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SEMICONTINUITY OF MULTIPLE INTEGRALS OF THE CALCULUS OF VARIATIONS IN PARAMETRIC FORM

BY

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In this paper, we study functionals of the type

$$\langle T, F \rangle = \int_{\operatorname{spt} T} F\left(x, \vec{T}(x)\right) dH^k(x),$$

where T is a k-dimensional rectifiable current in R^n, $\vec{T}(x)$ is a k-vector associated with T, H^k is k-dimensional Hausdorff measure in R^n, $F\left(x_0 \vec{T}\right)$ is a real valued continuous map in $R^n \times \Lambda_k(R^n)$, positively homogeneous in $\vec{T}$ and quasi convex in the following sense : if $F_0\left(\vec{T}\right) = F\left(x_0, \vec{T}\right)$, the solution of the problem of minimum of $\langle T, F_0 \rangle$, with given planar boundary C is the planar current S with $\partial S = C$. It is then shown that relatively to convergence in Whitney's flat norm, $\langle T, F_0 \rangle$ is a lower semicontinuous functional. Theorems of existence of minimum are then easily obtained.

1. Introduction.

Rectifiable currents were introduced in [3], and subsequently applied to the calculus of variations and area theory. The theory of integral currents has been applied to the problem of Plateau, allowing for powerful existence and continuity theorems ([3], [5]), and to the study of continuous maps of finite area from a compact k-manifold into n-space ([4]). In this paper, we study general variational problems formulated in terms of rectifiable currents, specifically the problem of minimum of multiple integrals in parametric form. By considering integrands which are

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quasi-convex in the sense first introduced by C. B. MORREY, Jr. ([7]), we are able to give a very general semicontinuity theorem when convergence of the rectifiable currents is understood with respect to a norm similar to the one introduced by H. WHITNEY in [8] for polyhedral chains and there called flat norm. This norm was recently used by W. H. FLEMING as the basis of his theory of flat chains over a finite coefficient group ([6]).

The proof of the semi-continuity theorem proceeds by classical arguments, in a way similar to the procedure in [9], § 5, and then existence theorems follow immediately from well known results in the theory of integral currents ([3]).

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2. **Preliminaries.**

We will recall briefly some definitions and results from [3], and establish terminology and notations.

$E^k(U)$ is the real vector space of differential k-forms of class ∞ on the open subset U of R^n, with the topology of uniform convergence on each compact subset of U of each partial derivative of any order. $E_k(U)$ is the space of real valued continuous linear functionals on $E^k(U)$; these are called *k-dimensional currents.* spt T stands for the usual support of the functional $T \in E_k(U)$, and throughout this paper only currents with compact support will be considered.

The *mass* of T is defined as

$$M(T) = \sup \{ T(\varphi) : \varphi \in E^k(U) \text{ and } M(\varphi) \leqq 1 \}$$

where

$$M(\varphi) = \sup \{ \| \varphi_x \| : x \in U \}$$

and one recalls that a differential form of $E^k(U)$ may be regarded as an infinitely differentiable map

$$\varphi : \quad x \in U \to \varphi_x \in \Lambda^k(R^n)$$

where $\Lambda^k(R^n)$ stands for the space of k-covectors of R^n. Then,

$$\| \varphi_x \| = \sup \{ \varphi_x(\gamma) : \gamma \text{ is a simple } k\text{-vector of } R^n \text{ with norm } \leqq 1 \}$$

All the currents in this paper will be supposed with finite mass.

The boundary of T is the $(k-1)$-dimensional current ∂T defined by $\partial T = T \circ d$, where d is the exterior differentiation on $E^k(U)$. Also, if $T \in E_k(U)$ and $\omega \in E^m(U)$, $T \wedge \omega$ is the $(k\text{-}m)$-dimensional current in U defined by

$$T \wedge \omega(\varphi) = T(\omega \wedge \varphi) \quad \text{for all } \varphi \in E^{k-m}(U).$$

A *k-cell* A is the current $\int_A \varphi$, $\varphi \in E^k(R^n)$; finite linear combination of k-cells are called *k-polyhedral chains*.

The C^∞ map $f : R^m \to R^n$ induces a continuous linear transformation

$$f^\# : \quad \oplus_k E^k(R^n) \to \oplus_k E^k(R^m),$$

where $\oplus_k E^k(R^n)$ stands for the direct sum of $E^k(R^n)$, all k. If T if a k-dimensional current of R^m, the image of T by f is the k-dimensionas current of R^n defined by $f_\# T = T \circ f^\#$.

