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The general form of residuated algebraic structures

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THE GENERAL FORM OF RESIDUATED ALGEBRAIC STRUCTURES;

BY

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INTRODUCTION. — This paper consists basically of the results given in the first chapter of the author's thesis [1]. However, it also contains some supplementary results; in particular, we resolve the unsolved problem cited in [1] and give a generalisation of a recent result by McFADDEN [2].

We begin with the following necessary overture of general definitions and known results.

DEFINITIONS. — By an *ordered groupoid* we shall mean simply a set $\mathcal{G}$ endowed with a closed binary operation (multiplication) and a partial ordering ($\leq$) with respect to which the multiplication is *isotone* [i. e., $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$, $\forall z \in \mathcal{G}$]. The ordered goupoid $\mathcal{G}$ is said to be *residuated on the right* (*left*) if, given any two elements $a, b \in \mathcal{G}$, the set of elements $x \in \mathcal{G}$ satisfying $ax \leq b$ ($xa \leq b$) is not empty and has a maximum element, denoted by $b \,.\!\cdot\, a$ ($b \,\cdot\!.\, a$) and called the *right* (*left*) *residual of b by a*. The ordered groupoid $\mathcal{G}$ is said to be *residuated* if it is residuated on the right and on the left.

In a residuated groupoid $\mathcal{G}$ the following properties are easily shown to hold (*see*, for example, MOLINARO [3]) :

$$(a) \qquad a \leq b \Rightarrow \begin{cases} a \,.\!\cdot\, x \leq b \,.\!\cdot\, x \\ a \,\cdot\!.\, x \leq b \,\cdot\!.\, x \end{cases} \text{ and } \begin{cases} x \,.\!\cdot\, b \leq x \,.\!\cdot\, a, \\ x \,\cdot\!.\, b \leq x \,\cdot\!.\, a, \end{cases} \qquad \forall x \in \mathcal{G};$$

$$(b) \qquad \begin{Bmatrix} x(a \,.\!\cdot\, x) \leq a \\ (a \,\cdot\!.\, x)\, x \leq a \end{Bmatrix} \text{ with equality if, and only if, } \begin{cases} a = x\mu, \\ a = \mu x; \end{cases}$$

$$(c) \qquad \begin{Bmatrix} a \leq xa \,.\!\cdot\, x \\ a \leq ax \,\cdot\!.\, x \end{Bmatrix} \text{ with equality if, and only if, } \begin{cases} a = \mu \,.\!\cdot\, x, \\ a = \mu \,\cdot\!.\, x; \end{cases}$$

$$(d) \qquad \begin{Bmatrix} a \leq x \,.\!\cdot\, (x \,\cdot\!.\, a) \\ a \leq x \,\cdot\!.\, (x \,.\!\cdot\, a) \end{Bmatrix} \text{ with equality if, and only if, } \begin{cases} a = x \,.\!\cdot\, \mu, \\ a = x \,\cdot\!.\, \mu; \end{cases}$$

(*e*) the following three equalities are equivalent and are necessary and sufficient for $\mathcal{G}$ to be a semi-group :

(i) $(a .\cdot b) .\cdot c = a .\cdot bc, \quad \forall a, b, c \in \mathcal{G},$

(ii) $(a \cdot. b) \cdot. c = a \cdot. cb, \quad \forall a, b, c \in \mathcal{G},$

(iii) $(a \cdot. b) .\cdot c = (a .\cdot c) \cdot. b, \quad \forall a, b, c \in \mathcal{G}.$

Three types of equivalence introduced by MOLINARO [3] play an important rôle in the study of residuated structures; these are :

(α) *Equivalences of type A*, defined by

$$a \equiv b\ (\mathrm{A}_x) \iff x .\cdot a = x .\cdot b,$$
$$a \equiv b\ ({}_x\mathrm{A}) \iff x \cdot. a = x \cdot. b.$$

These equivalences have the following properties : the classes are convex; if $\mathcal{G}$ is $\cup$-semi-reticulated (*see* M. L. DUBREIL-JACOTIN [4], p. 128), A_x and ${}_xA$ are compatible with union; A_x and ${}_xA$ are strongly upper regular (*see* [4], p. 178); for any $\alpha \in \mathcal{G}$ the element $\bar{\alpha} = x \cdot. (x .\cdot \alpha)$ [resp. $\bar{\alpha} = x .\cdot (x \cdot. \alpha)$] is congruent to α modulo A_x [${}_xA$] and is the maximum element in the class of α; A_x and ${}_xA$ are closure equivalences. Whenever $\mathcal{G}$ is a semi-group, A_x [${}_xA$] is regular on the right [left] with respect to multiplication and $A_x \subseteq A_{x \cdot. \mu}$ [${}_xA \subseteq {}_{x .\cdot \mu}A$], $\forall x, \mu \in \mathcal{G}$.

(β) *Equivalences of type B*, defined by

$$a \equiv b\ (B_x) \iff a .\cdot x = b .\cdot x,$$
$$a \equiv b\ ({}_xB) \iff a \cdot. x = b \cdot. x.$$

These equivalences have the following properties: the classes are convex; B_x and ${}_xB$ are compatible with intersection (if it exists in $\mathcal{G}$); B_x and ${}_xB$ are strongly lower regular ([4], p. 180); for any $\beta \in \mathcal{G}$ the element $\bar{\beta} = x(\beta .\cdot x)$ [resp. $\bar{\beta} = (\beta \cdot. x)x$] is congruent to β modulo B_x [${}_xB$] and is the minimum element in the class of β; B_x and ${}_xB$ are anti-closure equivalences. Whenever $\mathcal{G}$ is a semi-group, B_x [${}_xB$] is regular on the right with respect to residuation on the left [right] and $B_x \subseteq B_{x\mu}$ [${}_xB \subseteq {}_{\mu x}B$], $\forall x, \mu \in \mathcal{G}$.

