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ON LOGARITHMIC DERIVATIVES

BY

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1. Introduction.

Let C be a ring, always commutative with identity and of prime characteristic $p > 0$. Let C^* denote the group of invertible elements of C. Given a derivation ∂ on C, the mapping

$$\delta_0 : \quad C^* \to C^+$$

defined by $\delta_0(u) = (\partial u)/u$ is a group-homomorphism. Now assume ∂ satisfies a polynomial

$$X = \alpha_0 t + \alpha_1 t^p + \ldots + \alpha_i t^{p^i} + \ldots + \alpha_n t^{p^n}$$

with coefficients in the ring A = kernel ∂. For any c in C, let Lc denote the map $C \to C$ produced by multiplication by c. From the formula

$$(\partial + Lc)^p = \partial^p + L(\partial^{p-1} c + c^p) \quad ([3], \text{ p. } 201),$$

it is easily seen that

$$X(\partial + Lc) = L(\delta_1 \, c),$$

where

$$\delta_1(c) = \sum_{i=0}^{n} \alpha_i([\partial^{p^i-1} c] + [\partial^{p^{i-1}-1} c]^p + \ldots + [\partial^{p^{i-j}-1} c]^{p^j} + \ldots + c^{p^i})$$

is an element in A. It is also immediately clear that

$$\delta_1 : \quad C^+ \to A^+$$

is again a group-homomorphism. Let u be an element of C^*. Then

$$\partial + L(\delta_0 u) = (Lu)^{-1} \, \partial(Lu),$$

and so

$$X(\partial + L(\delta_0 u)) = (Lu)^{-1} \, X(\partial) \, (Lu) = 0.$$

This means given ∂ and X, we have a complex :

$$0 \to A^* \underset{\varepsilon}{\to} C^* \underset{\delta_0}{\to} C^+ \underset{\delta_1}{\to} A^+ \to 0.$$

When C is a finite dimensional field extension over A and X is the characteristic polynomial for ∂, a theorem of N. JACOBSON ([7], theorem 15) states that the kernel of δ_1 coincides with the image of δ_0.

The purpose of this paper is to describe, for a general commutative ring C, the group (kernel δ_1)/(image δ_0) in terms of classes of rank one projective A-modules which are split by C. If C is a noetherian integrally closed domain, a description is also given in terms of divisor classes of A which become principal in C. These are done in the next section. In the final section, some examples are given.

2. The rank one projective class group.

LEMMA 2.1. — *Let* $\mathfrak{g}$ *be a set of derivations on a semi-local ring* C *of prime characteristic* $p > 0$, *and let* A *denote the kernel*

$$\{ x \in C \mid \partial x = 0 \text{ for all } \partial \in \mathfrak{g} \}$$

of $\mathfrak{g}$. *Assume* C *is a finitely generated projective* A*-module and* $\mathrm{Hom}_A(C, C) = C[\mathfrak{g}]$. *Then both* C *and* A *are finite ring direct sums of indecomposable semi-local rings*

$$C = C_1 + \ldots + C_m, \qquad A = A_1 + \ldots + A_m;$$

and for each i,

$$C_i \cong A_i[t_1, \ldots, t_r]/(t_1^p - a_1, \ldots, t_r^p - a_r),$$

where $a_1, \ldots, a_r$ *are in* A_i, $t_1, \ldots, t_r$ *are indeterminates, and* r *depends on* i.

Proof. — Given a prime ideal q in A, $\mathfrak{Q} = \{ x \in C \mid x^p \in \mathfrak{q} \}$ is a prime in C, and $\mathfrak{Q} \cap A = \mathfrak{q}$. If $\mathfrak{q}$ is maximal, so is $\mathfrak{Q}$, hence A must be semi-local. Let e be any idempotent in C. We have $\partial e = \partial e^p = p(\partial e)\, e^{p-1}$ is zero. This shows e is in A. The ring C being semi-local contains no more than finitely many indecomposable indempotents $\{ e_1, \ldots, e_m \}$. Put $C_i = Ce_i$ and $A_i = Ae_i$. We have

$$C = C_1 + \ldots + C_m, \qquad A = A_1 + \ldots + A_m.$$

Let N denote the radical of A_i, and put $\bar{A} = A_i/N$, $\bar{C} = C_i/NC_i$. Of course $\bar{A}$ is a finite direct sum $\sum_j F_j$ of fields. Accordingly $\bar{C}$ decomposes into a direct sum $\sum_j R_j$, where R_j is a finite dimensional local F_j-algebra. Now C_i is a finitely generated projective module over a semi-local ring A_i with connected spectrum, so must be free ([1], p. 143). This shows the dimension of R_j over F_j is equal to the rank of C_i over A_i and hence is independent of j. If we denote by $\bar{\partial}$ the derivation on R_j induced by $\partial|_{C_i}$, and by $\bar{\mathfrak{g}}$ the set $\{\bar{\partial} \mid \partial \in \mathfrak{g}\}$, then $\mathrm{Hom}_{F_j}(R_j, R_j) = R_j[\bar{\mathfrak{g}}]$ because

$$\bar{A} \otimes_{A_i} \mathrm{Hom}_{A_i}(C_i, C_i) = \mathrm{Hom}_{\bar{A}}(\bar{C}, \bar{C}).$$