A curent Q is an integral Lipschitzian chain in R^n if it is the image ol a polyhedral chain with integer coefficients in R^m by a Lipschitzian function $f : R^m \to R^n$. The Lipschitzian function f induces a map in the space of polyhedral chains in the following way. Consider C^∞ maps $f_i : R^m \to R^n$ such that $f_1, f_2, f_3, \ldots$ converge to f uniformly on spt T and whose differentials are uniformly bounded on spt T. This guarantees that $\lim_{i\to\infty} f_{i\#}(T)$ exists and is independent of the choice of the f_i's (*see* [3], § 3.5). Then put

$$f_\# T = \lim_{i\to\infty} f_{i\#} T.$$

A k-dimensional current T in R^n is termed *rectifiable* if for every $\varepsilon > 0$ there exists an integral Lipschitzian chain Q of R^n such that $M(T - Q) < \varepsilon$.

With each T we can associate the non-negative measure $\|T\|$ on Borel sets defined by the formula

$$\|T\|(A) = M(T \cap A)$$

where $T \cap A$ stands, in general, for $T \wedge \chi_A$, χ_A being the characteristic function of A. The operation $T \wedge \chi_A$ is defined in the following way. By usual methods, the current T has a unique extension, which we will also denote by T, to the class of all k-forms whose coefficients are bounded Baire functions, and such that a given sequence of forms ω_i, with equally bounded coefficients in spt T, if there is a form ω such that $\lim_{i\to\infty} \omega_i = \omega$, then $T \wedge \omega = \lim_{i\to\infty} T \wedge \omega_i$. In particular, one can define $T \wedge \chi_A$ (*see* [3], § 2.4).

For $\| T \|$ almost every $x \in \mathrm{R}^n$ we can consider the approximate tangent unit k-vector $\vec{\xi}(x)$ associated with T by the formula

$$\varphi(x)\big(\vec{\xi}(x)\big) = \frac{d(T \wedge \varphi)}{d\| T \|}(x) \quad \text{for all } \varphi \in E^k(R^n) \qquad (\textit{see } [3], \S\, 8.8).$$

If γ is a Carathéodory measure over R^n, $A \subset R^n$, $\alpha(k)$ the volume of the unit k-ball, and $x \in R^n$, *the k-dimensional γ density of A at x* is

$$\theta^k(\gamma, A, x) = \lim_{r \to 0+} \alpha(k)^{-1} r^{-k} \gamma(A \cap \{ y : |x - y| < r \}).$$

Then, if T is a k-dimensional rectifiable current in R^n, $\theta^k(\| T \|, R^n, x)$ is an integer for H^k-almost all x in R^n, where H^k denotes k-dimensional Hausdorff measure ([3], § 8.16). We will assume throughout this paper $\theta^k(\| T \|, R^n, x) > 0$, for all the currents under consideration.

We will put

$$\vec{T}(x) = \vec{\xi}(x)\, \theta^k(\| T \|, R^n, x)$$

and consider functionals of the form

$$\langle T, F \rangle = \int_{\operatorname{spt} T} F\big(x, \vec{T}(x)\big)\, dH^k(x)$$

where the admissible integrands are maps

$$F : \quad R^n \times \Lambda_k(R^n) \to R$$

which are continuous, non-negative and positively homogeneous in the second argument. $\Lambda_k(R^n)$ stands for the space of k-vectors of R^n.

3. The semicontinuity theorem.

We will say that F is *quasi-convex* if, putting

$$F_0\big(\vec{T}\big) = F\big(x_0, \vec{T}\big)$$

for each x_0, the solution of the problem of minimizing $\langle T, F_0 \rangle$ with a given planar boundary C is the planar current S with $\partial S = C$.