(γ) *Equivalences of type F*, defined by

$$a \equiv b\ (F_x) \iff xa = xb,$$
$$a \equiv b\ ({}_xF) \iff ax = bx.$$

These equivalences have the following properties : the classes are convex; F_x and ${}_xF$ are compatible with union if $\mathcal{G}$ is $\cup$-semi-reticulated; F_x and ${}_xF$ are strongly upper regular; for any $\nu \in \mathcal{G}$ the element $\bar{\nu} = x\nu .\cdot x$ [resp. $\bar{\nu} = \nu x \cdot. x$] is congruent to ν modulo F_x [${}_xF$] and is the maximum element in the class of ν; F_x and ${}_xF$ are closure equivalences. Whenever $\mathcal{G}$ is a semi-group, F_x [${}_xF$] is regular on the right [left] with respect to multiplication and

$$F_x \subseteq F_{\mu x}, \quad F_x \subseteq A_{\mu .\cdot x} \quad [{}_xF \subseteq {}_{x\mu}F,\ {}_xF \subseteq {}_{\mu \cdot. x}A] \quad \forall x, \mu \in \mathcal{G}.$$

1. — Consider now the following remarks :

(*a*) There exist partially ordered sets on which no multiplication can be defined such that they be residuated [for example, $\mathcal{T} = \{ a, b, c \}$ with $b < a$, $c < a$, $b \parallel c$ ([1])].

(*b*) There exist partially ordered sets on which several multiplications can be defined such that they be residuated [for example, $\mathcal{T} = \{ a, b, c \}$ ordered by $c < b < a$ can be endowed with 13 different multiplications in order that it be residuated].

(*c*) There exist groupoids which, even though they can be ordered in an isotone manner, can nevertheless not be residuated [for example, $\mathcal{G} = \{ a, b \}$ with $aa = ab = a$, $ba = bb = b$ and $b < a$].

(*d*) There exist groupoids which can be ordered in several ways such that they be residuated [for example, $\mathcal{G} = \{ a, b, c, d \}$ with $xy = d$, $\forall x, y \in \mathcal{G}$, can be ordered in 12 different ways in order that it be residuated].

This being the case, we ask the following question : *regarded as a partially ordered set, what is the general form of a residuated structure?*

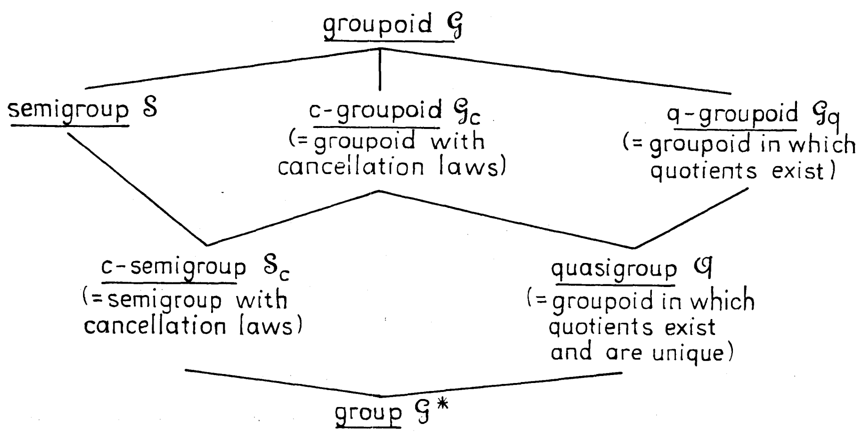


Fig. 1.

In seeking an answer to this question, we shall impose several restrictions on the multiplication and see, in this way, the different forms permissible for each structure. We shall in fact be concerned with the structures represented in the diagram of figure 1 and we shall give examples of each type occurring.

([1]) We use the notation $x \parallel y$ to denote $x \not\leq y$ and $x \not\geq y$ (x not comparable to y); similarly, $x \nparallel y$ will denote $x \leq y$ or $x \geq y$ (x comparable to y).

2. — Now we know that any partially ordered set can be represented by a *Hasse diagram*, the interpretation of which is : $b < a$ if, and only if, b can be joined to a in the diagram by an increasing line segment. Such a diagram is that of figure 2.

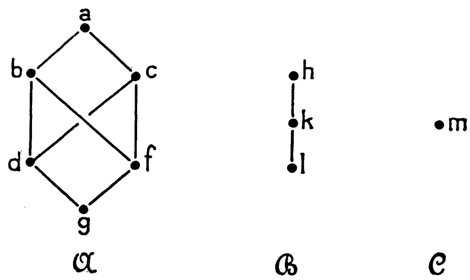


Fig. 2.

In general, such a diagram will consist of several mutually disjoint parts (for example $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of figure 2). We shall be particularly interested in these, and characterise them in the following useful way.

Let $\mathcal{T}$ be any partially ordered set. Define an equivalence $\mathcal{R}$ in $\mathcal{T}$ as follows :

> $a \equiv b$ $(\mathcal{R})$ if, and only if, there exists a finite number of elements $a = a_1, a_2, \ldots, a_n = b$ of $\mathcal{T}$ such that $a_i \between a_{i+1}$ for $i = 1, 2, \ldots, n-1$.

This binary relation is easily seen to be an equivalence relation and $\mathcal{T}$ is the set-theoretic union of the classes of $\mathcal{T}/\mathcal{R}$, say $\mathcal{T} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \ldots$, these classes being nothing else than the disjoint portions in question.

