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A property of A-sequences


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A PROPERTY OF $A$-SEQUENCES

BY

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Let $A$ be a noetherian local ring with maximal ideal $m$, containing a field $k$ (not necessarily its residue field). Recall ([1]; [7]) that an $A$-sequence is a finite set $x_1, \ldots, x_r$ of elements of $A$, contained in the maximal ideal $m$, such that $x_i$ is not a zero-divisor in $A$, and for each $i = 2, \ldots, r$, $x_i$ is not a zero-divisor in $A/(x_1, \ldots, x_{i-1})$. We will show that for many purposes, the elements of an $A$-sequence behave just like the variables in a polynomial ring over a field. In particular, the sum, product, intersection and quotient of ideals generated by monomials in a given $A$-sequence are just what one would expect (see Corollary 1 below for a precise statement).

**Proposition 1.** — Let $A$ be a noetherian local ring containing a field $k$, and let $x_1, \ldots, x_r$ be an $A$-sequence. Then the natural map

$$
\varphi : \ T = k[X_1, \ldots, X_r] \rightarrow A
$$

of $k$-algebras, which sends $X_i$ into $x_i$ for each $i$, is injective, and $A$ is flat as a $T$-module.

**Proof.** — We show $\varphi$ is injective by induction on $r$, the case $r = 0$ being trivial. Let $r > 0$ be given. Then $x_1, \ldots, x_r$ is an $(A/A)$-sequence, so by the induction hypothesis, we may assume that

$$
\overline{\varphi} : \ k[X_1, \ldots, X_r] \rightarrow A/x_1A
$$

is injective. Now let $t \in T$ be given and write

$$
t = \sum_{n=0}^{\infty} X_1^n f_n(X_2, \ldots, X_r),
$$

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where each $f_s(X_1, \ldots, X_r) \in k[X_1, \ldots, X_r]$. Suppose that $\varphi(t) = 0$. If $t \neq 0$, let $f_s$ be the first of the $f_n$ which is non-zero. Then

$$\varphi(t) = x_1^i \left( \sum_{n=0}^\infty x_1^{n-s} f_n(x_2, \ldots, x_r) \right).$$

Since $x_1$ is a non-zero-divisor in $A$, we have

$$\sum_{n=0}^\infty x_1^{n-s} f_n(x_2, \ldots, x_r) = 0.$$

Reducing modulo $x_1$, we find $f_s(x_2, \ldots, x_r) = 0$ in $A/x_1 A$. Now since $\overline{\varphi}$ is injective by the induction hypothesis, $f_s(X_1, \ldots, X_r) = 0$, which is a contradiction. Hence $t = 0$ and $\varphi$ is injective.

Now to show $A$ is flat over $T$, we use the local criterion of flatness ([3], chap. III, § 5, no 2, theorem 1, (iii)) applied to the ring $T$, the ideal $J = (x_1, \ldots, x_r)$, and the $T$-module $A$. We must verify the four following statements:

(a) $T$ is noetherian (well-known).

(b) $A$ is separated for the $J$-adic topology, i.e. $\bigcap J^n A = 0$. This is true since $JA$ is contained in the radical $m$ of $A$, and $\bigcap m^n = 0$ by Krull’s theorem ([3], chap. III, § 3, no 2).

(c) $A/JA$ is flat over $k = T/J$. This is true since anything is flat over a field.

(d) $\operatorname{Tor}^T_i (T/J, A) = 0$. To calculate this Tor, we use the Koszul complex $K.(X_1, \ldots, X_r; T)$ ([4], EGA, III, 1.1) which is a resolution of $T/J$ since $X_1, \ldots, X_r$ is a $T$-sequence. $\operatorname{Tor}_i (T/J, A)$ is the $i$th homology group of the complex

$$K.(X_1, \ldots, X_r; T) \otimes_T A = K.(x_1, \ldots, x_r; A).$$

But since $x_1, \ldots, x_r$ is an $A$-sequence, this homology is zero in degrees $i > 0$ ([4], EGA, III, 1.1.4). In particular $\operatorname{Tor}^T_1 (T/J, A) = 0$, which completes the proof of the proposition.

**Corollary 1.** — With the notations of the proposition, let $a$ and $b$ be any two ideals in $T$. For any ideal $c$ in $T$, denote by $c A$ its extension to $A$. Then

(i) $(a + b) A = a A + b A$;

(ii) $(a . b) A = (a A) . (b A)$;

(iii) $(a \cap b) A = (a A) \cap (b A)$;

(iv) $(a : b) A = (a A) : (b A)$.

(Recall that for any two ideals $a, b$ in a ring $R$, $a : b = \{ x \in R \mid x.b \subseteq a \}$.)
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Proof. — (i) and (ii) are trivially true for any ring extension and are repeated here for convenience. (iii) and (iv) are true for any flat ring extension. (iii) is proved in ([3], chap. I, § 2, n° 6, Prop. 6).

To prove (iv), let $y_1, \ldots, y_r$ be a set of generators for $b$. Then

$$a : b = \bigcap (a : (y_i)),$$

and so using (iii) we are reduced to the case where $b$ is generated by a single element $y$. Now $a : (y)$ is characterized by the exact sequence of $T$-modules

$$0 \to a : (y) \to T \to T/a,$$

where the last map is multiplication by $y$. Tensoring with $A$ we have an exact sequence of $A$-modules

$$0 \to (a : (y)) A \to A \to A/aA$$

from which we deduce that $(a : (y)) A = aA : yA$ (Note that for any ideal $b$ in $T$, the natural map $b \otimes_T A \to bA$ is an isomorphism, since $A$ is flat over $T$, so we identify the two).

