BULLETIN DE LA S. M. F.

TOSHIKO KOYAMA

On quasi-closed groups and torsion complete groups

Bulletin de la S. M. F., tome 95 (1967), p. 89-94
<http://www.numdam.org/item?id=BSMF_1967__95__89_0>

Bull. Soc. math. France,
95, 1967, p. 89 à 94.

ON QUASI-CLOSED GROUPS AND TORSION COMPLETE GROUPS

BY

TOSHIKO KOYAMA.

1. Introduction.

Every group, in this paper, is an abelian p-group. We will observe some properties of abelian p-groups using topological methods. Notation and terminology follow FUCHS [1] except that we use the word " torsion complete " instead of " closed " following [4]. Let G be a p-group. Then we can introduce the p-adic topology in G. If G has no elements of infinite height, this topology is a metric topology.

Let G be a p-group without elements of infinite height. If every bounded Cauchy sequence of G has a limit in G with respect to the p-adic topology, G is called torsion complete. A torsion complete group G has following properties.

(I) Let $B = C_1 \oplus C_2$ be a basic subgroup of G. Then $G = C_1^- \oplus C_2^-$, where C_1^- and C_2^- are closures of C_1 and C_2 in G.

(II) Let H be a pure subgroup of G. Then H^- is a direct summand of G.

(III) Let H be a pure subgroup of G. Then H^- is again pure.

(III)′ Let H be a pure subgroup of G. $\left(\frac{G}{H}\right)^1$ is divisible.

(IV) (Strong Purification Property). For a given subgroup P of $G[p]$ and for a given pure subgroup H of G such that $H[p] \subset P$, there exists a pure subgroup K containing H such that $K[p] = P$.

After considering these properties a natural question arises : Is the reduced p-group which satisfies (I) or (II) necessarily torsion complete ? We will give an affirmative answer to this question. This gives rise to a nice characterization of torsion complete groups.

P. Hill and C. Megibben [2] called the reduced p-group which satisfies (III) quasi-closed group. They have showed an example which is quasi-closed but not torsion complete in [2]. They have also proved in [2] that a quasi-closed group which is not torsion complete is essentially indecomposable. We will show that properties (III), (III)′ and (IV) are equivalent. That is, a reduced p-group is quasi-closed if and only if G satisfies " Strong Purification Property ". Since unbounded direct sum of cyclic groups is neither essentially indecomposable, nor torsion complete, it is not quasi-closed. We will construct a pure subgroup H in unbounded direct sum of cyclic groups such that H^- is not pure.

This paper is a part of the author's doctoral dissertation which was submitted to the Graduate Division of Wayne State University. The author wishes to express her sincere gratitude to her advisor, Dr. John M. Irwin for his invaluable suggestions and encouragement.

2. Topological Preliminaries.

Let G be a p-group and x be an element of G. $h(x)$ denotes the height of x. Set $d(x, y) = p^{-h(x-y)}$ for $x, y \in G$. d defines a pseudo-metric in G. Since d is invariant, G is a topological group with this pseudo-metric. This topology is called p-adic topology. If we assume the condition $G^1 = 0$, then d defines a metric in G. Let H be a subset of G, then we write H^- for the closure of H in G with respect to the p-adic topology of G.

Lemma 1. — *$\mathcal{B} = \{ p^n G, n = 0, 1, 2, \ldots \}$ is a local base at 0 for the p-adic topology of G. Hence $\{ 0 \}^- = G^1$, the p-adic topology in a bounded group is discrete and the p-adic topology in a divisible group is trivial.*

Lemma 2. — *If H is a pure subgroup of G, then the p-adic topology of H coincides with the relative topology, since $p^n H = H \cap p^n G$. Hence we need not distinguish the relative topology and the p-adic topology in H whenever H is pure in G.*

Lemma 3. — *Let $G = \sum_{i=1}^{m} G_i$. Then the p-adic topology of G is the product of p-adic topologies in G_i's. Hence a direct summand of G is closed in G whenever $G^1 = 0$.*

Lemma 4. — *A direct sum of a finite number of torsion complete groups is torsion complete and a direct summand of a torsion complete group is torsion complete.*

Lemma 5. — *Let G be a torsion complete group and let $B = C_1 \oplus C_2$ be a basic subgroup of G. Then $G = C_1^- \oplus C_2^-$. Hence, if H is a pure subgroup of G, H^- is a direct summand of G.*

LEMMA 6. — $G[p^n] = \{x \in G : p^n x = 0\}$ $(n = 1, 2, 3, \ldots)$ *is closed in G with respect to any compatible Hausdorff topology in G.*

Proof. — $f(x) = p^n x$ is continuous in any topological group, $\{0\}$ is closed in any Hausdorff topological group and $G[p^n]$ is the inverse image of 0 by $f(x)$. Hence $G[p^n]$ is closed.