Suppose now that $\mathcal{G}$ is an ordered groupoid, which we shall not necessarily suppose to be commutative; we have :

THEOREM 1. — *The equivalence $\mathcal{R}$ is compatible multiplication.*

Let $a_1 \equiv a_2$ $(\mathcal{R})$; then a_1 and a_2 are connected by at least one finite zig-zag chain, say

$$a_1 = x_1 \between x_2 \between x_3 \between \ldots \between x_n = a_2.$$

By the isotone property of multiplication we then have, for all $b \in \mathcal{G}$,

$$a_1 b = x_1 b \between x_2 b \between \ldots \between x_n b = a_2 b,$$

whence $a_1 b \equiv a_2 b$ $(\mathcal{R})$, $\forall b \in \mathcal{G}$. The equivalence $\mathcal{R}$ is therefore regular on the right with respect to multiplication. In a similar way, it can be shown that $\mathcal{R}$ is regular on the left with respect to multiplication; $\mathcal{R}$ is thus compatible.

THEOREM 2. — *If the ordered groupoid $\mathcal{G}$ is residuated, then $\mathcal{R}$ is compatible with residuation.*

Let $a \equiv a^\star\ (\mathcal{R})$ and $b \equiv b^\star\ (\mathcal{R})$; then there exist $a_i (i = 1, 2, \ldots, n)$ and $b_j (j = 1, 2, \ldots, m)$ such that

$$a = a_1 \pitchfork a_2 \pitchfork a_3 \pitchfork \ldots \pitchfork a_n = a^\star,$$
$$b = b_1 \pitchfork b_2 \pitchfork b_3 \pitchfork \ldots \pitchfork b_m = b^\star.$$

It follows from the property (a) mentioned in the introduction that (for residuals on the right, for example)

$$a \,.\!\cdot\, b = a_1 \,.\!\cdot\, b \pitchfork a_2 \,.\!\cdot\, b \pitchfork \ldots \pitchfork a_n \,.\!\cdot\, b = a^\star \,.\!\cdot\, b,$$
$$a^\star \,.\!\cdot\, b = a^\star \,.\!\cdot\, b_1 \pitchfork a^\star \,.\!\cdot\, b_2 \pitchfork \ldots \pitchfork a^\star \,.\!\cdot\, b_m = a^\star \,.\!\cdot\, b^\star,$$

whence we have $a \,.\!\cdot\, b \equiv a^\star \,.\!\cdot\, b^\star (\mathcal{G})$. A similar proof holds for left residuals.

THEOREM 3. — *If the ordered groupoid $\mathcal{G}$ is residuated then the groupoid $\mathcal{G}/\mathcal{R}$ is a quasi-group homomorphic to $\mathcal{G}$.*

Since $\mathcal{G}$ is residuated, the general relations $a(b \,.\!\cdot\, a) \leqq b$ and $(b \,\cdot\!.\, a)\, a \leqq b$ imply that $a(b \,.\!\cdot\, a) \equiv b(\mathcal{R})$ and $(b \,\cdot\!.\, a)\, a \equiv b(\mathcal{R})$, so that, if $\mathcal{A}$ denotes the class of a modulo $\mathcal{R}$ and $\mathcal{B}$ denotes the class of b modulo $\mathcal{R}$, the class $\mathcal{X}$ of $b \,.\!\cdot\, a$ modulo $\mathcal{R}$ satisfies $\mathcal{A}\mathcal{X} = \mathcal{B}$ and the class $\mathcal{Y}$ of $b \,\cdot\!.\, a$ modulo $\mathcal{R}$ satisfies $\mathcal{Y}\mathcal{A} = \mathcal{B}$. It follows that quotients exist in $\mathcal{G}/\mathcal{R}$; we now show that these quotients are unique. Consider $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ $\mathcal{D} \in \mathcal{G}/\mathcal{R}$ such that $\mathcal{A}\mathcal{B} = \mathcal{A}\mathcal{C} = \mathcal{D}$ and let us show that $\mathcal{B} = \mathcal{C}$. Since $\mathcal{G}$ is residuated, $d \,.\!\cdot\, a$ exists, $\forall d \in \mathcal{D}$, $\forall a \in \mathcal{A}$, and by theorem 2 all these residuals belong to the same class modulo $\mathcal{R}$. Let this class be $\mathcal{I}$ and consider the product $ab = d$ where $a \in \mathcal{A}$, $b \in \mathcal{B}$, $d \in \mathcal{D}$. We have $b \leqq d \,.\!\cdot\, a \in \mathcal{I}$. But $b \in \mathcal{B}$; hence $\mathcal{B} = \mathcal{I}$ since $\mathcal{R}$ is an equivalence relation. Considering in a similar way the product $ac = d^\star$, where $a \in \mathcal{A}$, $c \in \mathcal{C}$, $d^\star \in \mathcal{D}$, an analogous proof shows that also $\mathcal{C} = \mathcal{I}$. We have thus established that the equation $\mathcal{A}\mathcal{X} = \mathcal{B}$ has a unique solution; in a similar way, so also does $\mathcal{Y}\mathcal{A} = \mathcal{B}$. $\mathcal{G}/\mathcal{R}$ is thus a quasi-group. Finally, it is clear that the mapping $\psi : a \to \mathcal{A}$, where $\mathcal{A}$ denotes the class of a modulo $\mathcal{R}$, is a homomorphism.