We will consider for a k-dimensional current T the following norm :

$$W(T) = \inf \{ M(U) + M(V) : T = U + \partial V,\ U \in E_k(R^n),\ V \in E_{k+1}(R^n) \}$$

and we will say that the sequence (T_i) of k-dimensional currents converges in Whitney's sense to the k-dimensional current T if

$$W(T_i - T) \to 0.$$

THEOREM. — *If* (T_i) *is a sequence of k-dimensional rectifiable currents of* R^n*-converging in Whitney's sense to the k-dimensional rectifiable current* T, *then*

$$\langle T, F\rangle \leq \min_{i\to\infty} \lim \langle T_i, F\rangle$$

for all admissible integrands F *that are non negative and quasi-convex.*

Proof. — Since we are assuming that T is rectifiable, at $\| T \|$ almost every x_0 there exists a hyperplane $\pi(x_0)$ tangent to T, in the sense of being the k-space of the k-vector $\vec{\xi}(x_0)$, and by [3], § 8.16, there exists a submanifold $Y(x_0)$ of R^n of class C^1, tangent to $\pi(x_0)$ at x_0 and such that

$$\lim_{r\to 0+} \alpha(k)^{-1} r^{-k} M(T \cap S(x_0, r) - \Theta^k(\| T \|, R^n, x_0)\, Y(x_0) \cap S(x_0, r)) = 0$$

where $S(x_0, r) = \{ x \in R^n : | x - x_0 | < r \}$.

Then, by this remark and by the hypothesis of quasiconvexity for the admissible integrands, we can find, for a given $\varepsilon > 0$, a r_1 such that

$$(1) \qquad \langle T(x_0, r), F_0\rangle - \langle D(x_0, r), F_0\rangle < \frac{\varepsilon}{3} \alpha(k)\, r^k.$$

whenever $r \leq r_1$, where

$$D(x_0, r) = Y(x_0) \cap S(x_0, r) \qquad \text{and} \qquad T(x_0, r) = T \cap S(x_0, r).$$

Consider now currents $U_i \in \mathrm{E}_k(R^n)$, $V_i \in \mathrm{E}_{k+1}(R^n)$ such that $T_i - T = U_i + \partial V_i$, $i = 1, 2, 3, \ldots$ and since $(T_i) \to \mathrm{T}$ in Whitney's sense,

$$\lim_{i\to\infty} (\mathrm{M}(U_i) + M(V_i)) = 0.$$

For each i, we have

$$T_i(x_0, r) - T(x_0, r) = U_i(x_0, r) + (\partial V_i)(x_0, r),$$

where in general $A(x_0, r) = A \cap S(x_0, r)$.

From this we have

$$T_i(x_0, r) - T(x_0, r) = U_i(x_0, r) + \partial(V_i(x_0, r)) - (\partial(V_i(x_0, r)) - (\partial V_i)(x_0, r)).$$

Now, letting p be an orthogonal projection of R^k into $\pi(x_0)$, we can use the same argument as in [3], § 8.12, to get

$$p_\# T_i(x_0, r) - p_\# T(x_0, r) = p_\# U_i(x_0, r) - p_\# [\partial(V_i(x_0, r)) - (\partial V_i)(x_0, r)];$$

in fact $V_i(x_0, r)$ is, as a $(k+1)$-dimensional current in R^k, the zero, current and $p_\# V_i(x_0, r) = 0$ and since ∂ and $p_\#$ commute, the above relation holds.

Then,

$$\int_0^{r_1} M\big(p_\# T_i(x_0, r) - p_\# T(x_0, r)\big)\, dr$$

$$\leq \int_0^{r_1} M\big(p_\# U_i(x_0, r)\big)\, dr + \int_0^{r_1} M\big(p_\#(\partial(V_i(x_0, r)) - (\partial V_i)(x_0, r))\big)\, dr$$

$$\leq \int_0^{r_1} M(U_i(x_0, r))\, dr + \int_0^{r_1} M(\partial(V_i(x_0, r)) - (\partial V_i)(x_0, r))\, dr$$

since projection does not increase mass.

But now, for each i the last integral remains bounded by $r_1 M(U_i) + M(V_i)$. So, applying Fatou's lemma and passing to a subsequence, which we will not distinguish in notation from the original sequence, we have, for sufficiently large indices of this subsequence, the following estimate :

$$M\big(p_\# T_i(x_0, r) - p_\# T(x_0, r)\big) < \frac{\varepsilon}{12K} \alpha(k)\, r^k$$

for μ-almost every $r \leq r_1$, where μ stands for Lebesgue measure on the line and $2K$ is an upper bound for $F_0\big(\overrightarrow{T}(x)\big)$, which is a continuous function on a compact subset of $\Lambda_k(R^n)$.

But then, in view of [3], § 8.16, we have

$$H^k\big(p_\# T_i(x_0, r) - p_\# T(x_0, r)\big) < \frac{\varepsilon}{12K} \alpha(k)\, r^k. \tag{2}$$

Since $\pi(x_0)$ is the tangent hyperplane to T at x_0, there exists a r_2 such that

$$H^k\big(p_\# T(x_0, r) - E(x_0, r)\big) < \frac{\varepsilon}{12K} \alpha(k)\, r^k \tag{3}$$

for $r \leq r_2$.