LEMMA 7. — *Let H be a subgroup of G. Then* $\left(\frac{G}{H}\right)^1 = \frac{H^-}{H}$.

Proof. — Let φ be a canonical homomorphism : $G \to G/H$. $h(\varphi(x)) = \infty$ if and only if $(x + p^n G) \cap H \neq \Phi$ for all n. That is, $x \in H^-$.

LEMMA 8. — *A subgroup H is dense in G if and only if G/H is divisible.*

LEMMA 9. — *Let H be a pure subgroup of G. Then H^- is pure if and only if* $\left(\frac{G}{H}\right)^1$ *is divisible* $\left(\textit{i. e. reduced part of } \frac{G}{H} \textit{ has no elements of infinite height}\right)$.

LEMMA 10. — *Let G be a p-group without elements of infinite height and let H be a pure subgroup of G. Then*

(1) $(H[p])^- = H^-[p]$;

(2) $H[p] = H^-[p]$ *if and only if* $H = H^-$, *i. e.* $H[p]$ *is closed if and only if H is closed.*

(3) $H^-[p] = G[p]$ *if and only if* $H^- = G$, *i. e.* $H[p]$ *is dense in* $G[p]$ *if and only if H is dense in G.*

Proof.

1. $(H[p])^- \subset (G[p])^- \cap H^- = G[p] \cap H^- = H^-[p]$, by Lemma 6.

Suppose $x \in H^-[p]$. $H \cap (x + p^n G) \neq \Phi$ for all n and $px = 0$. That is, there exist $h_n \in H$ and $g_n \in G$ such that

$$h_n = x + p^n g_n \qquad \text{and} \qquad ph_n = p^{n+1} g_n.$$

Since H is pure, there exists $h'_n \in H$ such that $ph_n = p^{n+1} h'_n$.

$$h_n - p^n h'_n = x + p^n (g_n - h'_n), \qquad \text{where} \quad h_n - p^n h'_n \in H[p].$$

That is, $H[p] \cap (x + p^n G) \neq \Phi$ for all n. Hence $x \in (H[p])^-$.

2. H is pure in H^-, since H is pure in G. By Lemma 12, KAPLANSKY [5],

$$H[p] = H^-[p] \quad \text{implies} \quad H = H^-.$$

3. Suppose $H^-[p] = G[p]$. It suffices to show that $pg \in H^-$ implies $g \in H^-$.

If $pg \in H^-$, then $(pg + p^n G) \cap H \neq \Phi$ for all n. Write

$$h_n = pg + p^n g_n, \qquad \text{where} \quad h_n \in H, \quad g_n \in G.$$

Since H is pure, there exists $h'_n \in H$ such that

$$h_n = ph'_n, \quad g + p^{n-1} g_n - h'_n \in G[p].$$

Since

$$G[p] = H^-[p], \qquad g + p^{n-1} g_n \in H^-,$$

i. e. $(g + p^{n-1} G) \cap H^- \neq \Phi$. Therefore $g \in H^-$.

3. **Main Results.**

The following is a characterization of quasi-closed groups.

THEOREM 1. — *Let G be a reduced p-group. Following three conditions are equivalent :*

(III) *Let H be a pure subgroup of G, then H^- is again pure;*

(III)′ *Let H be a pure subgroup of G, then $\left(\frac{G}{H}\right)^1$ is divisible;*

(IV) *Strong Purification Property.* (*See* Introduction.)

Proof. — (III) $\Leftrightarrow$ (III)′ by Lemma 9.

(III)′ $\Rightarrow$ (IV). Since G is reduced, $G^1 = \{o\}^- = o$ by the condition (III). By Zorn's Lemma there exists a maximal pure subgroup K such that $H \subset K$ and $K[p] \subset P$. Suppose $x \in P$ and $x \notin K[p]$. Let φ be a canonical homomorphism $G \to G/K$. $\varphi(x) \in \frac{G}{K}[p]$ and $\varphi(x) \neq o$. Suppose $h(\varphi(x)) = \infty$. Since $\left(\frac{G}{K}\right)^1$ is divisible, there exists K' containing K such that

$$\frac{K'}{K} \cong Z(p^\infty) \qquad \text{and} \qquad \frac{K'}{K}[p] = \langle \varphi(x) \rangle.$$

K' is pure by Lemma 2, KAPLANSKY [5].

$\varphi(K'[p]) = (\varphi K')[p] = \langle \varphi(x) \rangle$ by Lemma 1, KAPLANSKY [5]. Hence

$$K'[p] = \langle x \rangle \oplus K[p] \subset P.$$

This contradicts to the maximality of K. If $h(\varphi(x)) = n < \infty$. Then we can find K' such that $\frac{K'}{K} = \langle \bar{y} \rangle$, where $\varphi(x) = p^n \bar{y}$. Therefore $K[p] = P$.