COROLLARY. — *In order that $\mathcal{G}/\mathcal{R}$ be a loop, it is necessary and sufficient that $a \,.\!\cdot \equiv b \,\cdot\!.\, b\ (\mathcal{R})$, $\forall a, b \in \mathcal{G}$.*

3. — In what follows, $\mathcal{G}$ will denote a (not necessarily commutative) residuated groupoid; moreover, the following notation will be used :

$$\hat{x} = \{ y \in \mathcal{G};\ y \leqq x \}, \qquad \check{x} = \{ z \in \mathcal{G};\ z \geqq x \}.$$

LEMMA 1. — *If $\mathcal{G}$ contains a maximal element $\bar{a}$ then each element $a_\iota \in \mathcal{G}$ which is such that $\hat{a}_\iota \Cap \hat{\bar{a}} \neq \emptyset$ is necessarily less than or equal to $\bar{a}$.*

Let $\mathcal{A}$ be the class modulo $\mathcal{R}$ of the elements $\bar a$, a_t and let $\mathcal{B} = \mathcal{A}\mathcal{A}$. For any element $a \in \hat a_t \cap \hat{\hat{a}}$, we have

$$(\alpha) \qquad b \mathbin{\cdot .} \bar a \leqq b \mathbin{\cdot .} a \quad \text{and} \quad b \mathbin{\cdot .} a_t \leqq b \mathbin{\cdot .} a, \qquad \forall\, b \in \mathcal{B}.$$

Considering therefore the element $b^\star = \bar a . \bar a$ we have, since $\bar a$ is maximal in $\mathcal{A}$,

$$(\beta) \qquad \bar a = b^\star \mathbin{\cdot .} \bar a = b^\star \mathbin{\cdot .} a.$$

From (α) and (β) it then follows that $b^\star \mathbin{\cdot .} a_t \leqq b^\star \mathbin{\cdot .} \bar a$, whence $(b^\star \mathbin{\cdot .} a_t)\, \bar a \leqq b^\star$ and so, $\bar a$ being maximal, we have

$$\bar a = b^\star \mathbin{. \cdot} (b^\star \mathbin{\cdot .} a_t) \geqq a_t.$$

LEMMA 2. — *If $\mathcal{G}$ contains a maximal element then that element is maximum in its class modulo $\mathcal{R}$.*

Let $\bar a$ be maximal in $\mathcal{A}$ and consider any element $a^\star \in \mathcal{A}$ other than $\bar a$ Suppose that there exists, in any finite zig-zag chain connecting $\bar a$ to $a^\star$, a first element, a_k say, which satisfies $a_k \notin \hat{a}$; then we have necessarily that $\hat a_k \cap \hat{\hat{a}} \neq \emptyset$, since the element preceding a_k in the chain belongs to this set. Applying lemma 1, we thus have $a_k \leqq \bar a$, contrary to the hypothesis. It follows that, in any finite zig-zag chain connecting $\bar a$ to $a^\star$, there is no first element which is not less than or equal to a. Consequently, all elements in any such chain are less than or equal to $\bar a$, and in particular, $a^\star \leqq \bar a$. Since $a^\star$ is an arbitrary element of $\mathcal{A}$, we conclude that $\bar a$ is maximum in $\mathcal{A}$.

COROLLARY. — *If $\mathcal{A} \in \mathcal{G}/\mathcal{R}$ contains a maximal element, then $\mathcal{A}$ contains no ascending chain which is unbounded above.*

LEMMA 3. — *If $\mathcal{G}$ contains a descending chain which is unbounded below then each class modulo $\mathcal{R}$ contains at least one such chain.*

Let $a_1 \geqq a_2 \geqq a_3 \geqq \dots$ denote the descending chain, unbounded below, in the class $\mathcal{A}$. Let $\mathcal{B}$ be any class of $\mathcal{G}/\mathcal{R}$; then there exists one (and only one) class $\mathcal{C}$ of $\mathcal{G}/\mathcal{R}$ such that $\mathcal{C}\mathcal{B} = \mathcal{A}$ and for all $c \in \mathcal{C}$, we have $a_1 \mathbin{. \cdot} c \geqq a_2 \mathbin{. \cdot} c \geqq a_3 \mathbin{. \cdot} c \geqq \dots$. Let $b_i = a_i \mathbin{. \cdot} c$ and let us show that the chain $b_i \geqq b_{i+1}$ ($i = 1, 2, 3, \dots$) is unbounded below : suppose in fact that there existed $b \in \mathcal{B}$ such that $b \leqq b_n$, $\forall n$; then we would have $cb \leqq a_n$, $\forall n$, and the chain $a_i \geqq a_{i+1}$ ($i = 1, 2, 3, \dots$) would be bounded below (by cb), contrary to the hypothesis. It follows that $\mathcal{B}$ contains a descending chain which is unbounded below, and since $\mathcal{B}$ is arbitrary the same is true for all classes modulo $\mathcal{R}$.

LEMMA 4. — *If $\mathcal{G}$ contains an ascending chain which is unbounded above then each class modulo $\mathcal{R}$ contains an ascending chain unbounded above and a descending chain unbounded below.*

Let $a_1 \leqq a_2 \leqq a_3 \leqq \dots$ be the ascending chain unbounded above in the class $\mathcal{A}$ modulo $\mathcal{R}$. Let $\mathcal{C}$ be any class of $\mathcal{G}/\mathcal{R}$; then there exists one (and only one) class $\mathcal{B}$ modulo $\mathcal{R}$ such that $\mathcal{B}\mathcal{A} = \mathcal{C}$ and for all $b \in \mathcal{B}$, we have $ba_1 \leq ba_2 \leq ba_3 \leq \ldots$. Let $c_i = ba_i$ and let us show that the chain $c_i \leq c_{i+1}$ is unbounded above : suppose in fact that there existed $c \in \mathcal{C}$ such that $c_n \leq c$, $\forall n$; then we would have $ba_n \leq c$, $\forall n$, whence $a_n \leq c \,.\!\cdot\, b$, $\forall n$, and the chain $a_i \leq a_{i+1}$ would be bounded above by $c \,.\!\cdot\, b$, contrary to the hypothesis. It follows that every class modulo $\mathcal{R}$ contains an ascending chain unbounded above.

