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ON $N$-HIGH SUBGROUPS OF ABELIAN GROUPS

BY

JOHN M. IRWIN AND KHALID BENABDALLAH.

1. Introduction.

This paper is based on a curious property of $N$-high subgroups when $N$ is a subgroup of $G$ the subgroup of elements of infinite height of a group $G$. Let $G$ be a group, $N$ a subgroup of $G$, we say that a subgroup $H$ of $G$ is $N$-high if $H$ is maximal with respect to the property $H \cap N = \emptyset$. Our first result (theorem 2.4) is that given a group $G$ and $N$ a subgroup of $G'$, then $G = \langle H, K \rangle$ whenever $H$ is an $N$-high subgroup of $G$ and $K$ is a pure subgroup of $G$ containing $N$. A close look at the proof of this result shows that the assumption that $K$ is pure can be replaced by the weaker one that $N \subseteq K'$. An immediate consequence is the classical theorem that divisible subgroups of a group are absolute summands of the group.

$N$-high subgroups where $N \subseteq G'$ were first introduced and studied by IRWIN and WALKER in [3]. These authors proved that $N$-high subgroups are pure and that the factor groups they induce are divisible. It turns out (theorem 2.5) that $H$ is an $N$-high subgroup of a group $G$, where $N \subseteq G'$ if and only if $H$ is pure, $H \cap N = \emptyset$, $G = \langle H, K \rangle$ for all $K$ pure containing $N$ and $G/H$ is divisible. We use this property of $N$-high subgroups where $N \subseteq G'$, to generalize and simplify many results in [4]. In particular, we obtain a criterion for a pure subgroup of a group $G$ containing $N \subseteq G'$ to be a summand of $G$ (theorem 3.1).

In the fourth part, we define the concept of quasi-essential and strongly quasi-essential subsocles of a $p$-group (definition 4.1) and proceed to characterize those quasi-essential subsocles which are also centers of purity (theorem 4.4) and those which are strongly quasi-essential (theorem 4.8).

We use standard notation from [1]. The symbol $Z^+$ denotes the set of positive integers. If $G$ is a $p$-group, $R$ a subgroup of $G$ and $g \in R$,
the symbol $h_n(g)$ denotes the height of the element $g$ in the subgroup $R$. All groups considered are Abelian.

2. A characterization of $N$-high subgroups of a group $G$, with $N \triangleleft G$.

We need the following lemmas.

**Lemma 2.1.** Let $G$ be a $p$-group, $N$ subgroup of $G$, and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$, $G = \langle H, K \rangle$.

**Proof.** Clearly $\langle H, K \rangle \supset H[p] \oplus N[p] = G[p]$. By induction suppose $\langle H, K \rangle \supset G[p^n]$. Let $g \in G$, $o(g) = p^{n+1}$, if $g \notin H$, $\langle g, H \rangle \cap N \neq o$ thus there exists $h \in H$, $g' \in N$, and $m < n + 1$, such that

\[ p^m g + h = g' \neq o \]

since $K$ is pure, $g' \in K'$, thus there exists $k \in K$, such that $g' = p^m k$, or $h = p^m (g - k)$. If $h \neq o$, by purity of $H$ (see [3], theorem 5) there exists $h' \in H$, such that $p^m h' = h$, therefore $p^m (g - k - h') = o$. This implies $g - k - h' \in \langle H, K \rangle$, and $g \in \langle H, K \rangle$, thus $\langle H, K \rangle \supset G[p^{n+1}]$. By induction

\[ G = \langle H, K \rangle. \]

**Lemma 2.2.** Let $G$ be a torsion group, $N$ a subgroup of $G$, and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$, $G = \langle H, K \rangle$.

**Proof.** Let $G = \sum G_p$, $H = \sum H_p$, $K = \sum K_p$ and $N = \sum N_p$ then for each prime $p$, $H_p$ is $N_p$-high in $G_p$ (see [2], lemma 11) and since $K_p$ is pure containing $N_p$ lemma 2.1 holds and $G_p = \langle H_p, K_p \rangle$. Therefore

\[ G = \sum \langle H_p, K_p \rangle = \langle H, K \rangle. \]

**Lemma 2.3.** Let $G$ be a group, $N$ a subgroup of $G$, and $H$ an $N$-high subgroup of $G$. Then $H_t$ is $N_t$-high in $G_t$.