(IV) $\Rightarrow$ (III). Since G satisfies socle purification property, $G^1 = 0$. Let H be a pure subgroup of G. By the strong purification property, there exists a pure subgroup K such that $H \subset K$ and $K[p] = H^-[p]$. By Lemma 10, (1) and (2), K is closed. Hence $H^- \subset K$. Since K is pure, we can apply Lemma 10, (3). Therefore $H^- = K$.

DEFINITION. — Let $B = \sum_{n=1}^{\infty} \langle x_n \rangle$. If $o(x_n) = p^n$, B is called a standard group. If $\{ o(x_n) \}$ is a strictly increasing sequence, B is called a substandard group.

THEOREM 2. — *Let B be a substandard group. There exists a pure subgroup H of B such that $\left(\frac{B}{H}\right)^1 \simeq C(p)$, i. e. H^- is not pure.*

Remark. — The fact that B is not quasi-closed follows immediately from Theorem 4 of [2], since B can be decomposed into a direct sum of two unbounded components.

Proof. — Let $B = \sum_{i=1}^{\infty} \langle x_i \rangle$, $o(x_i) = p^{n_i}$, $1 \leqq n_1 < n_2 < n_3 \ldots$. Set

$$y_i = x_{2i} + p^{n_{2i+1} - n_{2i} + 1} x_{2i+1} - p^{n_{2i+2} - n_{2i}} x_{2i+2}.$$

Then $o(y_i) = p^{n_{2i}}$. Let H be a subgroup of B generated by $\{ y_i : i = 1, 2, \ldots \}$. This H is a desired subgroup. We can verify that H is pure and $H^- = \langle p^{n_2 - 1} x_2 \rangle \oplus H$.

Following theorem gives us a characterization of torsion complete groups. There is a direct proof for the corollary to this theorem in [6].

THEOREM 3. — *A reduced p-group G is torsion complete if and only if H^- is a direct summand of G whenever H is a pure subgroup of G.*

Proof. — The necessity follows from Lemma 5.

Suppose that H^- is a direct summand of G whenever H is pure. G is quasi-closed. Let $B = \sum_{n=1}^{\infty} B_n$, where $B_n \simeq \sum C(p^n)$ be a basic subgroup of G. We can decompose B into a direct sum of two unbounded components

$$C_1 = \sum_{n \in N_1} B_n \quad \text{and} \quad C_2 = \sum_{n \in N_2} B_n, \quad \text{where} \quad N_1 \cap N_2 = \Phi.$$

Set $G = \bar{C_1} \oplus K$. K must be unbounded since the basic subgroup of K is isomorphic to C_2. G is torsion complete by Theorem 4, HILL and MEGIBBEN [2].

COROLLARY. — *A reduced p-group G is torsion complete if and only if G satisfies following condition :*

(I) *If* $B = C_1 \oplus C_2$ *is any basic subgroup of G and its decomposition, then* $G = C_1^- \oplus C_2^-$, *where* C_1^- *and* C_2^- *are closures of* C_1 *and* C_2 *in G.*

Remark. — The exercise 16 in KAPLANSKY [5] shows us how a standard group does not satisfy the property (I) in above corollary.

Let $B = \sum_{i=1}^{\infty} \langle x_i \rangle$, where $o(x_i) = p^i$ and let

$$S_0 = \sum_{i=1}^{\infty} \langle y_{2i-1} \rangle \quad \text{and} \quad S_e = \sum_{i=1}^{\infty} \langle y_{2i} \rangle, \quad \text{where} \quad y_i = x_i - px_{i+1}.$$

Then $S = S_0 \oplus S_e$ is a basic subgroup of B. On the other hand S_0 and S_e are the direct summands of B. Hence $S_0^- \oplus S_e^- = S_0 \oplus S_e \neq B$.

BIBLIOGRAPHY.

[1] FUCHS (L.). — *Abelian groups.* — Budapest, Publ. House of the Hungarian Acad. of Sciences, 1958.

[2] HILL (P.) and MEGIBBEN (C.). — Quasi-closed primary groups, *Acta Math. Acad. scient. Hungar.*, t. 16, 1965, p. 271-274.

[3] HILL (P.) and MEGIBBEN (C.). — *On primary groups with countable basic subgroups* (to appear).

[4] IRWIN (J. M.), RICHMAN (F.) and WALKER (E. A.). — Countable direct sums of torsion complete groups, *Proc. Amer. math. Soc.*, t. 17, 1966, p. 763-766.

[5] KAPLANSKY (I.). — *Infinite Abelian groups.* — Ann Arbor, University of Michigan Press, 1954 (University of Michigan, Publications in Mathematics, 2).

[6] KOYAMA (T.). — *Topological methods in Abelian groups.* A Dissertation Wayne State University, 1966.

(Manuscrit reçu le 6 mars 1967.)

Mrs. Toshiko KOYAMA,
Department of Mathematics,
Ochanomizu University,
Ootsuka-machi, Bunkyo-ku,
Tokyo (Japon).