Thus no non-trivial ideal of R_j can be stable under $\bar{\mathfrak{g}}$, the structure of R_j is therefore known ([9], corollary 2.8) :

$$R_j \cong F_j[t_1, \ldots, t_r]/(t_1^p - f_1, \ldots, t_r^p - f_r),$$

where $f_1, \ldots, f_r$ are elements of F_j, $t_1, \ldots, t_r$ are indeterminates. But r is independent of j, so

$$\bar{C} = \sum R_j \cong \bar{A}[t_1, \ldots, t_r]/(t_1^p - \bar{a}_1, \ldots, t_r^p - \bar{a}_r) \qquad (\bar{a}_i \in \bar{A}).$$

By [1], p. 105, this shows C_i is isomorphic to

$$A_i[t_1, \ldots, t_r]/(t_1^p - a_1, \ldots, t_r^p - a_r)$$

for some $a_1, \ldots, a_r$ in A_i as desired.

LEMMA 2.2. — *Let A be a commutative ring of prime characteristic $p > 0$, let*

$$C = A[t_0, \ldots, t_n]/(t_0^p - a_0, \ldots, t_n^p - a_n),$$

where $a_0, \ldots, a_n$ are elements of A and $t_0, \ldots, t_n$ are indeterminates. Assume ∂ is an A-derivation on C such that $\mathrm{Hom}_A(C, C) = C[\partial]$. *Then the characteristic polynomial of ∂ is of the form*

$$\alpha_0 t + \alpha_1 t^p + \ldots + \alpha_i t^{p^i} + \ldots + \alpha_n t^{p^n} + t^{p^{n+1}} \qquad (\alpha_i \in A).$$

Proof. — Let $\partial_i = \dfrac{\partial}{\partial t_i}$ be the A-derivation on C given by $\partial_i t_j = \delta_{ij}$ (the Kronecker delta function). So

$$\partial^{p^i} = b_{i0}\partial_0 + \ldots + b_{in}\partial_n, \qquad b_{ij} = \partial^{p^i}(t_j),$$

because ∂^{p^i} as a derivation is completely determined by its actions on the t_j's. Now from $\mathrm{Hom}_A(C, C) = C[\partial]$, we know $\{\partial^i \mid 0 \leq i < p^{n+1}\}$ form a linearly independent C-basis for $\mathrm{Hom}_A(C, C)$. (Notice that ∂ as an A-endomorphism on the free A-module C of rank p^{n+1} has a characteristic polynomial of degree p^{n+1}. Therefore $\mathrm{Hom}_A(C, C) = C[\partial]$ implies that every A-endomorphism on C is a C-linear combination in

$\{ \partial^i \mid 0 \leqq i < p^{n+1} \}$. But $\mathrm{Hom}_A(C, C)$ is a free C-module of rank p^{n+1}, $\{ \partial^i \mid 0 \leqq i < p^{n+1} \}$ must be C-linearly independent.) So

$$\partial_i = c_{i0}\partial + c_{i1}\partial^p + \ldots + c_{in}\partial^{p^n} + \sum c'_{ij}\partial^j \qquad (c_{ij}, c'_{ij} \in C),$$

where the summation runs through all j, $0 < j < p^{n+1}$ and j is not a power of p. So we have the matrix equation

$$\begin{pmatrix} \partial \\ \partial^p \\ \vdots \\ \partial^{p^n} \end{pmatrix} = \begin{pmatrix} b_{00} & \ldots & b_{0n} \\ \vdots & & \vdots \\ b_{n0} & \ldots & b_{nn} \end{pmatrix} \left(\begin{array}{ccc|c} c_{00} & \ldots & c_{0n} & \\ \vdots & & \vdots & c'_{ij} \\ c_{n0} & \ldots & c_{nn} & \end{array} \right) \begin{pmatrix} \partial \\ \partial^p \\ \vdots \\ \partial^{p^n} \\ \vdots \end{pmatrix}.$$

