|^2 \right)^{1/2} \to 0$$

as $R \to \infty$ through $\mathcal{R}$, and the result is proved.

5.7. Theorem. — *Let f be a non-constant entire function of finite λ-type. Then there exist :*

(*a*) *a function $h \in \Lambda_E$, h not identically* 0,

(*b*) *an unbounded set $\mathcal{R}$ of positive numbers and a family $\{f_R : R \in \mathcal{R}\}$ of entire functions f_R of finite λ-type,*

(*c*) *constants A, B,*

such that

(i) *the zeros of f_R are the zeros of fh in the disc $\{z : |z| \leq R\}$;*

(ii) *$fh/f_R \to 1$ as $R \to \infty$ through $\mathcal{R}$, uniformly on compact sets;*

(iii) $\log M(r, F) \leq A\lambda(Br)$ *if F is any of the functions f, h, f_R, or fh/f_R.*

Moreover, if $\liminf r^{-k}\lambda(r) = \infty$ *as $r \to \infty$, for each $k = 1, 2, 3, \ldots$, then we may take $h(z) = 1$ for all z. If* $\log \lambda(e^x)$ *is a convex function, then we may take $h(z) = 1$ for all z, and $\mathcal{R} = \{R : R \geq R_0\}$ for some $R_0 > 0$.*

We call the family $\{f_R : R \in \mathcal{R}\}$ a generalized Hadamard product associated with f.

Proof. — As before, we may suppose that $f(0) = 1$. Let $Z = Z(f)$, and let Z', $\mathcal{R}$, and $c_k(r, R) = c_k(r; Z'(R) : \alpha(R))$ be as given in theorem 2.9. Then by theorem 5.2, there is an entire function $f^* \in \Lambda_E$ such that $Z(f^*) = Z'$. We may also suppose that $f^*(0) = 1$. Also, by theorem 5.1, there are entire functions $g_R \in \Lambda_E$ with $g_R(0) = 1$, $c_k(r, g_R) = c_k(r, R)$, and $Z(g_R) = Z'(R)$. Then by lemma 5.6, $\lim g_R(z) = 1$ as $R \to \infty$ uniformly on compact sets. Also,

$|c_k(r, g_R)| = |c_k(r, R)| = (|k| + 1)^{-1} o(\lambda(o(r)))$ uniformly in k and R.

Let $f_R = f^*/g_R$. Evidently (i) and (ii) hold for f_R. Further,

$$|c_k(r, f_R)| = |c_k(r, f^*) - c_k(r, g_R)|$$
$$= (|k|+1)^{-1} O(\lambda(O(r))), \quad \text{uniformly in } k \text{ and } R.$$

Hence, there are constants A, B such that

$$|c_k(r, F) \leq \frac{1}{|k|+1} A\lambda(Br)$$

if F is any of the functions f, h, $f^* = fh$, $g_R = fh/f_R$, or $f_R = fh/g_R$. Hence or each of these functions F we have

$$E_2(r, F) \leq A'\lambda(Br),$$

where

$$A' = A\left(\sum_{-\infty}^{\infty}\left(\frac{1}{|k|+1}\right)^2\right)^{1/2}.$$

Assertion (iii) then follows from lemma 5.5.

The final assertions of the theorem follow from the assertions of theorem 2.9, except for the choice of h as the constant function 1. However, if we take $Z' = Z$, then in the above argument we can replace $h(z)$ by 1.

Appendix.

In this appendix, we give an alternate proof of theorem 5.2, that does not depend on the Fourier series method, although it does use proposition 1.14. It was communicated to the authors by H. Delange, and we are grateful to him for his permission to publish it here.

We first recall the following well-known lemma (*see*, for example A. I. Markushevitch [4], p. 85).

Lemma. — *Let g be analytic in $\{z : |z| < R\}$, $g(0) = 0$, $g(z) = \sum \alpha_k z^k$. If $Re[g(z)] \leq M$ for $|z| < R$, then for all $k \geq 1$,*

$$|\alpha_k| \leq 2MR^{-k}.$$

Now to prove the theorem, suppose first that $f \in \Lambda_E$, and let $Z = Z(f) = \{z_n\}$ be the sequence of zeros of f, supposing without loss of generality that $f(0) = 1$. As usual, we let

$$M(r) = M(r, f) = \max\{|f(z)| : |z| \leq r\}.$$

We suppose that $f(z) = \exp\left[\sum \alpha_k z^k\right]$ in some neighborhood of $z = 0$.

