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On quasi-closed groups and torsion complete groups


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ON QUASI-CLOSED GROUPS
AND TORSION COMPLETE GROUPS

BY

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1. Introduction.

Every group, in this paper, is an abelian p-group. We will observe
some properties of abelian p-groups using topological methods. Notation
and terminology follow Fuchs [1] except that we use the word “torsion
complete” instead of “closed” following [4]. Let G be a p-group.
Then we can introduce the p-adic topology in G. If G has no elements
of infinite height, this topology is a metric topology.

Let G be a p-group without elements of infinite height. If every
bounded Cauchy sequence of G has a limit in G with respect to the
p-adic topology, G is called torsion complete. A torsion complete
group G has following properties.

(I) Let $B = C_1 \oplus C_2$ be a basic subgroup of G. Then $G = C_1 \oplus C_2$, where $C_1$ and $C_2$ are closures of $C_1$ and $C_2$ in G.

(II) Let $H$ be a pure subgroup of G. Then $H^-$ is a direct summand
of $G$.

(III) Let $H$ be a pure subgroup of G. Then $H^-$ is again pure.

(III') Let $H$ be a pure subgroup of G. \( \left( \frac{G}{H} \right) \) is divisible.

(IV) (Strong Purification Property). For a given subgroup $P$ of $G[p]$ and for a given pure subgroup $H$ of $G$ such that $H[p] \subset P$, there exists a pure subgroup $K$ containing $H$ such that $K[p] = P$.

After considering these properties a natural question arises: Is the
reduced p-group which satisfies (I) or (II) necessarily torsion complete?
We will give an affirmative answer to this question. This gives rise
to a nice characterization of torsion complete groups.
P. Hill and C. Megibben [2] called the reduced $p$-group which satisfies (III) quasi-closed group. They have showed an example which is quasi-closed but not torsion complete in [2]. They have also proved in [2] that a quasi-closed group which is not torsion complete is essentially indecomposable. We will show that properties (III), (III)$'$ and (IV) are equivalent. That is, a reduced $p$-group is quasi-closed if and only if $G$ satisfies "Strong Purification Property". Since unbounded direct sum of cyclic groups is neither essentially indecomposable, nor torsion complete, it is not quasi-closed. We will construct a pure subgroup $H$ in unbounded direct sum of cyclic groups such that $H^-$ is not pure.

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2. Topological Preliminaries.

Let $G$ be a $p$-group and $x$ be an element of $G$. $h(x)$ denotes the height of $x$. Set $d(x, y) = p^{-h(x-y)}$ for $x, y \in G$. $d$ defines a pseudo-metric in $G$. Since $d$ is invariant, $G$ is a topological group with this pseudo-metric. This topology is called $p$-adic topology. If we assume the condition $G' = 0$, then $d$ defines a metric in $G$. Let $H$ be a subset of $G$, then we write $H^-$ for the closure of $H$ in $G$ with respect to the $p$-adic topology of $G$.

**Lemma 1.** $\emptyset = \{ p^n G, n = 0, 1, 2, \ldots \}$ is a local base at $0$ for the $p$-adic topology of $G$. Hence $\{ 0 \} = G'$, the $p$-adic topology in a bounded group is discrete and the $p$-adic topology in a divisible group is trivial.

**Lemma 2.** If $H$ is a pure subgroup of $G$, then the $p$-adic topology of $H$ coincides with the relative topology, since $p^n H = H \cap p^n G$. Hence we need not distinguish the relative topology and the $p$-adic topology in $H$ whenever $H$ is pure in $G$.

**Lemma 3.** Let $G = \bigoplus_{i=1}^m G_i$. Then the $p$-adic topology of $G$ is the product of $p$-adic topologies in $G_i$'s. Hence a direct summand of $G$ is closed in $G$ whenever $G^1 = 0$.

**Lemma 4.** A direct sum of a finite number of torsion complete groups is torsion complete and a direct summand of a torsion complete group is torsion complete.

**Lemma 5.** Let $G$ be a torsion complete group and let $B = C_1 \oplus C_2$ be a basic subgroup of $G$. Then $G = C_1 \oplus C_2$. Hence, if $H$ is a pure subgroup of $G$, $H^-$ is a direct summand of $G$. 
Lemma 6. — $G[p^n] = \{ x \in G : p^n x = e \}$; $(n = 1, 2, 3, \ldots)$ is closed in $G$ with respect to any compatible Hausdorff topology in $G$.