The linear independency of $\{ \partial^i \mid 0 \leqq i < p^{n+1} \}$ therefore asserts that $(b_{ij})\ (c_{ij})$ is the identity $n+1$ by $n+1$ matrix and $(b_{ij})\ (c'_{ij})$ is a zero matrix. This shows $(c'_{ij}) = (c_{ij})\ (b_{ij})\ (c'_{ij})$ is a zero matrix. In other words,

$$\partial_i = c_{i0}\partial + c_{i1}\partial^p + \ldots + c_{in}\partial^{p^n} \quad \text{for all } i.$$

From $\partial^{p^{n+1}} = b_{n+1\,0}\partial_0 + \ldots + b_{n+1\,n}\partial_n$, we see that ∂ satisfies a polynomial

$$\alpha_0 t + \alpha_1 t^p + \ldots + \alpha_i t^{p^i} + \ldots + \alpha_n t^{p^n} + t^{p^{n+1}}.$$

That this polynomial must coincide with the characteristic polynomial of ∂ follows from the fact that $\{ \partial^i \mid 0 \leqq i < p^{n+1} \}$ are linearly independent over C. This completes the proof of the lemma.

REMARK 2.3. — Derivations satisfying the hypothesis $\mathrm{Hom}_A(C, C) = C[\partial]$ always exist. For example, let ∂ be given by $\partial t_0 = 1$ and $\partial t_i = (t_0 \ldots t_{i-1})^{p-1}$ for all $i > 0$. It is easy to verify that the characteristic polynomial of this derivation is just $t^{p^{n+1}}$.

THEOREM 2.4. — *Let ∂ be a derivation on a ring C of prime characteristic $p > 0$ with A as its kernel. Assume C is a finitely generated projective A-module of rank r and $\mathrm{Hom}_A(C, C) = C[\partial]$. Then ∂ satisfies a polynomial $X = \alpha_0 t + \alpha_1 t^p + \ldots + \alpha_{n-1} t^{p^{n-1}} + t^{p^n}$ with α_i in A and $r = p^n$. Moreover $XC[t] = \{ f \in C[t] \mid f(\partial) = 0 \}$.*

Proof. — Given a maximal ideal q in A, let Q denote the maximal ideal $\{ x \in C \mid x^p \in q \}$ in C. It is clear that $C_Q = C \otimes_A A_q$. So $\mathrm{Hom}_{A_q}(C_Q, C_Q) = A_q \otimes_A \mathrm{Hom}_A(C, C) = C_Q[\partial]$. Hence by lemma 2.1 $r = p^n$ for some n. Let M be the A-submodule of $\mathrm{Hom}_A(C, C)$ generated by ∂^{p^i}, $i = 0, 1, \ldots, n$, and denote by M' the A-submodule of M generated by ∂^{p^i}, $i = 0, \ldots, n-1$. In view of [1] (p. 112, cor. 1) to show the inclusion map $M' \to M$ is onto it suffices to show at each maximal ideal q in A the corresponding map $M'_q \to M_q$ is onto which according to lemma 2.2 is indeed the case. So there is a polynomial $X = \alpha_0 t + \alpha_1 t^p + \ldots + \alpha_{n-1} t^{p^{n-1}} + t^{p^n}$, with α_i in A and $X(\partial) = 0$. Given $f \in C[t]$, $f(\partial) = 0$, we may write $f = gX + h$, with $g, h \in C[t]$ and degree $h < p^n$. So $h(\partial) = 0$. Since $\{ \partial^i \mid 0 \leq i < p^n \}$ is linearly independent over C_Q at every maximal ideal Q in C, all coefficients of h must vanish because they vanish locally. So $f = gX$. This completes the proof of the theorem.

COROLLARY 2.5. — *Let ∂ be a derivation on a ring C of prime characterictic $p > 0$ with A as its kernel. Assume C is a finitely generated projective A-module and* $\mathrm{Hom}_A(C, C) = C[\partial]$. *Then*

$$\{ f \in C[t] \mid f(\partial) = 0 \} = XC[t]$$

for some $X(t) = \alpha_0 t + \alpha_1 t^p + \ldots + \alpha_i t^{p^i} + \ldots + \alpha_n t^{p^n}$ *with* $\alpha_i \in A$ *and α_n a non-zero idempotent.*