Moreover, for each class $\mathcal{C}$ there exists one (and only one) class $\mathcal{J}$ such that $\mathcal{A}\mathcal{J} = \mathcal{C}$ and, for all $c \in \mathcal{C}$, we have that $c \,.\!\cdot\, a_1 \geq c \,.\!\cdot\, a_2 \geq c \,.\!\cdot\, a_3 \geq \ldots$. Defining $f_i = c \,.\!\cdot\, a_i$, we have that the chain $f_i \geq f_{i+1}$ $(i = 1, 2, 3, \ldots)$ is not bounded below [for if it were, there would exist $f \leq c \,.\!\cdot\, a_n$, $\forall n$, whence $a_n \leq c \cdot\!. f$, $\forall n$, and the chain $a_i \leq a_{i+1}$ would be bounded above]. Since $\mathcal{J}$ has a descending chain which is unbounded below, the result follows from lemma 3.

Lemma 5. — *If $\mathcal{G}$ contains a maximal element then each class modulo $\mathcal{R}$ contains a maximum element.*

Suppose that $\bar{a}$ is maximal in the class $\mathcal{A}$ modulo $\mathcal{R}$; then by lemma 2, $\bar{a}$ is maximum in $\mathcal{A}$. It then follows by the corollary to lemma 2 and by lemma 4 that no class modulo $\mathcal{R}$ can contain an ascending chain which is unbounded above. By Zorn's lemma, each class therefore contains a maximal element which, by virtue of lemma 2, is necessarily maximum.

Lemma 6. — *If $\mathcal{G}$ contains a minimal element then every class modulo $\mathcal{R}$ is an upper directed set.*

Suppose that $\underline{x}$ be minimal in $\mathcal{G}$ and consider first of all any two elements a_ι, $a_{\iota'}$ of $\mathcal{G}$ such that $\hat{a}_\iota \cap \hat{a}_{\iota'} \neq \emptyset$. We know that there exists $y (= \underline{x} \,.\!\cdot\, a_\iota)$ such that $a_\iota y = \underline{x}$. Let therefore $a \in \hat{a}_\iota \cap \hat{a}_{\iota'}$; by the isotone property, and remembering that $\underline{x}$ is minimal, we have

$$\underline{x} = a_\iota y = ay \leq a_{\iota'} y,$$

whence $a_\iota \leq a_{\iota'} y \cdot\!. y$. But we know that $a_{\iota'} \leq a_{\iota'} y \cdot\!. y$; it follows that $\check{a}_\iota \cap \check{a}_{\iota'} \neq \emptyset$.

This being the case, let $\mathcal{B}$ be any class modulo $\mathcal{R}$ and let b_ι, $b_{\iota'}$ be any two elements in $\mathcal{B}$. By the definition of $\mathcal{R}$ there exists a finite zig-zag chain

$$b_\iota = b_1 \not\parallel b_2 \not\parallel b_3 \not\parallel \ldots \not\parallel b_n = b_{\iota'}.$$

Now amongst these elements there is a finite number, N say, of elements b_r such that $b_{r-1} \leq b_r$ and $b_{r+1} \leq b_r$. Denoting such elements by $b_{\bar{r}}$, we consider the finite sequence

$$b_{\bar{1}} \parallel b_{\bar{2}} \parallel b_{\bar{3}} \parallel \ldots \parallel b_{\bar{N}}.$$

Now from the definition of $b_{\bar{r}}$, we have $\hat{b}_{\bar{1}} \cap \hat{b}_{\bar{2}} \neq \emptyset$. The result in the first paragraph above therefore gives $\check{b}_{\bar{1}} \cap \check{b}_{\bar{2}} \neq \emptyset$. Consider therefore $b_{1^*} \in \check{b}_{\bar{1}} \cap \check{b}_{\bar{2}}$; since $\hat{b}_{\bar{2}} \cap \hat{b}_{\bar{3}} \neq \emptyset$, we have that $\hat{b}_{1^*} \cap \hat{b}_{\bar{3}} \neq \emptyset$ so that, by the above result, $\check{b}_{1\star} \Cap \check{b}_{\bar{3}} \neq \emptyset$ and consequently $\check{b}_{\bar{1}} \Cap \check{b}_{\bar{3}} \neq \emptyset$. Consider now $b_{2\star} \in \check{b}_{\bar{1}} \Cap \check{b}_{\bar{3}}$; since $\hat{b}_{\bar{3}} \Cap \hat{b}_{\bar{4}} \neq \emptyset$, we have $\hat{b}_{2\star} \Cap \hat{b}_{\bar{4}} \neq \emptyset$ so that $\check{b}_{2\star} \Cap \check{b}_{\bar{4}} \neq \emptyset$ and consequently $\check{b}_{\bar{1}} \Cap \check{b}_{\bar{4}} \neq \emptyset$. Consider $\check{b}_{3\star} \in \check{b}_{\bar{1}} \Cap \check{b}_{\bar{4}}$, etc. After a finite number (in fact $N - 1$) applications of this process, we arrive at $\check{b}_{\bar{1}} \Cap \check{b}_{\bar{N}} \neq \emptyset$. Since therefore $b_\iota \leqq b_{\bar{1}}$ and $b_{\iota'} \leqq b_{\bar{N}}$, it follows that $\check{b}_\iota \Cap \check{b}_{\iota'} \neq \emptyset$ and the proof is thus complete.