As before, Jensen's Theorem shows that Z has finite λ-density. Next for $r > 0$, let

$$f_r(z) = f(z) \prod_{|z_n| \leq r} \frac{1}{1 - \frac{z}{z_n}}.$$

Then f_r is an entire function with $f_r(0) = 1$ and $f_r(z) \neq 0$ for $|z| \leq r$. For each $r > 0$, there exists a function g_r, analytic in $|z| < r$, such that $g_r(0) = 0$ and $\exp[g_r(z)] = f_r(z)$ for $|z| < r$. Letting r_0 be the smallest modulus of the z_n, we see that for $|z| < r_0$ we have

$$g_r(z) = \sum_1^{\infty} [\alpha_k + S(r; k : Z)] z^k.$$

But, for $|z| = 2r$, and therefore for $|z| \leq 2r$, we have $|f_r(z)| \leq M(2r)$. Consequently,

$$Re[g_r(z)] \leq \log M(2r) \qquad \text{for} \quad |z| < r,$$

and the lemma yields

$$|\alpha_k + S(r; k : Z)| \leq \frac{2 \log M(2r)}{r^k},$$

which shows that Z is λ-poised, so that by proposition 1.14, Z is λ-balanced.

Suppose now that we are given a sequence $Z = \{z_n\}$ that has finite λ-density and is λ-balanced, hence λ-poised. We construct a function $f \in \Lambda_E$ such that $Z = Z(f)$. There exist positive constants A and B and a sequence $\{\alpha_k\}$ of complex numbers such that

$$\sum_{|z_n| \leq r} \log \frac{r}{|z_n|} \leq A\lambda(Br)$$

and

$$|\alpha_k + S(r; k : Z)| \leq \frac{A\lambda(Br)}{r^k}$$

for $k \geq 1$ and $r > 0$. Thus, for every positive r, the series $\sum [\alpha_k + S(r; k : Z)] z^k$ converges when $|z| < r$. Let r_0 be the smallest modulus of the z_n. For $r < r_0$, this series reduces to $\sum \alpha_k z^k$, so that $\sum \alpha_k z^k$ converges for $|z| < r_0$. We define functions g and F, for $|z| < r_0$, by $g(z) = \sum \alpha_k z^k$ and $F(z) = \exp[g(z)]$. We see that F can be continued to an entire function f, for we may define g_r and f_r in $|z| < r$ by

$$g_r(z) = \sum [\alpha_k + S(r; k : Z)] z^k$$

and

$$f_r(z) = \exp [g_r(z)].$$

For $|z| < r_0$, we obviously have

$$f_r(z) \prod_{|z_n| \leq r} \left(1 - \frac{z}{z_n}\right) = F(z).$$

Since $f_r(z) \neq 0$ for $|z| < r$, we see that Z is the sequence of zeros of f Now it is clear that for $|z| \leq r$,

$$|g_{2r}(z)| \leq \sum |\alpha_k + S(2r; k : Z)| r^k \leq \sum A\lambda(2Br) \left(\frac{1}{2}\right)^k = A\lambda(2Br).$$

Therefore, for $|z| = r$,

$$\begin{aligned} |F(z)| &= \left| f_{2r}(z) \prod_{|z_n| \leq 2r} \left(1 - \frac{z}{z_n}\right) \right| \\ &= |\exp [g_{2r}(z)]| \prod_{|z_n| \leq 2r} \left| 1 - \frac{z}{z_n} \right| \\ &\leq \{ \exp [A\lambda(2Br)] \} \prod_{|z_n| \leq 2r} \left[1 + \frac{r}{|z_n|} \right]. \end{aligned}$$

It follows that

$$\log M(r, f) \leq A\lambda(2Br) + \sum_{|z_n| \leq 2r} \log \left[1 + \frac{r}{|z_n|} \right].$$

Since for $|z_n| \leq 2r$ it follows that

$$1 + \frac{r}{|z_n|} \leq \frac{3r}{|z_n|},$$

we have

$$\sum_{|z_n| \leq 2r} \log \left[1 + \frac{r}{|z_n|} \right] \leq \sum_{|z_n| \leq 2r} \log \frac{3r}{|z_n|} \leq A\lambda(3Br).$$

Combining results, we have that

$$\log M(r, f) \leq 2A\lambda(3Br),$$

so that $f \in \Lambda_E$, and the proof is complete.

REFERENCES.

[1] Edrei (A.) and Fuchs (W. H. J.). — Meromorphic functions with several deficient values, *Trans. Amer. math. Soc.*, t. 93, 1959, p. 292-328.
[2] Hayman (W. K.). — *Meromorphic functions.* — Oxford, at the Clarendon Press, 1964 (*Oxford mathematical Monographs*).
[3] Lindelöf (E.). — Fonctions entières d'ordre entier, *Ann. scient. Éc. Norm. Sup.*, t. 41, 1905, p. 369-395.
[4] Markuševič (A. I.). — *Entire functions.* — New York, American Elsevier publishing Company, 1966.
[5] Rubel (L. A.). — A Fourier series method for entire functions, *Duke math. J.*, t. 30, 1963, p. 437-442.
[6] Taylor (B. A.). — *Duality and entire functions*, Thesis, University of Illinois, 1965.
[7] Zygmund (A.). — *Trigonometrical series*, First edition. — Warszawa, Seminar. Matem. Univ. Warsz., 1935 (*Monografie matematyczne*, 5).

(Manuscrit reçu le 13 juin 1967.)

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