Hence, we can find a r_0, depending on ε and x_0, such that whenever $r \leq r_0$, and for sufficiently large indices i, we have, in view of (2) and (3), the following estimate

$$H^k\big(p_\# T_i(x_0, r) - E(x_0, r)\big) < \frac{\varepsilon}{6K} \alpha(k)\, r^k. \tag{4}$$

Now, in view of the convergence of (T_i) to T in Whitney's sense, $T_i - T = U_i + \partial V_i$, with $U_i \in E_k(R^n)$, $V_i \in E_{k+1}(R^n)$ and $M(U_i) \to 0$, $M(V_i) \to 0$. Hence, we can add to $T_i(x_0, r)$ a k-dimensional current of small mass, such that the resulting current will have boundary $\partial E(x_0, r)$.

But then, considering the problem of minimizing $\langle T(x_0, r), F_0 \rangle$ with planar boundary $\partial E(x_0, r)$, the hypothesis of quasi-convexity yields

$$\langle E(x_0, r), F_0 \rangle < \langle T_i(x_0, r), F_0 \rangle + \frac{\varepsilon}{6} \alpha(k)\, r^k.$$

Clearly,

$$\langle E(x_0, r), F_0 \rangle > \langle p_{\#} T_i(x_0, r), F_0 \rangle$$

so we get the inequality

$$\langle T_i(x_0, r), F_0 \rangle > \langle p_{\#} T_i(x_0, r), F_0 \rangle - \frac{\varepsilon}{6} \alpha(k) r^k$$

and from this

$$\begin{aligned} &\langle E(x_0, r), F_0 \rangle - \langle T_i(x_0, r), F_0 \rangle \\ &< \langle E(x_0, r), F_0 \rangle - \langle p_{\#} T_i(x_0, r), F_0 \rangle + \frac{\varepsilon}{6} \alpha(k) r^k \end{aligned}$$

But, in view of (4),

$$\langle E(x_0, r), F_0 \rangle - \langle p_{\#} T_i(x_0, r), F_0 \rangle < K \frac{\varepsilon}{6K} \alpha(k) r^k$$

and so we have

$$\langle E(x_0, r), F_0 \rangle - \langle T_i(x_0, r), F_0 \rangle < \frac{\varepsilon}{3} \alpha(k) r^k.$$

Observe also that

$$\lim_{r \to 0+} r^{-k} | \langle E(x_0, r), F_0 \rangle - \langle D(x_0, r), F_0 \rangle | = 0$$

and hence

$$\langle D(x_0, r), F_0 \rangle - \langle T_i(x_0, r), F_0 \rangle < \frac{\varepsilon}{3} \alpha(k) r^k. \tag{5}$$

Then, using (1) we obtain the estimate

$$\langle T(x_0, r), F_0 \rangle - \langle T_i(x_0, r), F_0 \rangle < \frac{2\varepsilon}{3} \alpha(k) r^k. \tag{6}$$

Now, the continuity of the admissible integrands yield the following estimates

$$\begin{aligned} \langle T_i(x_0, r), F \rangle - \langle T_i(x_0, r), F_0 \rangle &< \delta(r) M(T_i(x_0, r)) \\ \langle T(x_0, r), F \rangle - \langle T(x_0, r), F_0 \rangle &< \delta(r) M(T(x_0, r)) \end{aligned}$$

where $\delta(r) \to 0$ as $r \to 0$.

But then

$$\begin{aligned} &\langle T(x_0, r), F \rangle - \langle T_i(x_0, r), F \rangle \\ &\quad < \delta(r) (M(T) - M(T_i)) + \langle T(x_0, r), F_0 \rangle - \langle T_i(x_0, r), F_0 \rangle \end{aligned}$$

and since M is semicontinuous with respect to convergence in Whitney's sense, we can, passing to another subsequence which again we will not distinguish in notation, estimate the first term in the right hand side by $\frac{\varepsilon}{3} \alpha(k) r^k$.

Now using (6) we obtain

$$\langle T(x_0, r), F \rangle - \langle T_i(x_0, r), F \rangle < \varepsilon \alpha(k) r^k \tag{7}$$

for sufficiently large i and almost all $r \leq r_0$; clearly this r_0 depends on x_0.

