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On quasi-closed groups and torsion complete groups


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ON QUASI-CLOSED GROUPS
AND TORSION COMPLETE GROUPS

BY

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1. Introduction.

Every group, in this paper, is an abelian $p$-group. We will observe some properties of abelian $p$-groups using topological methods. Notation and terminology follow Fuchs [1] except that we use the word "torsion complete" instead of "closed" following [4]. Let $G$ be a $p$-group. Then we can introduce the $p$-adic topology in $G$. If $G$ has no elements of infinite height, this topology is a metric topology.

Let $G$ be a $p$-group without elements of infinite height. If every bounded Cauchy sequence of $G$ has a limit in $G$ with respect to the $p$-adic topology, $G$ is called torsion complete. A torsion complete group $G$ has following properties.

(I) Let $B=C_1 \oplus C_2$ be a basic subgroup of $G$. Then $G=C_1 \oplus C_2$, where $C_1$ and $C_2$ are closures of $C_1$ and $C_2$ in $G$.

(II) Let $H$ be a pure subgroup of $G$. Then $H^-$ is a direct summand of $G$.

(III) Let $H$ be a pure subgroup of $G$. Then $H^-$ is again pure.

(IV) Let $H$ be a pure subgroup of $G$. ($\frac{G}{H}$) is divisible.

(IV) (Strong Purification Property). For a given subgroup $P$ of $G[p]$ and for a given pure subgroup $H$ of $G$ such that $H[p] \subset P$, there exists a pure subgroup $K$ containing $H$ such that $K[p] = P$.

After considering these properties a natural question arises: Is the reduced $p$-group which satisfies (I) or (II) necessarily torsion complete? We will give an affirmative answer to this question. This gives rise to a nice characterization of torsion complete groups.
P. Hill and C. Megibben [2] called the reduced $p$-group which satisfies (III) quasi-closed group. They have showed an example which is quasi-closed but not torsion complete in [2]. They have also proved in [2] that a quasi-closed group which is not torsion complete is essentially indecomposable. We will show that properties (III), (III)' and (IV) are equivalent. That is, a reduced $p$-group is quasi-closed if and only if $G$ satisfies "Strong Purification Property ". Since unbounded direct sum of cyclic groups is neither essentially indecomposable, nor torsion complete, it is not quasi-closed. We will construct a pure subgroup $H$ in unbounded direct sum of cyclic groups such that $H^-$ is not pure.

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2. Topological Preliminaries.

Let $G$ be a $p$-group and $x$ be an element of $G$. $h(x)$ denotes the height of $x$. Set $d(x, y) = p^{-h(x-y)}$ for $x, y \in G$. $d$ defines a pseudo-metric in $G$. Since $d$ is invariant, $G$ is a topological group with this pseudo-metric. This topology is called $p$-adic topology. If we assume the condition $G' = 0$, then $d$ defines a metric in $G$. Let $H$ be a subset of $G$, then we write $H^\sim$ for the closure of $H$ in $G$ with respect to the $p$-adic topology of $G$.

Lemma 1. — $\emptyset = \{ p^n G, n = 0, 1, 2, \ldots \}$ is a local base at $0$ for the $p$-adic topology of $G$. Hence $0 \wedge G$, the $p$-adic topology in a bounded group is discrete and the $p$-adic topology in a divisible group is trivial.

Lemma 2. — If $H$ is a pure subgroup of $G$, then the $p$-adic topology of $H$ coincides with the relative topology, since $p^n H = H \cap p^n G$. Hence we need not distinguish the relative topology and the $p$-adic topology in $H$ whenever $H$ is pure in $G$.

Lemma 3. — Let $G = \sum_{i=1}^{m} G_i$. Then the $p$-adic topology of $G$ is the product of $p$-adic topologies in $G_i$'s. Hence a direct summand of $G$ is closed in $G$ whenever $G_i = 0$.

Lemma 4. — A direct sum of a finite number of torsion complete groups is torsion complete and a direct summand of a torsion complete group is torsion complete.

Lemma 5. — Let $G$ be a torsion complete group and let $B = C_1 \oplus C_2$ be a basic subgroup of $G$. Then $G = C_1 \oplus C_2$. Hence, if $H$ is a pure subgroup of $G$, $H^-$ is a direct summand of $G$. 
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Lemma 6. — \( G[p^n] = \{ x \in G : p^n x = 0 \} \) \((n = 1, 2, 3, \ldots)\) is closed in \( G \) with respect to any compatible Hausdorff topology in \( G \).

Proof. — \( f(x) = p^n x \) is continuous in any topological group, \( \{ 0 \} \) is closed in any Hausdorff topological group and \( G[p^n] \) is the inverse image of \( 0 \) by \( f(x) \). Hence \( G[p^n] \) is closed.