Proof. — Since C is finitely generated and projective as A-module, the map $\rho : q \to$ (rank of C_q over A_q) is locally constant on $\Omega = \mathrm{Spec}\, A$. For any positive integer r_i write $\Omega_i = \{ q \in \Omega \mid \rho(q) = r_i \}$. So Ω_i is both open and closed in Ω and we have a finite disjoint union $\Omega = \bigcup \Omega_i$ because Ω is quasi-compact. If $\tilde{A} = (\Omega, \tilde{A})$ is the sheaf of local rings associated to A and $\tilde{A}_i = \tilde{A} \mid \Omega_i$, then $A = \tilde{A}(\Omega)$ decomposes into a finite ring direct sum $\bigoplus \tilde{A}_i(\Omega_i)$. So $A = \bigoplus Ae_i$ and $C = \bigoplus Ce_i$ where e_i is the identity element of $\tilde{A}_i(\Omega_i)$. Since Ce_i is a finitely generated projective Ae_i-module of finite rank and $\mathrm{Hom}_{Ae_i}(Ce_i, Ce_i) = Ce_i[e_i \partial]$. An application of the theorem completes the proof of the corollary.

Hereafter we shall always denote by X the polynomial given by corollary 2.5.

THEOREM 2.6. — *Let ∂ be a derivation on a ring C of prime characteristic $p > 0$ with A as its kernel. Assume C is a finitely generated projective module over A and* $\mathrm{Hom}_A(C, C) = C[\partial]$. *Then the group $P(C/A)$ of classes of rank one projective A-modules split by C is isomorphic to the homology group $L(C/A)$* = (kernel δ_1)/(image δ_0) *of the complex*

$$C^* \underset{\delta_0}{\to} C^+ \underset{\delta_1}{\to} A^+$$

defined by ∂ and X.

Proof (1). — Let M be a rank one projective A-module such that the C-module $M \otimes C$ is free on one generator b. Let F be a finite subset

(8) Henceforth all tensor-product signs without subscripts will denote tensor product over A.

of A such that the ideal in A generated by F is A and such that for any $f \in F$, the A_f-module $M \otimes A_f$ is free on one generator b_f ([1], p. 138). Given $f \in F$, $b = b_f(1 \otimes u_f)$ for some invertible element u_f of A_f. Now let $\mathfrak{Q}$ be a prime ideal of C, and let $\mathfrak{q}$ denote the prime $\mathfrak{Q} \cap A$ in A. To any generator $b_{\mathfrak{Q}}$ for the free $A_{\mathfrak{q}}$-module $M \otimes A_{\mathfrak{q}}$, there is a unique invertible element $u_{\mathfrak{Q}}$ in $C_{\mathfrak{Q}}$ given by the equation $b = b_{\mathfrak{Q}}(1 \otimes u_{\mathfrak{Q}})$. It is easily seen that the correspondence $\mathfrak{Q} \to (\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$ is independent of the choice of $b_{\mathfrak{Q}}$. In particular, if $f \in F$ is not in $\mathfrak{q}$, then $(\partial u_f)/u_f$ goes to $(\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$ under the canonical homomorphism $C_f \to C_{\mathfrak{Q}}$. This shows $\mathfrak{Q} \to (\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$ is a section for the structural sheaf of Spec$C$. By [4], p. 86, there is a unique element $z \in C$ such that for all $\mathfrak{Q} \in \operatorname{Spec} C$, the canonical image of z in $C_{\mathfrak{Q}}$ is $(\partial u_{\mathfrak{Q}})/u_{\mathfrak{Q}}$. Now $\delta_1 z$ must be trivial because at each $\mathfrak{Q}$,

$$X(\partial + Lz) = (Lu_{\mathfrak{Q}})^{-1} X(\partial) (Lu_{\mathfrak{Q}}) = 0 \quad ([1], \text{ p. } 112).$$

If b' is another generator for the free C-module $M \otimes C$, and z' is the element in C to correspond, then $z' = z$ modulo image δ_0. So we have a well-defined mapping $\lambda : P(C/A) \to L(C/A)$.

Obviously λ is a group-homomorphism. To show it is one-to-one, assume $z = \partial u/u$ for some $u \in C^*$. Then for any $\mathfrak{Q} \in \operatorname{Spec} C$, $u_{\mathfrak{Q}} = ua_{\mathfrak{Q}}$ for some $a_{\mathfrak{Q}} \in A^*_{\mathfrak{q}}$, $(1 \otimes \partial)(b[1 \otimes u^{-1}])$ must be zero in $M \otimes C$ because at every $\mathfrak{Q}$,

$$(1 \otimes \partial)(b[1 \otimes u^{-1}]) = (1 \otimes \partial)(b_{\mathfrak{Q}}[1 \otimes a_{\mathfrak{Q}}]) = 0.$$

But the sequence $0 \to M \otimes A \to M \otimes C \xrightarrow[1 \otimes \partial]{} M \otimes C$ is exact, $b(1 \otimes u^{-1})$ therefore is already contained in M. Let m be any element of M. Then $m \otimes 1 = b(1 \otimes u^{-1}c)$ for some $c \in C$. Therefore c must be an element of A because $0 = (1 \otimes \partial)(m \otimes 1) = b(1 \otimes u^{-1}[\partial c])$. This shows M is free over A and hence λ is one-to-one (2).