**Proof.** Clearly $H_t \cap N_t = o$, let $g \in G_t$, $o(g) = b, g \notin H$ then $\langle g, H \rangle \cap N \neq o$, thus there exists $h \in H, n \in N$, and a positive integer $a$ such that $ag + h = n \neq o$. Clearly $a \neq b$. Now $baq + bh = bn$, thus $bh = bn = o$, and $h \in H_t, n \in N_t$, therefore $\langle g, H_t \rangle \cap N_t \neq o$. This implies that $H_t$ is $N_t$-high in $G_t$.

**Theorem 2.4.** Let $G$ be a group, $N$ a subgroup of $G$, and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G,

\[ G = \langle H, K \rangle. \]
Proof. — Suppose $g \in G$, $g \notin H$, then $\langle g, H \rangle \cap N \neq \emptyset$, thus there exists $h \in H$, $n \in N$ and a positive integer $a$, such that

$$ag + h = n \neq 0.$$ 

By an argument similar to the one used in lemma 2.1, there exists $h' \in H$ and $k \in K$ such that $a(g + h' - k) = 0$. Thus $g + h' - k \in G$. But, by lemmas 2.3 and 2.2, we know that $G = \langle H, K \rangle$, therefore $g \in \langle H, K \rangle$ and $G = \langle H, K \rangle$.

A classical theorem follows immediately from theorem 2.4.

Corollary. — If $D$ is a divisible subgroup of a group $G$, then $D$ is an absolute summand of $G$.

Proof. — Let $D = N$ in theorem 2.4, since $D$ is divisible it is pure in $G$. Thus $G = D \oplus H$, for any $D$-high subgroup $H$ of $G$.

Theorem 2.5. — Let $G$ be a group, $N$ a subgroup of $G$ and $H$ a subgroup of $G$ disjoint from $N$. Then $H$ is $N$-high in $G$ if and only if $H$ is pure, $G/H$ is divisible and $G = \langle H, K \rangle$ for any pure subgroup $K$ of $G$ containing $N$.

Proof. — The necessity follows from theorem 2.4. Suppose then that $H$ satisfies the conditions of the theorem. Since $H \cap N = \emptyset$ there exists an $N$-high subgroup $H'$ of $G$ containing $H$. Since $H'$ is pure in $G$, $H'/H$ is pure in $G/H$ which is divisible, therefore $H'/H$ is divisible and $G/H = (H'/H) \oplus (R/H)$ where $R$ can be chosen to contain $N$. Since $H$ is pure in $G$ and $R/H$ is pure in $G/H$, $R$ is pure in $G$, and since $R \supseteq N$,

$$R = \langle R, H \rangle = G.$$ 

Therefore

$$H = R \cap H' = G \cap H' = H'$$

and $H$ is $N$-high in $G$.

3. Some applications.

We first obtain a criterion for pure subgroups of a group $G$ to be summands of $G$.

Theorem 3.1. — Let $G$ be a group, $K$ a pure subgroup of $G$ containing a subgroup $N$ of $G$. Then $K$ is a direct summand of $G$ if and only if there exists an $N$-high subgroup $H$ of $G$ such that $H \cap K$ is a direct summand of $H$.

Proof. — Suppose $G = K \oplus L$, let $M$ be any $N$-high subgroup of $K$, then it is easy to see that $H = L \oplus M$ is $N$-high in $G$ and $H \cap K = M$ is a summand of $H$. 


Suppose now that there exists an $N$-high subgroup $H$ of $G$ such that $H = (H \cap K) \oplus R$, by theorem 2.4:

$$G = \langle H, K \rangle = \langle (H \cap K) \oplus R, K \rangle = \langle R, K \rangle$$

and since $R \cap K = o$, $G = R \oplus K$.

The following corollary contains theorem 2 in [4].

**Corollary.** — A reduced group $G$ splits over its maximal torsion subgroup $G_t$ if and only if some $N$-high subgroup of $G$ splits, where $N \subset G' \cap G$.

**Proof.** — If $G$ is reduced and $G = G_i \oplus L$ then $G' \subset G_i$ and since $G_i$ is pure theorem 3.1 implies there exists an $N$-high subgroup such that $H \cap G_i = H_i$ is a summand of $H$. Now if $H$ is $N$-high and $H = H_i \oplus L$ since $N \subset G_i \cap G'$ by theorem 3.1, $G_i$ is a summand of $G$.

For what follows we need the following lemmas.

**Lemma 3.2.** — Let $G$ be a group, $H$ a subgroup of $G$ then if $K|H_i$ is an $(H/H_i)$-high subgroup of $G|H_i$, then $K$ is pure in $G$ and $K|G_i$.