Lemma 7. — Let \( H \) be a subgroup of \( G \). Then \( \left( \frac{G}{H} \right)^{-1} = \frac{H^-}{H} \).

Proof. — Let \( \varphi \) be a canonical homomorphism : \( G \to G/H \). \( h(\varphi(x)) = \infty \) if and only if \( (x + p^n G) \cap H \neq \Phi \) for all \( n \). That is, \( x \in H^- \).

Lemma 8. — A subgroup \( H \) is dense in \( G \) if and only if \( G/H \) is divisible.

Lemma 9. — Let \( H \) be a pure subgroup of \( G \). Then \( H^- \) is pure if and only if \( \left( \frac{G}{H} \right)^{-1} \) is divisible (i.e. reduced part of \( \frac{G}{H} \) has no elements of infinite height).

Lemma 10. — Let \( G \) be a \( p \)-group without elements of infinite height and let \( H \) be a pure subgroup of \( G \). Then

1. \((H[p])^- = H^-[p] \);
2. \( H[p] = H^-[p] \) if and only if \( H = H^- \), i.e. \( H[p] \) is closed if and only if \( H \) is closed.
3. \( H^-[p] = G[p] \) if and only if \( H^- = G \), i.e. \( H[p] \) is dense in \( G[p] \) if and only if \( H \) is dense in \( G \).

Proof.

1. \((H[p])^- \subseteq (G[p])^- \cap H^- = G[p] \cap H^- = H^-[p] \), by Lemma 6.

Suppose \( x \in H^-[p] \). \( H \cap (x + p^n G) \neq \Phi \) for all \( n \) and \( px = 0 \). That is, there exist \( h_n \in H \) and \( g_n \in G \) such that

\[
h_n = x + p^n g_n \quad \text{and} \quad ph_n = p^{n+1} g_n.
\]

Since \( H \) is pure, there exists \( h'_n \in H \) such that \( ph_n = p^{n+1} h'_n \).

\[
h_n - p^n h'_n = x + p^n (g_n - h'_n), \quad \text{where} \quad h_n - p^n h'_n \in H[p].
\]

That is, \( H[p] \cap (x + p^n G) \neq \Phi \) for all \( n \). Hence \( x \in (H[p])^- \).

2. \( H \) is pure in \( H^- \), since \( H \) is pure in \( G \). By Lemma 12, Kaplansky [5],

\[
H[p] = H^-[p] \quad \text{implies} \quad H = H^-.
\]

3. Suppose \( H^-[p] = G[p] \). It suffices to show that \( pg \in H^- \) implies \( g \in H^- \).
If $pg \in H^-$, then $(pg + p^n G) \cap H \neq \Phi$ for all $n$. Write
\[ h_n = pg + p^n g_n, \quad \text{where} \quad h_n \in H, \quad g_n \in G. \]
Since $H$ is pure, there exists $h' \in H$ such that
\[ h_n = ph', \quad g + p^{n-1} g_n = h' \in G[p]. \]
Since
\[ G[p] = H-[p], \quad g + p^{n-1} g_n \in H^-, \]
i.e. $(g + p^{n-1} G) \cap H^- \neq \Phi$. Therefore $g \in H^-.$

3. Main Results.

The following is a characterization of quasi-closed groups.

**Theorem 1.** — *Let $G$ be a reduced $p$-group. Following three conditions are equivalent:

(III) Let $H$ be a pure subgroup of $G$, then $H^-$ is again pure;

(III)' Let $H$ be a pure subgroup of $G$, then \( \left( \frac{G}{H} \right)^1 \) is divisible;

(IV) Strong Purification Property. (See Introduction.)

**Proof.** — (III) $\iff$ (III)' by Lemma 9.

(III)' $\implies$ (IV). Since $G$ is reduced, $G' = 1$ by the condition (III). By Zorn's Lemma there exists a maximal pure subgroup $K$ such that $H \subset K$ and $K[p] \subset P$. Suppose $x \in P$ and $x \in K[p]$. Let $\varphi$ be a canonical homomorphism $G \to G/K$. $\varphi(x) \in \frac{G}{K}[p]$ and $\varphi(x) \neq 0$.

Suppose $h(\varphi(x)) = \infty$. Since $(\frac{G}{K})^1$ is divisible, there exists $K'$ containing $K$ such that
\[ \frac{K'}{K} \cong Z(p^n) \quad \text{and} \quad \frac{K'}{K}[p] = \langle \varphi(x) \rangle. \]

$K'$ is pure by Lemma 2, Kaplansky [5].