It remains to show λ is onto. So let $C[t; \partial]$ be the non-commutative ring of differential polynomials with coefficients in C defined by $tc = ct + \partial c$. An inductive argument shows that

$$t^r c = ct^r + \binom{r}{1}(\partial c)t^{r-1} + \binom{r}{2}(\partial^2 c)t^{r-2} + \ldots + (\partial^r c),$$

and so X is in the center of $C[t; \partial]$ because $t^p c = ct^p + \partial^p c$.

Now to any z in the kernel of $\delta_1 : C^+ \to A^+$, we associate a ring-homomorphism

$$\rho_z : \; C[t; \partial] \to \operatorname{Hom}_A(C, C) \quad \text{given by } \rho_z(g) = g(\partial + Lz).$$

(2) Note that the hypotheses C over A being finitely generated projective and $\operatorname{Hom}_A(C, C) = C[\partial]$ are not needed for the existance and the injectivity of λ. Similar remark applies to theorem 2.9.

If g is in the kernel of ρ_0, then $g\,(\partial + Lz)$ is the zero endomorphism on C. This shows the kernel of ρ_0 is contained in the kernel of ρ_z. So we have a ring-homomorphism $\rho_z\rho_0^{-1} : C[\partial] \to \mathrm{Hom}_A(C, C)$. In other words, $X(\partial + Lz) = 0$ means that C is made into a $C[\partial]$-module with ∂ acting on C as $\partial + Lz$. But if $C[\partial] = \mathrm{Hom}_A(C, C)$, the modules over the latter are well-known. Write $E = \mathrm{Hom}_A(C, C)$, then the formula is $\mathrm{Hom}_E(C, C) \otimes C \simeq C$ ([1], p. 181, exercice 18). Now each element of $\mathrm{Hom}_E(C, C)$ is determined by its action on $1 \in C$ which must go to an element of C annihilated by the new operation of ∂ since in the old operation of ∂, $\partial 1 = 0$. Thus $\mathrm{Hom}_E(C, C) \cong$ kernel $(\partial + Lz)$ and so $C = C.$kernel $(\partial + Lz)$. But C over A is a faithfully flat module : given any prime ideal q in A, $Q = \{x \in C \mid x^p \in q\}$ is a prime in C, and $Q \cap A = q$; if q is maximal, so is Q ([1], p. 51). $\mathrm{Hom}_E(C, C) \otimes C = C$ therefore implies that $\mathrm{Hom}_E(C, C)$ and hence kernel $(\partial + Lz)$ is a rank one projective A-module ([1], p. 53, 142). Write $\pi_z =$ kernel $(\partial + Lz)$, and let b be the element $\sum m_i \otimes c_i$ in $\pi_z \otimes C$ such that $\sum m_i c_i = 1$ in C. For each $Q \in \mathrm{Spec}\, C$, pick $m_Q \in \pi_z$ such that $b_Q = m_Q \otimes 1$ is a generator for the rank one free A_q-module $\pi_z \otimes A_q$. We have, for all i, $m_i \otimes 1 = m_Q \otimes a_i$ for some $a_i \in A_q$. Now with the notations introduced earlier in this proof, $u_Q = \sum a_i c_i$. But in C_Q $m_Q \sum a_i c_i = \sum m_i c_i = 1$. So

$$0 = (\partial m_Q)\left(\sum a_i c_i\right) + m_Q \sum a_i (\partial c_i)$$

$$= - m_Q\left(z \sum a_i c\right) + m_Q \sum a_i (\partial c_i).$$

This shows $(\partial u_Q)/u_Q = \left(\partial \sum a_i c_i\right) \Big/ \left(\sum a_i c_i\right) = z$, and hence λ is onto. This completes the proof of the theorem.

We list some special cases of theorem 2.6. When C is a field, the following is the well-known theorem of Jacobson ([7], theorem 15].

COROLLARY 2.7. — *Let C be a semi-local ring of prime characteristic $p > 0$. Let ∂ be a derivation on C with A as its kernel such that C is a finitely generated projective module over A and* $\mathrm{Hom}_A(C, C) = C[\partial]$ (3). *Then the sequence*

$$0 \to A^* \underset{\varepsilon}{\to} C^* \underset{\partial_0}{\to} C^+ \underset{\partial_1}{\to} A^+$$

is exact.

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(3) When C is a finite dimensional field extension of A, this is always satisfied.