**Proof.** — Suppose $n g \in K$ where $g \in G$. Let $o \neq h = ag + k \in \langle K, g \rangle \cap H$ then $nag + nk = nh \in K \cap H = H_i$, therefore $h \in H_i$, thus $\langle K, g \rangle \cap H=H_i$, which implies $\langle K, g \rangle = K$, therefore $g \in K$ and thus $K$ is pure in $G$. Now if $g \in G$, then letting $n = o(g)$ in the above argument we see that $K \supset G_i$.

**Lemma 3.3.** — Let $G$ be a group, $N$ a subgroup of $G'$, $H$ an $N$-high subgroup of $G$ and $K$ a pure subgroup of $G$ containing $\langle N, G' \rangle$ and such that $K \cap H = H_i$. Then for any $N$-high subgroup $H'$ we have $K \cap H' = H_i$.

**Proof.** — Such $K$ do exist (lemma 3.2). Clearly $K \cap H' \supset H_i$. Let $h' \in K \cap H'$ and suppose $h' \notin H$ then there exists $h \in H, g \in N$ and a positive integer $a$ such that $ah' + h = g \neq o$, thus $h \in K \cap H = H_i$, let $b = o(h)$, then

$$bah' = bah' + bh = bg \in H' \cap N = o$$

thus $bah' = o$ and consequently $h' \in H_i$. Therefore $K \cap H' = H_i$.

**Corollary 1 ([4], lemma).** — If $G$ is a group, $N$ a subgroup of $G'$ and $H$ is an $N$-high subgroup of $G$, then $H|H_i$ is a summand of $G|H_i$.

**Proof.** — Let $K|H_i$ be $H_i|H_i$-high in $G|H_i$. Choose $K \supset N$. Then, since $K$ is pure in $G$ (lemma 3.2), it follows from theorem 2.4 that $G = \langle K, H \rangle$. Therefore $G|H_i = (H|H_i) \oplus (K|H_i)$.

**Corollary 2 ([4], theorem 4).** — Let $H$ and $H'$ be two $N$-high subgroups of a reduced group $G$ where $N$ is a subgroup of $G'$. Then $H|H_i \simeq H'|H_i$ and $G|H_i \simeq G|H_i$. 


Proof. — From corollary 1, \( G/H_t = (H/H_t) \oplus (K/H_t) \). From lemma 3.3, \( K \cap H' = H_t \), therefore \( G/H_t = (H'/H_t) \oplus (K/H_t) \). The result follows from this and the fact that \( G/H \simeq G/H' \) (see [3]).

Corollary 3 ([4], theorem 1). — Let \( G \) be a reduced group, \( N \) a subgroup of \( G \) and \( H \) an \( N \)-high subgroup of \( G \) then if \( H = H_t \oplus L \), we have \( G = K \oplus L \) where \( K/G_t \) is the divisible part of \( G/G_t \).

Proof. — \( G/H_t = H/H_t \oplus K/H_t \) from corollary 1.

Now \( K/H_t \) is divisible since \( K/H_t \cong G/H_t \), and \( H/H_t \cong L \) is reduced. Thus \( K/G_t \) is the divisible part of \( G/H_t \). Now \( K \cap H = H_t \) implies \( K \cap L = 0 \) and \( \langle K, H \rangle = G \) implies \( \langle K, L \rangle = G \), therefore

\[ G = K \oplus L. \]


It is natural to ask, what kind of subgroups of a group \( G \) have properties similar to subgroups of \( G \). We consider first \( p \)-groups. It is trivial to verify that two subgroups of a \( p \)-group are disjoint if and only if their socles are. Thus it suffices to consider subgroups of the socle of a \( p \)-group which we will call subsocles.

Definition 4.1. — Let \( G \) be a \( p \)-group, a subsocle \( S \) of \( G \) is said to be quasi-essential (q.e.) if \( G = \langle H, K \rangle \) whenever \( H \) is an \( S \)-high subgroup of \( G \) and \( K \) a pure subgroup of \( G \) containing \( S \). \( S \) is said to be strongly quasi-essential (s.q.e.) if every subgroup of \( S \) is q.e.

We now proceed to characterize those quasi-essential subsocles of \( G \) a \( p \)-group \( G \) which are also centers of purity (see [7] and [6]).