$\varphi(K'[p]) = \langle \varphi K' \rangle[p] = \langle \varphi(x) \rangle$ by Lemma 1, Kaplansky [5]. Hence
\[ K'[p] = \langle x \rangle \oplus K[p] \subset P. \]

This contradicts to the maximality of $K$. If $h(\varphi(x)) = n < \infty$. Then we can find $K'$ such that $\frac{K'}{K} = \langle \bar{y} \rangle$, where $\varphi(x) = p^n \bar{y}$. Therefore $K[p] = P$. 

(IV) \Rightarrow (III). Since \( G \) satisfies socle purification property, \( G' = 0 \). Let \( H \) be a pure subgroup of \( G \). By the strong purification property, there exists a pure subgroup \( K \) such that \( H \subset K \) and \( K[p] = H[p] \).

By Lemma 10, (1) and (2), \( K \) is closed. Hence \( H^- \subset K \). Since \( K \) is pure, we can apply Lemma 10, (3). Therefore \( H^- = K \).

**Definition.** — Let \( B = \sum_{n=1}^{\infty} \langle x_n \rangle \). If \( o(x_n) = p^n \), \( B \) is called a standard group. If \( \{ o(x_n) \} \) is a strictly increasing sequence, \( B \) is called a substandard group.

**Theorem 2.** — Let \( B \) be a substandard group. There exists a pure subgroup \( H \) of \( B \) such that \( \left( \frac{B}{H} \right)^{\prime} \cong C(p) \), i.e. \( H^- \) is not pure.

**Remark.** — The fact that \( B \) is not quasi-closed follows immediately from Theorem 4 of [2], since \( B \) can be decomposed into a direct sum of two unbounded components.

**Proof.** — Let \( B = \sum_{i=1}^{\infty} \langle x_i \rangle \), \( o(x_i) = p^{n_i} \), \( 1 \leq n_1 < n_2 < \ldots \). Set

\[
y_i = x_{2i} + p^{n_{i+1} - n_i - 1} x_{2i+1} - p^{n_{i+1} - n_i} x_{2i+2}.
\]

Then \( o(y_i) = p^{n_{i+1}} \). Let \( H \) be a subgroup of \( B \) generated by \( \{ y_i : i = 1, 2, \ldots \} \). This \( H \) is a desired subgroup. We can verify that \( H \) is pure and \( H^- = \langle p^{n_{i+1}} x_{2i} \rangle \oplus H \).

Following theorem gives us a characterization of torsion complete groups. There is a direct proof for the corollary to this theorem in [6].

**Theorem 3.** — A reduced \( p \)-group \( G \) is torsion complete if and only if \( H^- \) is a direct summand of \( G \) whenever \( H \) is a pure subgroup of \( G \).

**Proof.** — The necessity follows from Lemma 5.

Suppose that \( H^- \) is a direct summand of \( G \) whenever \( H \) is pure. \( G \) is quasi-closed. Let \( B = \sum_{n=-1}^{\infty} B_n \), where \( B_n \cong \sum_{p} C(p^n) \) be a basic subgroup of \( G \). We can decompose \( B \) into a direct sum of two unbounded components

\[
C_1 = \sum_{n \in N_1} B_n \quad \text{and} \quad C_2 = \sum_{n \in N_2} B_n, \quad \text{where} \quad N_1 \cap N_2 = \emptyset.
\]

Set \( G = C_1 \oplus K \). \( K \) must be unbounded since the basic subgroup of \( K \) is isomorphic to \( C_2 \). \( G \) is torsion complete by Theorem 4, HILL and MEGIBBEN [2].
COROLLARY. — A reduced $p$-group $G$ is torsion complete if and only if $G$ satisfies following condition:

(I) If $B = C_1 \oplus C_2$ is any basic subgroup of $G$ and its decomposition, then $G = C_1 \oplus C_2$, where $C_1$ and $C_2$ are closures of $C_1$ and $C_2$ in $G$.

Remark. — The exercise 16 in KAPLANSKY [5] shows us how a standard group does not satisfy the property (1) in above corollary.

Let $B = \sum_{i=1}^{\infty} \langle x_i \rangle$, where $o(x_i) = p^i$ and let

$$S_0 = \sum_{i=1}^{\infty} \langle y_{2i-1} \rangle \quad \text{and} \quad S_e = \sum_{i=1}^{\infty} \langle y_{2i} \rangle,$$

where $y_i = x_i - px_{i+1}$.

Then $S = S_0 \oplus S_e$ is a basic subgroup of $B$. On the other hand $S_0$ and $S_e$ are the direct summands of $B$. Hence $S_0 \oplus S_e = S_0 \oplus S_e \neq B$.

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