Proof. — Since A is also semi-local, we have $L(C/A) \cong P(C/A) = 0$ ([1], p. 143) hence the corollary.

Of particular interest is the following corollary.

COROLLARY 2.8. — *Let C be either a noetherian ring or an integral domain of prime characteristic $p > 0$. Let ∂ be a derivation on C with A as its kernel such that C is a finitely generated projective A-module and $\mathrm{Hom}_A(C, C) = C[\partial]$. Let L be the total ring of fractions of C, and denote by $L(C/A)$ the group*

$$[\delta_0(L^*) \cap C^+]/\delta_0(C^*) = \{\partial x/x \mid x \in L^*;\ \partial x/x \in C\}/\{\partial x/x \mid x \in C^*\}.$$

Then there is an isomorphism

$$\pi : \quad L(C/A) \to P(C/A)$$

which takes class z to class kernel $(\partial + Lz)$.

Proof. — Consider the commutative diagram given by ∂ and X,

$$\begin{array}{ccccc} C^* & \xrightarrow{\delta_0} & C^+ & \xrightarrow{\delta_1} & A^+ \\ \cap & & \cap & & \cap \\ L^* & \xrightarrow[\delta_0]{} & L^+ & \xrightarrow[\delta_1]{} & K^+ \end{array} \qquad (K = \text{the total ring of fractions of } A),$$

the lower sequence is exact by corollary 2.7. So z belongs to kernel $\{C^+ \xrightarrow{\delta_1} A^+\}$ if and only if $z = \partial x/x$ for some $x \in L^*$. By theorem 2.6, this shows π is an isomorphism as asserted.

In the above corollary, if C is a noetherian integrally closed domain, the hypothesis that C over A is finitely generated and projective can be relaxed to C over A is finitely presented, that is, there is an exact sequence of A-modules

$$F_2 \to F_1 \to C \to 0,$$

where F_1 and F_2 are finitely generated free A-modules. But instead of rank one projectives, we now have to describe $L(C/A)$ in terms of divisor classes.

The definition of Krull domain can be found in [2]. Noetherian integrally closed domains form the main example of Krull domains. If g is a set of derivations on a field L, and z a non-zero element in L, we shall denote by $\zeta_z : g \to L$ the map defined by $\partial \to (\partial z/z)$.

THEOREM 2.9. — *Let g be a finite set of derivations on a Krull domain C of characteristic $p \neq 0$, and let A be the Krull domain*

$$\{x \in C \mid \partial x = 0 \quad \text{for all } \partial \in g\}.$$

Denote by L and K the fields of fractions of C and A respectively. Assume C is finitely presented as A-module and $\mathrm{Hom}_A(C, C) = C[g]$. *Then the group* $\Gamma(C/A)$ *of divisor classes in A which become principal in C is isomorphic to*

$$L(C/A) = \{\zeta_z \mid z \in L^* \text{ and } \zeta_z(\partial) \in C \text{ for all } \partial \in g\}/\{\zeta_z \mid z \in C^*\}.$$

Proof. — Let d be a divisor in A which becomes a principal divisor (z) in C. Then for each prime ideal Q of height one in C, there is some z_Q in K such that $|z|_Q = |z_Q|_Q$, where $|\ \ |_Q$ is the discrete valuation on C given by Q. So $z = u_Q z_Q$ for some invertible element u_Q in C_Q. This shows for any ∂ in g, $\partial z/z = \partial u_Q/u_Q$ is an element of C_Q for all prime Q of height one. So $\partial z/z$ is an element of C because C is a Krull domain. Since $\zeta_z = \zeta_u$ ($z \in L^*$, $u \in C^*$) is equivalent to $\partial(z/u) = 0$ for all ∂ in g, or in other words $z/u \in K^*$, the correspondence $d \to \zeta_z$ gives rise to a one-to-one group-homomorphism $\lambda : \Gamma(C/A) \to L(C/A)$.

To prove the map is onto, let z be an element of L^* such that $\partial z/z \in C$ for all ∂ in g. We claim that if $|z|_Q \neq 0$ modulo p, then the ramification index $e(Q)$ of Q over A must be one. Let $t \in Q$ be a uniformizing variable for Q, that is, $tC_Q = QC_Q$. So $z = ut^n$ for some invertible element u in C_Q, and

$$(\partial u/u) + n(\partial t/t) = \partial z/z \in C \quad \text{for all } \partial \text{ in } g.$$

This shows if $n \not\equiv 0\ (p)$, then tC_Q is stable under g. Now C is finitely presented as A-module, if $q = Q \cap A$, then

$$A_q \otimes_A \mathrm{Hom}_A(C, C) \cong \mathrm{Hom}_{A_q}(C_Q, C_Q) \quad ([1], \text{p. } 98).$$

