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F. RICHMAN
C.P. WALKER

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ON A CERTAIN PURIFICATION PROBLEM FOR PRIMARY ABELIAN GROUPS

BY

FRED RICHMAN AND CAROL P. WALKER.

1. **Introduction.** — MITCHELL has shown in [4] that if G is an abelian p-group and K is a neat subgroup of $G^1 = \bigcap nG$ then there exists a pure subgroup P of G such that $P \cap G^1 = K$. He then raises the question whether the converse holds, i. e. if P is pure in G is $P \cap G^1$ neat in G^1? This question is one of the important family of questions dealing with purification. The general purification problem is to ascertain precisely which subgroups of a subgroup A of an abelian p-group G are the intersections of A with a pure subgroup of G. It is the purpose of this note to solve the purification problem for $A = G^1$.

Terminology and notation will not deviate sharply from [1]. All groups are abelian p-groups. Cardinal numbers are identified with the least ordinal number of that cardinality.

2. **Quasi-neatness, high subgroups and the main theorem.** — A subgroup K of a group G is *neat* if $pG \cap K = pK$. In any event $pG \cap K \supseteq pK$. If K is not neat in G the quotient $(pG \cap K)/pK$ gives some measure as to how neat K is in G. If α is a cardinal number, we shall say that K is *α-quasi-neat* in G if $|(pG \cap K)/pK| \leqq \alpha$.

Recall that a high subgroup of G is a subgroup which is maximal with respect to disjointness from G^1 [2]. Since two high subgroups of G are pure with the same socle in G/G^1 they have the same final rank. We can now state the main theorem of this note.

THEOREM. — *Let G be an abelian p-group, K a subgroup of G^1 and α the final rank of a high subgroup of G. There exists a pure subgroup P of G such that $P \cap G^1 = K \Leftrightarrow K$ is α-quasi-neat in G^1.*

In the sequel K will be a subgroup of G^1, H will be a high subgroup of G, and α will be the final rank of H. The phrase "can purify K" will signify that there exists a pure subgroup P of G such that $P \cap G^1 = K$.

3. **The dirty work.** — We make the first simplification.

LEMMA 1. — *Can purify* $K \Leftrightarrow$*. There exists a* $P \subseteq G$ *such that* $P^1 = P \cap G^1 = K$.

Proof. $\Rightarrow$ Clear.

$\Leftarrow$ Choose a maximal such P. We shall show that P is pure. Suppose $p^n g = x \in P$ for some $g \in G$ and some positive integer n. By induction on n, we show that $x \in p^n P$. If $g \notin P$, then $p^t g + y = g_1 \in G^1 - K$ for some $y \in P$ and non-negative integer $t < n$ by the maximality of P Therefore $y = p^t z$ for some $z \in P$ by induction. Multiplying by p^{n-t} we get $x + p^n z \in G^1$ and so $x + p^n z \in P^1$ by hypothesis. Thus $x \in p^n P$ as claimed.

Bounded summands often make no difference. This is the case in our endeavors.

LEMMA 2. — *Let* $G = A \oplus B$ *where* B *is bounded. Can purify* K *in* $G \Leftrightarrow$ *can purify* K *in* A.

Proof. $\Leftarrow$ Trivial.

$\Rightarrow$ Let P purify K in G. Then $(P \cap A)^1 = K = (P \cap A) \cap G^1$ and we are done by Lemma 1.

Half of the theorem is now relatively painless.

LEMMA 3. — *Can purify* $K \Rightarrow |(pG^1 \cap K)/pK| \leq \alpha =$ *final rank of* H.

Proof. — Using Lemma 2 to chop off a bounded piece of G, we may assume that the final rank of H is the rank of H. Suppose that $|(pG^1 \cap K)/pK| = \delta > \alpha$ and P purifies K. Let $\{x_i\}$ be a set of elements of $pG^1 \cap K$ independent mod pK and indexed by a set I of cardinal δ. There exist $y_i \in P$ such that $py_i = x_i$. Now $x_i = pg_i$ for some $g_i \in G^1$. Thus

$$y_i - g_i \in G[p] = G^1[p] \oplus H[p].$$

By adjusting g_i, we may assume that $y_i - g_i \in H[p]$. Therefore there exist indices $i \neq j$ such that $y_i - g_i = y_j - g_j$ since rank $H < \delta$. Hence

$$p(y_i - y_j) = x_i - x_j \notin pK$$

and so $y_i - y_j \notin K$. But $y_i - y_j = g_i - g_j \in G^1$ and $y_i - y_j \in P$ and so $y_i - y_j$ is in K, a contradiction.

For the other half of the theorem, it is convenient to reduce the problem to direct sums of cyclic groups.

LEMMA 4. — *Let* B *be a basic subgroup of* K. *Then*

$$(pG^1 \cap K)/pK \cong (pG^1 \cap B)/pB$$

and K *can be purified if* B *can.*

Proof. — The isomorphism is clear. Let P purify B. Then $G/P = D \oplus T$ where D, the image of K, is divisible. The inverse image of D purifies K.