Proof. — $f(x) = p^n x$ is continuous in any topological group, $\{ e \}$ is closed in any Hausdorff topological group and $G[p^n]$ is the inverse image of $e$ by $f(x)$. Hence $G[p^n]$ is closed.

Lemma 7. — Let $H$ be a subgroup of $G$. Then $\left( \frac{G}{H} \right)^1 = H^{-}$.

Proof. — Let $\varphi$ be a canonical homomorphism : $G \to G/H$. $h(\varphi(x)) = \infty$ if and only if $(x + p^n G) \cap H \neq \Phi$ for all $n$. That is, $x \in H^{-}$.

Lemma 8. — A subgroup $H$ is dense in $G$ if and only if $G[H]$ is divisible.

Lemma 9. — Let $H$ be a pure subgroup of $G$. Then $H^{-}$ is pure if and only if $\left( \frac{G}{H} \right)^1$ is divisible (i.e. reduced part of $\frac{G}{H}$ has no elements of infinite height).

Lemma 10. — Let $G$ be a $p$-group without elements of infinite height and let $H$ be a pure subgroup of $G$. Then

(1) $(H[p])^{-} = H^{-}[p]$;

(2) $H[p] = H^{-}[p]$ if and only if $H = H^{-}$, i.e. $H[p]$ is closed if and only if $H$ is closed.

(3) $H^{-}[p] = G[p]$ if and only if $H^{-} = G$, i.e. $H[p]$ is dense in $G[p]$ if and only if $H$ is dense in $G$.

Proof.

1. $(H[p])^{-} \subset (G[p])^{-} \cap H^{-} = G[p] \cap H^{-} = H^{-}[p]$, by Lemma 6.

Suppose $x \in H^{-}[p]$. $H \cap (x + p^n G) \neq \Phi$ for all $n$ and $px = e$. That is, there exist $h_n \in H$ and $g_n \in G$ such that

$$h_n = x + p^n g_n \quad \text{and} \quad ph_n = p^{n+1} g_n.$$ 

Since $H$ is pure, there exists $h'_n \in H$ such that $ph_n = p^{n+1} h'_n$.

$$h_n - p^n h'_n = x + p^n (g_n - h'_n), \quad \text{where} \quad h_n - p^n h'_n \in H[p].$$ 

That is, $H[p] \cap (x + p^n G) \neq \Phi$ for all $n$. Hence $x \in (H[p])^{-}$.

2. $H$ is pure in $H^{-}$, since $H$ is pure in $G$. By Lemma 12, Kaplansky [5],

$$H[p] = H^{-}[p] \quad \text{implies} \quad H = H^{-}.$$

If \( pg \in H^- \), then \( (pg + p^n G) \cap H \neq \Phi \) for all \( n \). Write
\[
h_n = pg + p^n g_n, \quad \text{where} \quad h_n \in H, \quad g_n \in G.
\]
Since \( H \) is pure, there exists \( h'_n \in H \) such that
\[
h_n = ph'_n, \quad g + p^{n-1} g_n - h_n' \in G[p].
\]
Since
\[
G[p] = H^-[p], \quad g + p^{n-1} g_n \in H^-,
\]
i.e. \( (g + p^{n-1} G) \cap H^- \neq \Phi \). Therefore \( g \in H^- \).

3. Main Results.

The following is a characterization of quasi-closed groups.

**Theorem 1.** — Let \( G \) be a reduced \( p \)-group. Following three conditions are equivalent:

(I) Let \( H \) be a pure subgroup of \( G \), then \( H^- \) is again pure;

(II) Let \( H \) be a pure subgroup of \( G \), then \( \left( \frac{G}{H} \right) \) is divisible;

(IV) Strong Purification Property. (See Introduction.)

**Proof.** — (III) \( \iff \) (III)' by Lemma 9.

(III)' \( \implies \) (IV). Since \( G \) is reduced, \( G' = \cdot o = o \) by the condition (III). By Zorn’s Lemma there exists a maximal pure subgroup \( K \) such that \( H \subseteq K \) and \( K[p] \subseteq P \). Suppose \( x \in P \) and \( x \notin K[p] \). Let \( \varphi \) be a canonical homomorphism \( G \to G/K \). \( \varphi(x) \in G/K[p] \) and \( \varphi(x) \neq o \).

Suppose \( h(\varphi(x)) = \infty \). Since \( \left( \frac{G}{K} \right) \) is divisible, there exists \( K' \) containing \( K \) such that
\[
\frac{K'}{K} \cong Z(p^n) \quad \text{and} \quad \frac{K'}{K}[p] = \langle \varphi(x) \rangle.
\]

\( K' \) is pure by Lemma 2, Kaplansky [5].