Theorem 4.2. — Let \( G \) be a \( p \)-group, \( S \) a center of purity, \( S \subset G[p] \). If \( S \) is not quasi-essential in \( G \) then there exists \( n \in \mathbb{Z} \), \( g \in G[p] \), \( g \notin S \) and \( s \in S \) such that

\[ h(s) = h(g) = n \quad \text{and} \quad h(s + g) = n + 1. \]

Proof. — Set \( P_n = (p^n G)[p] \), \( P_n = G'[p] \) and \( P_{n+1} = 0 \) then it is known (see [6]) that \( S \) is a center of purity if and only if

\[ P_n \ni S \ni P_{n+2} \quad \text{for some} \quad n \in \{ 1, 2, \ldots, \infty, \infty + 1 \}. \]

From lemma 2.1, we see that if \( n = \infty \), i.e. \( S \subset G^1 \), \( S \) is q.e. Thus if \( S \) is not q.e. there exists \( n \in \mathbb{Z}^+ \), such that

\[ P_n \ni S \ni P_{n+2}. \]

Also \( S \) is not q.e. implies that there exist a pure subgroup \( K \) of \( G \) containing \( S \) and an \( S \)-high subgroup \( H \), of \( G \) such that \( \langle H, K \rangle \neq G \). Let
\( \langle H, K \rangle = R. \) Since \( R \triangleright G[p] \) and \( R \not\subseteq G, R \) is not pure in \( G \) (see [5], lemma 12). Therefore there exists an element \( x \in R[p] \) such that \( h(x) > h_R(x). \) \( H \) and \( K \) being both pure in \( G \) implies that \( x \in H \) and \( x \notin K \) Therefore there exists \( g \in H[p] \) and \( s \in S \) such that \( x = g + s, \ g \neq 0 \neq s. \) It is easy to verify that

\[
h_R(g) = h_H(g) = h(g) \quad \text{and} \quad h_R(s) = h_K(s) = h(s),
\]

therefore

\[
h(g) = h(s) \leq h_R(g + s) < h(g + s).
\]

Now \( s \in S \) implies \( h(s) \geq p^n, g \notin S \) implies \( h(g) \leq n + 1 \) and since \( S \supset P_{n+1} \) we conclude that \( h(s) = h(g) = n \) and \( h(g + s) = n + 1 \) as stated.

**Corollary 1.** — Let \( G \) be a \( p \)-group, \( S \) a subsocle of \( G \) such that

\[
P_n \triangleright S \triangleright P_{n+1}
\]

then \( S \) is quasi-essential.

**Proof.** — \( S \) is a center of purity, thus theorem 4.2 applies and clearly there exists no pair \( g \in G[p], g \notin S \) and \( s \in S \) that satisfy the conditions of the theorem. Thus \( S \) is q. e.

**Corollary 2.** — Let \( G \) be a \( p \)-group, \( S \) subsocle of \( G \) such that \( S \) supports an absolute summand \( A \) of \( G \) then \( S \) is quasi-essential.

**Proof.** — \( S \) is a center of purity, thus theorem 4.2 holds and again if \( g \notin S \) and \( s \in S \) and \( h(g) = h(s) \) then, since \( g \) can be embedded in a complementary summand of \( A \) in \( G, h(g + s) = h(g) = h(s). \) Therefore the condition of the theorem cannot be satisfied and \( S \) must be q. e.

**Corollary 3.** — Let \( G \) be a \( p \)-group, \( K \) a pure subgroup of \( G \) containing \( P_n \) for some \( n \in \mathbb{Z}^+ \), then \( K \) is a direct summand containing \( p^n G. \)

**Proof.** — Since \( P_n \) is q. e., \( G = \langle K, H \rangle, \) where \( H \) is a \( P_n \)-high subgroup of \( G. \) Now \( H \) is bounded, in fact \( p^n H = 0, \) and \( G/H \cong H/H \cap K, \) therefore \( K \) is a direct summand of \( G \) and \( p^n G \subseteq K. \)

In fact, it turns out that the conditions on \( S \) in corollary 1 and 2 as well as the condition that \( S \) be quasi-essential and a center of purity, are equivalent provided \( S \notin G'. \) To prove this, we need the following lemma.