**Corollary 2 (Theorem of Rees).** — *Let $A$ be a noetherian local ring containing a field, and let $I$ be an ideal generated by an $A$-sequence $x_1, \ldots, x_r$. Then the images $\bar{x}_1, \ldots, \bar{x}_r$ of the $x_i$ in the graded ring

$$\text{gr}_x(A) = \sum_{n=0}^{\infty} J^n/J^{n+1}$$

are algebraically independent, so that $\text{gr}_x(A)$ is isomorphic to the polynomial ring $A/J[x_1, \ldots, x_r]$.*

**Proof (see also [7], Appendix 6, theorem 3).** — It is sufficient to show that for each $n$, $J^n/J^{n+1}$ is a free $A/J$-module, with the images of the monomials in $x_1, \ldots, x_r$ of degree $n$ for basis. It is clear that these monomials generate $J^n/J^{n+1}$. To show they are linearly independent, let $z$ be a monomial of degree $n$ in $x_1, \ldots, x_r$, and let $J'$ be the ideal generated by all the other monomials of degree $n$ and by $J^{n+1}$. Then we must show that $J' : z = J$, which follows from Corollary 1.

**Corollary 3.** — *Let $A$ be a noetherian local ring containing a field $k$, and let $x_1, \ldots, x_r$ be an $A$-sequence. Then any ideal of $A$ generated by polynomials in the $x_i$, with coefficients in $k$, is of finite homological dimension over $A$.*

**Proof.** — Using the notations of the proposition, any such ideal can be written as $\alpha A$, where $\alpha$ is an ideal in the polynomial ring $T = k[x_1, \ldots, x_r]$. Over $T$, $\alpha$ has a finite projective resolution ([$7$, chap. VII, § 13, theorem 43])

$$0 \to L_n \to \ldots \to L_1 \to L_0 \to \alpha \to 0.$$
Tensoring with $A$ gives an exact sequence

$$0 \rightarrow L_n \otimes A \rightarrow \ldots \rightarrow L_1 \otimes A \rightarrow L_0 \otimes A \rightarrow aA \rightarrow 0$$

which is a finite projective resolution of $aA$.

**Remark.** — A refinement of the proof of proposition 1 due to D. Quillen allows one to dispense with the hypothesis that $A$ contains a field, provided that one is interested only in ideals of $A$ generated by monic monomials in the $x_i$. In particular this is sufficient for the result of Corollary 2, and of Proposition 2 below.

As an application we give the following:

**Proposition 2.** — Let $A$ be a noetherian local ring containing a field. Let $I$ be a radical ideal in $A$ (i.e. an ideal which is a finite intersection of prime ideals), and let $J$ be any ideal generated by an $A$-sequence whose radical is $I$. Then, to within isomorphism, the $A/I$-module

$$M = \text{Hom}_A(A/I, A/J)$$

is independent of $J$.

**Example.** — An interesting case (already known [2]) is that of a local Cohen-Macaulay ring $A$, with $I = \mathfrak{m}$ the maximal ideal. Then there are ideals $J$ generated by an $A$-sequence with radical $\mathfrak{m}$, so that $M$ is defined. Its dimension as an $A/\mathfrak{m}$-vector space is an invariant of $A$, which is equal to 1 if and only if $A$ is a Gorenstein ring. (See [2], where if $n$ is the dimension of $M$, then $A$ is called a $MCn$-ring. This number is also the "vordere Loewysche Invariante" of $A/J$ in [6], p. 28, and is the number $e$ of the exercises in [5], § 4, p. 67.)

**Proof of Proposition.** — Let $J$ be generated by the $A$-sequence $x_1, \ldots, x_r$. Then $r$ is the height of $I$, and so is independent of $J$. We consider the $r$th local cohomology group (see [5] for definition and methods of calculation)

$$H = H^r(A) = \varprojlim \text{Ext}^r(A/J^{[n]}, A),$$

where $J^{[n]} = (x_1^n, \ldots, x_r^n)$. Using the Koszul complex $K. (x_1^n, \ldots, x_r^n; A)$ to calculate the $\text{Ext}^r$, we find an isomorphism

$$\varphi_n : \text{Ext}^r(A/J^{[n]}, A) \cong A/J^{[n]}$$

which transforms the maps of the direct system into the maps

$$f_n : A/J^{[n]} \rightarrow A/J^{[n+1]}$$

which are defined by multiplication by $x_1 \cdots x_r$.

I claim that the maps $f_n$ are all injective. Indeed, it is sufficient to see that

$$J^{[n+1]} : (x_1 \cdots x_r) = J^{[n]}.$$
This follows from Corollary 1 and the fact that the analogous relation holds in a polynomial ring. Therefore we can write $H$ as an increasing union

$$H = \bigcup_{n=1}^{\infty} E_n,$$

where $E_n$ is the isomorphic image of $A/J^{(n)}$ in $H$. Furthermore, I claim that for each $n$, $E_n$ is the set of elements of $H$ annihilated by $J^{(n)}$. Indeed, we have only to observe that for each $n$, $k > 0$,

$$J^{(n+k)} : J^{(n)} = (x_1 \cdots x_r)^k$$

which follows from Corollary 1 and the analogous formula in a polynomial ring. Now since $J \subseteq I$, anything in $H$ annihilated by $I$ is annihilated by $J$. Hence

$$M = \text{Hom}_A(A/I, A/I) = \text{Hom}_A(A/I, E_n) = \text{Hom}_A(A/I, H).$$

But by definition, $H$ depends only on the radical of $J$ [5], so we are done.

BIBLIOGRAPHY.


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