LEMMA 7. — *If $\mathcal{G}$ contains a minimal element then that element is minimum in its class modulo $\mathcal{R}$.*

Suppose that $\underline{a}$ be minimal in the class $\mathcal{A}$ modulo $\mathcal{R}$, and let x be any other element of $\mathcal{A}$. Since $\mathcal{A}$ is an upper directed set (lemma 6) there exists $z \in \check{x} \Cap \check{\underline{a}}$, and since there exists an element $t (= \underline{a} \cdot . z)$ such that $tz = \underline{a}$, we have $tx = t\underline{a} = \underline{a}$. In other words, for any $x \in \mathcal{A}$ there exists an element, which we shall denote by t_x, such that $t_x x = \underline{a}$ and $t_x \leqq \underline{a} \cdot . \underline{a}$.

Consider now any element $y \in \mathcal{A}$; since $\mathcal{G}$ is residuated, there exists $x \in \mathcal{A}$ such that $(\underline{a} \cdot . \underline{a})x \leqq y$. It follows by the isotone property that $px \leqq y$ for all $p \leqq \underline{a} \cdot . \underline{a}$, and in particular that $\underline{a} = t_x x \leqq y$. Since y was chosen arbitrarily in $\mathcal{A}$, it follows that $\underline{a}$ is the minimum element of $\mathcal{A}$.

LEMMA 8. — *If $\mathcal{G}$ contains a minimal element the every class modulo $\mathcal{R}$ contains a maximum element and a minimum element.*

If $\underline{a} \in \mathcal{A}$ is minimal in $\mathcal{G}$ then $\underline{a}$ is minimum in $\mathcal{A}$ by virtue of the preceding result. It follows that $\mathcal{A}$ contains no descending chains which are unbounded below and so, by lemma 4, no class modulo $\mathcal{R}$ contains an ascending chain which is unbounded above (since the existence of such a chain would imply that $\mathcal{A}$ had a descending chain unbounded below). In other words, all ascending chains in $\mathcal{G}$ are bounded above. It follows by Zorn's lemma that each class modulo $\mathcal{R}$ contains maximal elements and so, by lemma 2, we can assert that each class has a maximum element.

Moreover, since $\underline{a}$ is minimum in $\mathcal{A}$, it follows by lemma 3 that no class modulo $\mathcal{R}$ can contain descending chains which are unbounded below. Consequently, each class contains minimal elements. The proof is then completed by appealing to lemma 7.

The results of the previous lemmata thus give us the general form of residuated groupoids which we resume in the following :

THEOREM 4. — *If the ordered groupoid $\mathcal{G}$ is residuated, then each class modulo $\mathcal{R}$ contains either :*

1. *a maximum element and a minimum element,*

or 2. *a maximum element and no minimal elements,*

or 3. *no maximal elements and no minimal elements.*

We now give examples of each of the above types.

EXAMPLE $\mathcal{G}_1$. — Consider the groupoid whose Hasse diagram is that of figure 2, and whose multiplication table is as follows :

		$\mathcal{A}$						$\mathcal{B}$			$\mathcal{C}$
	$\bullet$	a	b	c	d	f	g	h	k	l	m
$\mathcal{A}$	a	d	d	g	g	g	g	k	k	l	m
	b	d	d	g	g	g	g	k	k	l	m
	c	g	g	g	g	g	g	k	k	l	m
	d	g	g	g	g	g	g	k	k	l	m
	f	g	g	g	g	g	g	k	k	l	m
	g	g	g	g	g	g	g	l	l	l	m
$\mathcal{B}$	h	h	h	h	h	h	l	m	m	m	g
	k	h	h	h	h	h	l	m	m	m	g
	l	l	l	l	l	l	l	m	m	m	g
$\mathcal{C}$	m	m	m	m	m	m	m	g	g	g	l

Since $ah = k$ and $ha = h$, we have that $\mathcal{G}_1$ is non-commutative; moreover, it is not a semi-group since $a(ha) = ah = k$ and $(ah)a = ka = h$ and neither do quotients exist for all pairs of elements nor do the cancellation laws hold. The multiplication is, however, isotone and $\mathcal{G}_1$ is residuated. The table of left residuals, for example, is the following :

$\cdot .$	a	b	c	d	f	g	h	k	l	m
a	a	a	a	a	a	a	m	m	m	h
b	a	a	a	a	a	a	m	m	m	h
c	a	a	a	a	a	a	m	m	m	h
d	a	a	a	a	a	a	m	m	m	h
f	c	c	a	a	a	a	m	m	m	h
g	c	c	a	a	a	a	m	m	m	h
h	h	h	h	h	h	h	a	a	a	m
k	l	l	l	l	l	h	g	g	a	m
l	l	l	l	l	l	h	g	g	a	m
m	m	m	m	m	m	m	h	h	h	a

and the table of right residuals is the same, except that we have $k \cdot\!\!\cdot \{a, b, c, d, f\} = h$.