Then, for each $x \in \operatorname{spt} T$, given $\varepsilon < 0$, we can find $r(x)$ such that for almost all $r \leq r(x)$ we have (7). For a given ρ suppose for each $x \in \operatorname{spt} T$ we have $r(x) < \rho$, which is not restrictive since otherwise we could consider $r'(x) < \min\{\rho, r(x)\}$ in place of $r(x)$ for what follows. Then, each $x \in \operatorname{spt} T$ will be the center of arbitrarily small balls for which (7) holds for a given $\varepsilon < 0$. Using a covering theorem due to BESICOVITCH [1], we can select a sequence of pairs (x_j, r_j) where $x_j \in \operatorname{spt} T$ and $r_j < \rho$ such that

$$S(x_i, r_i) \cap S(x_j, r_j) = \emptyset \qquad \text{with} \quad i \neq j$$

and $\bigcup_j S(x_j, r_j)$ covers H^k-almost all of spt T.

Now we apply the above procedure for the pair (x_1, r_1) getting a sub) sequence for which (7) is valid for almost all $r < r_1$; apply now the same procedure for this subsequence and, relatively to the pair $(x_2, r_2$- obtain another subsequence. Repeating the process, we obtain a subsequence of the original sequence of currents such that

$$\langle T(x_j, r_j), F \rangle - \langle T_i(x_j, r_j), F \rangle < \frac{\varepsilon}{2^j} \alpha(k) r_j^k \tag{8}$$

for almost every real number $r_j < \rho$, and $j = 1, 2, 3, \ldots$.

Now, take a finite number of spheres $S(x_j, r_j)$, $j = 1, 2, \ldots, m$, from the above sequence of spheres, apply (8) and sum, getting

$$\left\langle T \cap \bigcup_{j=1}^{m} S(x_j, r_j), F \right\rangle - \left\langle T_i \cap \bigcup_{=1}^{m} S(x_j, r_j), F \right\rangle < \varepsilon \alpha(k) \rho^k \sum_{j=1}^{m} \frac{1}{2^j}. \tag{9}$$

Since F is non negative we have, for every m,

$$\langle T_i, F \rangle \geq \left\langle T_i \cap \bigcap_{j=1}^{m} S(x_j, r_j), F \right\rangle;$$

now letting $m \to \infty$ in (9), and having in mind that

$$H^k\left(\operatorname{spt} T - \bigcup S(x_j, r_j)\right) = 0$$

we have

$$\langle T, F \rangle \leq \min \lim_{i \to \infty} \langle T_i, F \rangle$$

which finishes the proof.

4. Existence theorem.

The above theorem allows for a general existence theorem. Before we give this theorem, let us recall some more definitions and results from [3].

An integral current is a current T such that both T and ∂T are rectifiable. Calling *norm* of T the number

$$N(T) = M(T) + M(\partial T)$$

we will call *normal* a current T such that $N(T) < +\infty$.

Integral currents are rather smooth, in the sense that they are limit, in convergence in the above N-norm, of a sequence of currents $f_{i\#}(P_i)$, where P_i are polyhedral chains with integer coefficients and f_i ar diffeomorphisms of class 1 converging to the identity map ([3], § 8.22). Call $I_k(U)$ and $N_k(U)$ respectively the class of integral and normal k-dimensional currents on $U \subset R^n$.

From [3], § 8.13, we have the following compactness theorem :

If A is a compact subset of R^n and c is a positive number, $I_k(A) \cap \{ T : N(T) \leq c \}$ is compact.

This theorem, allied to the semicontinuity theorem given in § 3, and the fact that every normal rectifiable current is integral ([3], § 8.14), yields the following :

THEOREM. — *The problem of minimizing $\langle T, F \rangle$ admits a solution in the class of all k-dimensional rectifiable currents which have equibounded norm, provided the admissible integrands are non-negative and quasi-convex.*

5. Remarks.

The same proof given in § 3 for the semicontinuity theorem applies for flat chains over a finite coefficient group [6]. Also, this result generalizes a previous one, in the case $k = n - 1$, obtained in the setting of the theory of sets of finite perimeter, convergence in the Lebesgue measure of symmetric difference and F a norm [2].

In the case $F(x, \vec{T}(x)) = |\vec{T}(x)|$, which corresponds to the problem of minimal surfaces, existence theorems, as well as some regularity theorems, have been given in [3] and [5].

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