**Lemma 4.3.** — Let \( G \) be a \( p \)-group, \( H \) a pure subgroup of \( G \) such that \( G/H \) is pure-complete. Let \( S \) be a subsocle of \( G \) such that \( H[p] \subset S. \) Then \( S \) supports a pure subgroup \( K \) of \( G \) containing \( H. \)

**Proof.** — Since \( G/H \) is a pure-complete group, by definition, every subsocle of \( G/H \) supports a pure subgroup of \( G/H. \) Now \( \langle S, H \rangle/H \) is
clearly a subsocle of $G/H$, therefore there exists $K/H$ a pure subgroup of $G/H$ such that
\[(K/H)[p] = \langle S, H \rangle /H.\]
Since $H$ is pure in $G$, $K$ is pure in $G$ (see [5], lemma 2). Clearly $K[p] \supset S,$ let $k \in K[p]$, then $k + H \in (K/H)[p] = \langle S, H \rangle /H$, thus there exists $s \in S$ and $h \in H$ such that $k - s = h$, but $ph = p(k - s) = 0$, and since $S \supset H[p]$, we conclude that $k \in S$. Therefore $K[p] = S$.

**Corollary.** — Let $G$ be a $p$-group, $S$ a subsocle containing $P^n$ (see theorem 4.2) for some $n \in \mathbb{Z}^+$, then $S$ supports a pure subgroup of $G$ containing $P^n G$.

**Proof.** — Let $G_n$ be as in [1], p. 98. Then $G_n$ is pure in $G$, $G_n[p] = P_n$ and $G/G_n$ is bounded and therefore pure complete. Thus lemma 4.3 holds, and $S$ supports a pure subgroup of $G$ containing $G_n$.

**Theorem 4.4.** — Let $G$ be a $p$-group, $S$ subsocle of $G$ not contained in $G'$ then the following are equivalent:

(i) $S$ is both a center of purity and a quasi-essential subsocle of $G$;

(ii) $S$ supports an absolute direct summand of $G$;

(iii) There exists $n \in \mathbb{Z}^+$ such that $P_n \supset S \supset P_{n+1}$.

**Proof.** — (i) implies (ii). Suppose $S$ satisfies (i), then since $S$ is a center of purity $S \supset P_m$ for some $m \in \mathbb{Z}^+$ and by the corollary to lemma 4.3, $S$ supports a pure subgroup $K$ of $G$. Since $S$ is also quasi-essential $K$ is an absolute summand of $G$.

(ii) implies (i). Suppose $S$ supports an absolute summand $K$. Then $S$ is clearly a center of purity and by corollary 2 to theorem 4.2, $S$ is q. e.

(i) implies (iii). Suppose $S$ satisfies (i), then $S$ supports an absolute summand $K$ of $G$. Since $S$ is a center of purity, we know there exists $m \in \mathbb{Z}^+ \ni P_m \supset S \supset P_{m+2}$. Suppose $P_{m+1} \not\subset S$, we will show, by contradiction, that $S \supset P_{m+1}$. Indeed, suppose not, i.e. there is $x \in G[p]$ such that $x \not\in S$ and $h(x) = m + 1$. Now $P_{m+1} \not\subset S$ implies there exists $s \in S$, $h(s) = m$, otherwise $P_m \subset S \subset P_{m+1}$, and we would be done. Let $y = x - s$ then $h(y) = m$, $y \in S$ and $h(y + s) = m + 1$.

Since $y \not\in S$ there is an $S$-high subgroup $H$ of $G$ such that $y \in H$. But, $G = K \oplus H$ and $h(y) = h(s)$ imply that $h(y + s) = h(y) = h(s)$ which is a contradiction. Therefore $S \supset P_{m+1}$.

(iii) implies (i). If $S$ satisfies (iii) it is a center of purity (see theorem 4.2, proof) and by corollary 1 to theorem 4.2 it is also q. e.

At this point we have completely characterized those quasi-essential subsocles of a $p$-group which are also centers of purity. An immediate consequence is the following.
COROLLARY. — Let $G$ be a $p$-group, $A$ a pure subgroup of $G$, then $A$ is an absolute direct summand of $G$ if and only if $A$ is divisible or $P_n \triangleright A[p] \triangleright P_{n+1}$ for some $n \in \mathbb{Z}^+$. The strongly quasi-essential subsocles have also a simple characterization which can be obtained from the previous result. We need the following lemmas.

**Lemma 4.5.** — Let $G$ be a group; $A$, $B$, $C$ three subgroups of $G$ then
\[ \langle A \cap B, C \cap B \rangle = \langle A \cap B, C \rangle \cap B = \langle A, C \cap B \rangle \subseteq B. \]

**Lemma 4.6.** — Let $G$ be a group, $N$ a subgroup of $M$ a subgroup of $G$, if a subgroup $H$ is $N$-high in $G$ then $H \cap M$ is $N$-high in $M$. Conversely if $H'$ is an $N$-high subgroup of $M$ then $H' = H \cap M$ for any $N$-high subgroup $H$ of $G$ containing $H'$.