EXAMPLE $\mathcal{G}_2$. — Consider the partially ordered set defined by

$$\begin{cases} \mathcal{T} = \{ a_i^\lambda, b_j^\lambda; \quad \lambda = 0, 1, \ldots, 5; \quad i, j, = 1, 2, 3, \ldots \}, \\ a_i^\lambda \leq b_j^\mu \leq a_k^\nu \quad \Leftrightarrow \quad \lambda = \mu = \nu, \quad i > j \geq k. \end{cases}$$

Let us endow $\mathcal{T}$ with the following multiplication :

(*a*) choose as the groupoid $\mathcal{T}/\mathcal{R}$ the following quasi-group :

(★)

	$\mathcal{C}^0$	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^3$	$\mathcal{C}^4$	$\mathcal{C}^5$
$\mathcal{C}^0$	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^4$	$\mathcal{C}^5$	$\mathcal{C}^0$	$\mathcal{C}^3$
$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^5$	$\mathcal{C}^0$	$\mathcal{C}^4$	$\mathcal{C}^3$	$\mathcal{C}^1$
$\mathcal{C}^2$	$\mathcal{C}^4$	$\mathcal{C}^0$	$\mathcal{C}^3$	$\mathcal{C}^2$	$\mathcal{C}^1$	$\mathcal{C}^5$
$\mathcal{C}^3$	$\mathcal{C}^5$	$\mathcal{C}^3$	$\mathcal{C}^1$	$\mathcal{C}^0$	$\mathcal{C}^4$	$\mathcal{C}^2$
$\mathcal{C}^4$	$\mathcal{C}^0$	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^3$	$\mathcal{C}^5$	$\mathcal{C}^4$
$\mathcal{C}^5$	$\mathcal{C}^3$	$\mathcal{C}^4$	$\mathcal{C}^5$	$\mathcal{C}^1$	$\mathcal{C}^2$	$\mathcal{C}^0$

(*b*) for the multiplication of the elements of $\mathcal{T}$, define

$$\begin{cases} a_i^\lambda . a_j^\mu = a_i^\lambda . b_j^\mu = a_{ij}^{[\lambda\mu]}, \\ b_i^\lambda . a_j^\mu = b_i^\lambda . b_j^\mu = b_{ij}^{[\lambda\mu]}, \end{cases}$$

where $\mathcal{C}^{[\lambda\mu]}$ is the product $\mathcal{C}^\lambda \mathcal{C}^\mu$ as determined by the table (★).

It is an easy matter to verify that the multiplication is isotone and that the ordered groupoid so constructed is not a semi-group, that quotients do not exist for all pairs of elements, and that the cancellation laws do not hold. $\mathcal{G}_2$ is in fact residuated; we have the following formulae

$$\begin{cases} a_i^\lambda .\cdot a_j^\mu = a_i^\lambda .\cdot b_j^\mu = b_i^\lambda .\cdot b_j^\mu = a_{\{i/j\}}^{[\lambda .\cdot \mu]}, \\ b_i^\lambda .\cdot a_j^\mu = a_{\{i/j\}+1}^{[\lambda .\cdot \mu]}; \end{cases}$$

$$\begin{cases} a_i^\lambda \cdot. a_j^\mu = a_i^\lambda \cdot. b_j^\mu = a_{\{i/j\}}^{[\lambda \cdot. \mu]}, \\ b_i^\lambda \cdot. a_j^\mu = b_i^\lambda \cdot. b_j^\mu = b_{\{i/j\}}^{[\lambda \cdot. \mu]}; \end{cases}$$

where $\{ i/j \}$ denotes the integer N such that $N - 1 < i/j \leq N$, $\mathcal{C}^{[\lambda .\cdot \mu]}$ denotes the class $\mathcal{C}^\rho$ such that $\mathcal{C}^\mu \mathcal{C}^\rho = \mathcal{C}^\lambda$, and $\mathcal{C}^{[\lambda \cdot. \mu]}$ the class $\mathcal{C}^\sigma$ such that $\mathcal{C}^\sigma \mathcal{C}^\mu = \mathcal{C}^\lambda$.

EXAMPLE $\mathcal{G}_3$. — Consider the partially ordered set defined by

$$\begin{cases} \mathcal{T} = \{ (x, y)^\lambda; \quad \lambda = 0, 1, \ldots, 5; \quad x, y \text{ integers} \leq 0 \}, \\ (x, y)^\lambda \leq (x', y')^\mu \quad \Leftrightarrow \quad \lambda = \mu, \quad x \leq x', \quad y \leq y'. \end{cases}$$

Endow $\mathcal{T}$ with the following multiplication :

(*a*) as the groupoid $\mathcal{T}/\mathcal{R}$ choose the quasigroup (★) of example $\mathcal{G}_2$.

(*b*) for the elements of $\mathcal{T}$, define

$$(x, y)^\lambda . (u, v)^\mu = (\min(x, u), y + v^\star)^{[\lambda\mu]},$$

where $[\lambda\mu]$ is given as in example $\mathcal{G}_2$ and $v^\star$ denotes the greatest multiple of a fixed integer $N > 1$, given in advance, which is less than or equal to v i. e., $v^\star = kN \leqq v < (k+1)N$.

As in example $\mathcal{G}_2$, it is easily seen that this multiplication is non-commutative and non-associative; moreover, quotients do not exist and the cancellation laws do not hold. This multiplication is, however, isotone and using the following three easily-established properties

$$d^\star = (N - 1 + d^\star)^\star; \qquad b^\star + d^\star = (b + d^\star)^\star; \qquad b \leqq b^\star + N - 1$$

it is easy to show that $\mathcal{G}_3$ is residuated with the following formulae

$$\begin{cases} (c, d)^\lambda \cdot . (a, b)^\mu = \begin{cases} (0, d - b^\star)^{[\lambda \cdot . \mu]} & \text{if } a \leqq c, \\ (c, d - b^\star)^{[\lambda \cdot . \mu]} & \text{if } a > c; \end{cases} \\ (c, d)^\lambda . \cdot (a, b)^\mu = \begin{cases} (0, (d - b)^\star + N - 1)^{[\lambda . \cdot \mu]} & \text{if } a \leqq c, \\ (c, (d - b)^\star + N - 1)^{[\lambda . \cdot \mu]} & \text{if } a > c. \end{cases} \end{cases}$$

LEMMA 9. — *The fundamental equivalences of types A, B and F are finer than the equivalence* $\mathcal{R}$.