But A_q is a discrete valuation ring, C_Q as a finitely generated torsion-free A_q-module must be free, so

$$\hat{C}_Q[g] \cong \hat{A}_q \otimes_A \mathrm{Hom}_A(C, C) \cong \hat{A}_q \otimes_{A_q} [A_q \otimes_A \mathrm{Hom}_A(C, C)]$$
$$\cong \hat{A}_q \otimes_{A_q} \mathrm{Hom}_{A_q}(C_Q, C_Q) \cong \mathrm{Hom}_{\hat{A}_q}(\hat{C}_Q, \hat{C}_Q),$$

where $\wedge$ means taking completion. Now the ramification index of $t\hat{C}_Q$ is either 1 or p. If it is p, then there is an $\hat{A}_q$-derivation Δ on $\hat{C}_Q$ such that $\Delta t = 1$. From $\hat{C}_Q(g] = \mathrm{Hom}_{\hat{A}_q}(\hat{C}_Q, \hat{C}_Q)$, we see that $\partial t \notin t\hat{C}_Q$ for some ∂ in g. This shows that if $QC_Q = tC_Q$ is stable under g, then $e(Q) = 1$. Let d denote the divisor $\sum_Q \frac{|z|_Q}{e(Q)} (Q \cap A)$. Clearly λ maps class d to class $\partial z/z$. This completes the proof of the theorem.

REMARK 2.10. — When L is a field extension over K of dimension p, g has only one element ∂, and $\partial(C)$ contained in no prime ideal of height one, theorem 2.9 is given by SAMUEL ([8], theorem 2). The monomorphism part of theorem 2.9 is also given by HALLIER ([6], p. 3924). That this monomorphism in general is by no means onto is clear from the following.

REMARK 2.11. — The hypothesis $\mathrm{Hom}_A(C, C) = C[\partial]$ cannot be dropped from theorems 2.6 and 2.9. Consider the polynomial ring $C = E[x, y, z]$ where E is a field of characteristic 2. Let ∂' be the E-derivation on C given by

$$\partial' x = y^4, \qquad \partial' y = x^2 \qquad \text{and} \qquad \partial' z = xyz.$$

Then C is a free module over $A = \text{kernel } \partial' = E[x^2, y^2, z^2]$. The latter is a unique factorization domain, so both $P(C/A)$ and $\Gamma(C/A)$ are trivial. $L(C/A, \partial')$ however is not trivial : $\partial' z/z = xy$ is an element of C while C^* is just E^*, the image of C^* in C^+ is trivial.

If instead of ∂', we consider the E-derivation ∂ on C given by $\partial x = 1$, $\partial y = x$ and $\partial z = xy$, then $\mathrm{Hom}_A(C, C) = C[\partial]$. The sequence $C^* \to C^+ \to A^+$ given by ∂ and its characteristic polynomial t^8 is exact, and

$$L(C/A) = L(C/A, \partial) = 0.$$

3. Examples.

3.1. *Counter-example for a conjecture of Samuel.* — Let C be the polynomial ring $E[x, y]$ where E is a field of characteristic 2. Let ∂ be the E-derivation on C given by $\partial x = 1$ and $\partial y = y^2$. Then C is a free module over $A = \text{kernel } \partial = E[x^2, y^2, xy^2 + y]$ and $\mathrm{Hom}_A(C, C) = C[\partial]$. The characteristic polynomial for ∂ is t^2, and the map $\delta_1 : C^+ \to A^+$ given by ∂ and t^2 is $c \to \partial c + c^2$. Now C^* is just E^*, so $\delta_0(C^*)$ is trivial. The kernel of δ_1 is $\{0, y\}$. So $P(C/A) = \Gamma(C/A)$ [$= P(A) = \Gamma(A)$ because C is a unique factorization domain] is cyclic of order 2. The non-trivial rank one projective A-module is the ideal $y^2A + (xy^2 + y)A$. Since $\partial y/y = y$ is an element of $C = (\partial C)C$, we get a counter-example for the following conjecture of Pierre SAMUEL ([8], p. 88) :

Let ∂ be a derivation on an integral domain of characteristic $p > 0$. If Q is the ideal in C generated by the image of ∂, then $\partial c/c \in Q$ $(c \in C)$ implies $\partial u/u = \partial c/c$ for some $u \in C^*$.

Some special cases of this statement have been verified by HALLIER [5] and also by SAMUEL [8], and was used by SAMUEL to compute the divisor class group of the following example when the characteristic of C is 2, 3 and 5.