**Proof.** — Let $H$ be $N$-high in $G$ then for all $x \notin H$, we have
\[ \langle H, x \rangle \cap N \neq \emptyset. \]
Suppose $m \in M$, $m \notin H$, then
\[ \langle H \cap M, m \rangle \cap N = \langle H, m \rangle \cap M \cap N = \langle H, m \rangle \cap N \neq \emptyset. \]
and since $(H \cap M) \cap N = \emptyset$, $H \cap M$ is $N$-high in $M$.

Let $H'$ be an $N$-high subgroup of $M$, and let $H$ be any $N$-high subgroup of $G$ then $H \cap M \triangleright H'$ and $(H \cap M) \cap N = \emptyset$. The maximality of $H'$ implies $H \cap M = H'$.

**Lemma 4.7.** — Let $G$ be a $p$-group, $S$ a quasi-essential subsocle of $G$. Let $K$ be a pure subgroup of $G$ containing $S$. Then $S$ is quasi-essential in $K$.

**Proof.** — Let $M$ be a pure subgroup of $K$ containing $S$ and let $H$ be an $S$-high subgroup of $K$. Let $H'$ be an $S$-high subgroup of $G$ containing $H$ then, since $S$ is q.e. and $M$ is also pure in $G$ we have $\langle M, H' \rangle = G$, thus by lemma 4.5 and 4.6,
\[ K = \langle M, H' \rangle \cap K = \langle M \cap K, H' \rangle \cap K = \langle M \cap K, H' \cap K \rangle = \langle M, H \rangle, \]
and $S$ is q.e. in $K$ as stated.

**Theorem 4.8.** — Let $G$ be a $p$-group, $S$ a subsocle of $G$ then $S$ is strongly quasi-essential if and only if either $S \subseteq G'$ or there exists $n \in \mathbb{Z}^+$, such that $p^n G = o$ and $(p^{n-1} G)[p] \triangleright S$.

**Proof.** — If $S \subseteq G'$, $S$ is s.q.e. follows from lemma 2.1. If there exists $n \in \mathbb{Z}^+$ such that $p^{n-1} G[p] \triangleright S \triangleright p^n G = o$ then $S$ is s.q.e. as a
consequence of corollary 1 to theorem 4.2. Suppose now that $S$ is s. q. e. and $S \not\leq G$ then there exists $s \in S$ such that $h(s) < \infty$. By corollary 24.2 in [1], $S$ can be embedded in a finite pure subgroup $K$ such that $K[p] = \langle s \rangle$. Since $S$ is s. q. e., $K$ is an absolute summand of $G$. Thus by theorem 4.4, there exists $m \in \mathbb{Z}^+$, such that $(p^m G)[p] \leq \langle s \rangle \leq (p^{m-1} G)[p]$ but $\langle s \rangle$ is a cyclic group of order $p$, therefore $G$ is a bounded group. This implies that $S$ supports a pure subgroup $M$ of $G$, and since $S$ is q. e., $M$ is an absolute summand of $G$. From lemma 4.7, we see that every subsocle of $M$ is q. e. in $M$, and thus every summand of $M$ is an absolute summand.

By problem 11 (b), p. 93 in [1], $M = \sum C(p^n)$ for some $n \in \mathbb{Z}^+$ and $S = M[p] \leq (p^{n-1} G)[p]$. Clearly $M[p] \leq (p^n G)[p]$, therefore $(p^n G)[p] \leq S \leq (p^{n-1} G)[p]$, and since $M$ is pure $P^n G \leq M$. Thus $p^n G = (p^n G) \cap M = p^n M = 0$,

and the proof is complete.

The following characterization follows immediately from theorem 4.8.

**Theorem 4.9.** — Let $G$ be a $p$-group, every subsocle of $G$ is quasi-essential if and only if $G$ is divisible or $G$ is a direct sum of cyclic groups of same order.

We have not been able to decide whether a quasi-essential subsocle is necessarily a center of purity or not. But in the next theorem, we have a case where quasi-essential subsocles are centers of purity.

**Theorem 4.10.** — Let $G$ be a $p$-group, if $G$ is pure-complete then every quasi-essential subsocle of $G$ is a center of purity.

**Proof.** — Let $S$ be q. e. Since $G$ is pure-complete, $S$ supports a pure subgroup $K$ of $G$. This $K$ is an absolute summand and therefore the result follows from corollary 2 to theorem 4.2

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