\( \varphi(K'[p]) = (\varphi K')[p] = \langle \varphi(x) \rangle \) by Lemma 1, Kaplansky [5]. Hence
\[
K'[p] = \langle x \rangle \oplus K[p] \subseteq P.
\]

This contradicts to the maximality of \( K \). If \( h(\varphi(x)) = n < \infty \). Then we can find \( K' \) such that \( \frac{K'}{K} = \langle \tilde{y} \rangle \), where \( \varphi(x) = p^n \tilde{y} \). Therefore \( K[p] = P \).
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(IV)⇒(III). Since $G$ satisfies socle purification property, $G^1=0$. Let $H$ be a pure subgroup of $G$. By the strong purification property, there exists a pure subgroup $K$ such that $H\subseteq K$ and $K[p]=H[p]$. By Lemma 10, (1) and (2), $K$ is closed. Hence $H^-\subseteq K$. Since $K$ is pure, we can apply Lemma 10, (3). Therefore $H^-=K$.

**DEFINITION.** Let $B=\bigoplus_{n=1}^{\infty} \langle x_n \rangle$. If $o(x_n)=p^n$, $B$ is called a standard group. If $\{o(x_n)\}$ is a strictly increasing sequence, $B$ is called a substandard group.

**THEOREM 2.** Let $B$ be a substandard group. There exists a pure subgroup $H$ of $B$ such that $B/H\cong C(p)$, i.e. $H^-$ is not pure.

**Remark.** The fact that $B$ is not quasi-closed follows immediately from Theorem 4 of [2], since $B$ can be decomposed into a direct sum of two unbounded components.

**Proof.** Let $B=\bigoplus_{i=1}^{\infty} \langle x_i \rangle$, $o(x_i)=p^{n_i}$, $1\leq n_1<n_2<n_3$. . . . . Set

$$y_i=x_{2i}+p^{n_{i+1}-n_{i}+1}x_{2i+1}+p^{n_{i+1}-n_{i}}x_{2i+2}.$$ 

Then $o(y_i)=p^{n_i}$. Let $H$ be a subgroup of $B$ generated by $\{y_i: i=1, 2, \ldots \}$. This $H$ is a desired subgroup. We can verify that $H$ is pure and $H^-=\langle p^{n_i-1}x_i \rangle \oplus H$.

Following theorem gives us a characterization of torsion complete groups. There is a direct proof for the corollary to this theorem in [6].

**THEOREM 3.** A reduced $p$-group $G$ is torsion complete if and only if $H^-$ is a direct summand of $G$ whenever $H$ is a pure subgroup of $G$.

**Proof.** The necessity follows from Lemma 5.

Suppose that $H^-$ is a direct summand of $G$ whenever $H$ is pure. $G$ is quasi-closed. Let $B=\bigoplus_{n=1}^{\infty} B_n$, where $B_n\cong \sum C(p^n)$ be a basic subgroup of $G$. We can decompose $B$ into a direct sum of two unbounded components

$$C_1=\bigoplus_{n\in N_1} B_n \quad \text{and} \quad C_2=\bigoplus_{n\in N_2} B_n$$

where $N_1\cap N_2=\emptyset$.

Set $G=C_1\oplus K$. $K$ must be unbounded since the basic subgroup of $K$ is isomorphic to $C_2$. $G$ is torsion complete by Theorem 4, HILL and MEGIBBEN [2].
Corollary. — A reduced $p$-group $G$ is torsion complete if and only if $G$ satisfies following condition:

(I) If $B = C_1 \oplus C_2$ is any basic subgroup of $G$ and its decomposition, then $G = C_1 \oplus C_2$, where $C_1$ and $C_2$ are closures of $C_1$ and $C_2$ in $G$.

Remark. — The exercise 16 in Kaplansky [5] shows us how a standard group does not satisfy the property (I) in above corollary.

Let $B = \sum_{i=1}^{\infty} \langle x_i \rangle$, where $o(x_i) = p^i$ and let

\[ S_0 = \sum_{i=1}^{\infty} \langle y_i^{p^i} \rangle \quad \text{and} \quad S_\infty = \sum_{i=1}^{\infty} \langle y_i \rangle, \]

where $y_i = x_i - px_{i+1}$.

Then $S = S_0 \oplus S_\infty$ is a basic subgroup of $B$. On the other hand $S_0$ and $S_\infty$ are the direct summands of $B$. Hence $S_0 \oplus S_\infty = S_0 \oplus S_\infty \neq B$.

BIBLIOGRAPHY.


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