To prove the next lemma, we use the high subgroup to escort elements out of G^1.

LEMMA 5. — *Let K a direct sum of cyclic groups contained in G^1 such that $|K| \leqq \alpha =$ final rank of H. Then there exists a subgroup P of G such that $|P| \leqq \alpha$ and $P^1 = P \cap G^1 = K$.*

Proof. — Well order the cyclic generators of K by $\{k_\beta\}_{\beta<\alpha}$. Let $p^n k_\beta^n = k_\beta$, n a positive integer. *Claim* : There exist $h_\beta^n \in H$, $\beta < \alpha$ n a positive integer such that :

(i) order of $(k_\beta^n + h_\beta^n + G^1) = p^n$;

(ii) $\{k_\beta^n + h_\beta^n + G^1\}$ are independent, $\beta < \alpha$, n a positive integer.

To see this, well order the pairs (β, n) by α, and use transfinite induction There is clearly no trouble at limit ordinals. To advance one step, we note that there are α possible h_β^n at our disposal which will satisfy (i) and which yield distinct $p^{n-1}(k_\beta^n + h_\beta^n + G^1)$ since the final rank of H is α and $H \cap G^1 = 0$. But there are less than α things for $p^{n-1}(k_\beta^n + h_\beta^n + G^1)$ to avoid to insure (ii). Letting P be generated by $\{k_\beta^n + h_\beta^n\}_{(\beta, n)<\alpha}$ brings us home.

We have reduced the problem to K a direct sum of cyclics. A further reduction allows us to assume that $K[p] = G^1[p]$. This follows upon writing $G^1[p] = K[p] \oplus L$ and replacing G by a subgroup S containing $H \oplus K$ and maximal with respect to disjointness from L. The subgroup S is pure in G ([3], Theorem 5) and so $K \subseteq S^1$. Clearly $S^1[p] = K[p]$ and H is high in S. Since purifying K in S will purify K in G, we have achieved the desired reduction.

We now take care of the elements that need no escort and so finish off the other half of the theorem.

LEMMA 6. — *Let K be a direct sum of cyclic groups contained in G^1 such that $K[p] = G^1[p]$ and $|(pG^1 \cap K)/pK| \leqq \alpha =$ final rank of H. Then there exists a P in G such that $P^1 = P \cap G^1 = K$.*

Proof. — Let $|K| = \gamma$. If $\gamma \leqq \alpha$, we are done by Lemma 5. Let A be generated by those cyclic summands of K (relative to a given decomposition) for which some element of $pG^1 \cap K$ has a height-0 coordinate. From the hypothesis, it is easily seen that $|A| \leqq \alpha$. Let B be generated by the remaining cyclic summands of K. By Lemma 5, wa can find a subgroup Q of G such that $Q^1 = Q \cap G^1 = A$.

Claim : There exists a subgroup C of G^1 such that $A \subseteq C$, $|C| \leqq \alpha$ and $C + B = G^1$. It will suffice to show that $|(G^1/B)[p]| = |A[p]|$ for then $|G^1/B| \leqq \alpha$, and we let C be generated by A and representatives of G'/B. But if $p(x+B)=0$, $x \in G'$, then $px \in B$ and so $px = pb$ for some $b \in B$ by the construction of B. Thus

$$x - b \in G'[p] = A[p] \oplus B[p]$$

and hence $x + B = a + B$ for some $a \in A[p]$.

Now let the cyclic generators of B be $\{b_\beta\}_{\beta<\gamma}$. *Claim* : There exist $b_\beta^n \in G$, $\beta < \gamma$, n a positive integer such that :

(1) $p^n b_\beta^n = b_\beta$;

(2) $\left(Q + \sum \{b_\beta^n\}\right) \cap C \subseteq K$.

We prove this by induction on (β, n) well ordered by γ. Again, there is no trouble at limit ordinals. To advance one step, we note that there exist γ elements which satisfy (1) with pairwise intersection $\{b_\beta\}$, e. g. alter an element z such that $p^n z = b_\beta$ by elements g such that $p^{n-1} g \in G'[p]$. That the g yield the required elements is assured by the fact that $p^{n-1} z \notin G'$ and that $|G'[p]| = \gamma$. To show that (2) is preserved upon adjoining one of these elements z we need only worry about $p^j z$ where $j < n$, since $Q \cap G' = A$. But we can insure that for some such z, $p^j z \notin Q + C$ for all $j < n$ since we have γ such z with all $p^j z$ distinct, for $j < n$, and $|Q + C| \leqq \alpha$.

Finally, let $P = \left(Q + \sum \{b_\beta^n\}\right)$ and all is well.

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Fred RICHMAN and Carol P. WALKER,
Dept. of Mathematical Sciences,
College of Arts and Sciences,
New Mexico State University,
University Park, N. M. 88 070 (États-Unis).