3.2. — Let $C = E[[x, y]]$ be the formal power series ring over a field E of characteristic $p > 0$. Let ∂ be the E-derivation on C given by $\partial x = x$ and $\partial y = -y$. So $A = \text{kernel } \partial = E[[x^p, y^p, xy]]$. Both A and C are noetherian integrally closed. Since C is finitely generated as A-module, C is finitely presented also [1], p. 36. The rank one projective class group $P(A)$ is trivial because A is a local ring. We propose to verify the following statements :

(i) $C[\partial] = \mathrm{Hom}_A(C, C)$;

(ii) $\Gamma(A) = \Gamma(C/A)$ is cyclic of order p;

(iii) the A-module C is not flat, and hence not projective.

Given f in $\mathrm{Hom}_A(C, C)$, we have $f = x_0 + x_1\partial + \ldots + x_{p-1}\partial^{p-1}$ with $x_i \in L$ because $\mathrm{Hom}_K(L, L) = L[\partial]$ and $[L : K] = p$. Now $x_0 = f(1) \in C$, so we may assume $x_0 = 0$ and

$$f = x_1\partial + \ldots + x_{p-1}\partial^{p-1}.$$

From $\partial^i(x^j) = j^i x^j$, $\partial^i(y^j) = (-j)^i y^j$, we get two systems of linear equations in x_i :

$$\text{(I)} \qquad ix_1 + i^2x_2 + \ldots + i^{p-1}x_{p-1} = f(x^i)/x^i \qquad (0 < i < p);$$

$$\text{(II)} \quad (-i)x_1 + (-i)^2x_2 + \ldots + (-i)^{p-1}x_{p-1} = f(y^i)/y^i \qquad (0 < i < p).$$

The first system of equations shows x_i is a polynomial in $1/x$ with coefficients in C, while the second system shows x_i is a polynomial in $1/y$ also with coefficients in C. So $x_i \in C$ and $f \in C[\partial]$.

The divisor class group $\Gamma(A)$ is just $\Gamma(C/A)$ because C is a unique factorization domain. So $\Gamma(A) = [\delta_0(L^*) \cap C^+]/\delta_0(C^*)$. Now the minimal polynomial for ∂ is $t^p - t$. The mapping $\delta_1 : C^+ \to A^+$ with respect to ∂ and $t^p - t$ is given by $\delta_1(s) = \partial^{p-1}s - s + s^p (s \in C)$.

Assume z is an element of kernel δ_1, and write

$$z = \alpha + \beta + \sum_{i=1}^{p-1}(u_i x^i + v_i y^i),$$

where $\alpha \in E$, β, u_i, $v_i \in A$, and β has no constant term. We have

$$(\alpha^p - \alpha) + (\beta^p - \beta) + \sum_{i=1}^{p-1}(u_i x^i + v_i y^i)^p = 0.$$

So $\alpha = \alpha^p$, which implies α is an element of $\{0, 1, \ldots, p-1\}$, and

$$\begin{aligned} \beta &= \sum_{i=1}^{p-1} (u_i x^i + v_i y^i)^p + \beta^p \\ &= \sum_{i=1}^{p-1}(u_i x^i + v_i y^i)^p + \sum_{i=1}^{p-1}(u_i x^i + v_i y^i)^{p^2} + \beta^{p^2} \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{p-1}(u_i x^i + v_i y^i)^{p^n}. \end{aligned}$$

This shows z is an element of kernel δ_1 if and only if

$$z = \alpha + \sum_{i=0}^{\infty} \sum_{n=1}^{p-1} (u_i x^i + v_i y^i)^{p^n},$$

with $\alpha \in \{ 0, 1, \ldots, p-1 \}$, $u_i, v_i \in A$. But given $u \in A$, $0 < i < p$, the element $ux^i + (ux^i)^p + (ux^i)^{p^2} + \ldots$ always lies in the image of $\delta_0 : C^* \to C^+$ because the equation

$$\partial\left(\sum_{j=0}^{p-1} s_j x^j\right) = \left(\sum_{j=0}^{p-1} s_j x^j\right) \sum_{n=0}^{\infty} (ux^i)^{p^n} \qquad (s_j \in A)$$

is solvable in s_j. This proves $\Gamma(A)$ is cyclic of order p since elements in the image of $\delta_0 : C^* \to C^+$ has no constant terms.

Finally, C is finitely presented as A-module, if C is flat over A, C would be projective over A ([1], p. 140); according to corollary 2.8, that would imply $P(C/A) = L(C/A) = \Gamma(C/A)$ is cyclic of order p. But A is a local ring, $P(C/A)$ must be trivial, therefore the A-module C is not flat.

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