In fact, we have that

$$\begin{aligned} a \equiv b(A_x) &\Rightarrow x . \cdot a = x . \cdot b \\ &\Rightarrow a \leqq x \cdot . (x . \cdot a) = x \cdot . (x . \cdot b) \geqq b \Rightarrow a \equiv b \ (\mathcal{R}). \end{aligned}$$

Similarly,

$$\begin{aligned} a \equiv b(B_x) &\Rightarrow a . \cdot x = b . \cdot x \\ &\Rightarrow a \geqq x(a . \cdot x) = x(b . \cdot x) \leqq b \Rightarrow a \equiv b \ (\mathcal{R}). \end{aligned}$$

Finally,

$$\begin{aligned} a \equiv b(F_x) \Rightarrow xa = xb &\Rightarrow \mathcal{X}\mathcal{A} = \mathcal{X}\mathcal{B} \text{ (theorem 1)} \\ &\Rightarrow \mathcal{A} = \mathcal{B} \text{ (theorem 3)} \Rightarrow a \equiv b \ (\mathcal{R}). \end{aligned}$$

If now $\mathcal{H}$ is any equivalence relation defined on $\mathcal{G}$ and $\mathcal{A}$ is any class modulo $\mathcal{R}$, we shall denote by $(\mathcal{H})_\mathcal{A}$ the *restriction of* $\mathcal{A}$ *to* $\mathcal{H}$, defined by

$$x \equiv y \ (\mathcal{H})_\mathcal{A} \iff x, y \in \mathcal{A} \quad \text{and} \quad x \equiv y \ (\mathcal{H}).$$

In particular, if $\mathcal{H} = \mathcal{R}$, we have obviously $x \equiv y \ (\mathcal{R})_\mathcal{A} \iff x, y \in \mathcal{A}$.

LEMMA 10. — $\mathcal{G}$ *contains a maximal element if, and only if, it contains an element* x *such that* $(A_x)_\mathcal{A} = (\mathcal{R})_\mathcal{A}$ $[({}_xA)_\mathcal{A} = (\mathcal{R})_\mathcal{A}]$ *for some class* $\mathcal{A}$ *modulo* $\mathcal{R}$.

Suppose that $\bar{x}$ is maximal (hence maximum) in the class $\mathcal{X}$ modulo $\mathcal{R}$. Let $\mathcal{A}$ be any class modulo $\mathcal{R}$; then there exists one (and only one) class $\mathcal{B}$ modulo $\mathcal{R}$ such that $\mathcal{A}\mathcal{B} = \mathcal{X}$. Let $\bar{b}$ be the maximum element in $\mathcal{B}$; then we have $a\bar{b} \leqq \bar{x}$, $\forall a \in \mathcal{A}$, whence $\bar{b} = \bar{x} .\!\cdot a$, $\forall a \in \mathcal{A}$. It follows from this that $a \equiv a^\star\ (\mathcal{R}) \Rightarrow a \equiv a^\star\ (A_{\bar{x}})$; in other words, $(\mathcal{R})_{\mathcal{A}} \subseteq (A_{\bar{x}})_{\mathcal{A}}$. But by lemma 9, we have $(A_x)_{\mathcal{A}} \subseteq (\mathcal{R})_{\mathcal{A}}$, $\forall x \in \mathcal{G}$, $\forall \mathcal{A} \in \mathcal{G}/\mathcal{R}$. The required property is thus established.

Conversely, suppose there exists $x \in \mathcal{G}$ and $\mathcal{A} \in \mathcal{G}/\mathcal{R}$ such that $(A_x)_{\mathcal{A}} = (\mathcal{R})_{\mathcal{A}}$. Then since each class modulo A_x contains a maximum element, each class modulo $\mathcal{R}$ in $\mathcal{A}$ also contains a maximum element. But $\mathcal{A}$ itself is a class modulo $\mathcal{R}$; $\mathcal{A}$ therefore contains a maximum element, which is necessarily maximal in $\mathcal{G}$.

LEMMA 11. — *$\mathcal{G}$ contains a minimal element if, and only if, it contains an element x such that, for some class $\mathcal{A}$ modulo $\mathcal{R}$, either*

$$(F_x)_{\mathcal{A}} = (\mathcal{R})_{\mathcal{A}}\ [({}_xF)_{\mathcal{A}} = (\mathcal{R})_{\mathcal{A}}] \qquad or \qquad (B_x)_{\mathcal{A}} = (\mathcal{R})_{\mathcal{A}}\ [({}_xB)_{\mathcal{A}} = (\mathcal{R})_{\mathcal{A}}].$$

Suppose that $\mathcal{G}$ contains a minimal element; then each class modulo $\mathcal{R}$ contains a maximum element and a minimum element. Consider $\mathcal{B}\mathcal{A} = \mathcal{C}$ with $\underline{b}$ minimum in $\mathcal{B}$, $\bar{a}$ maximum in $\mathcal{A}$, $\underline{c}$ minimum in $\mathcal{C}$. We have necessarily that $\underline{b}\,\bar{a} = \underline{c}$ [for if we had $\underline{b}\,\bar{a} = c > \underline{c}$ then we would have, by the isotone property, $b\bar{a} \geqq c > \underline{c}$, $\forall b \in \mathcal{B}$, and consequently there would not exist $b \in \mathcal{B}$ such that $b\bar{a} \leqq \underline{c}$, so that $\mathcal{G}$ would not be residuated]. It then follows by the isotone property that $\underline{b}a = \underline{c}$, $\forall a \in \mathcal{A}$, whence $a \equiv a^\star\ (\mathcal{R}) \Rightarrow a \equiv a^\star\ (F_{\underline{b}})$ and so $(\mathcal{R})_{\mathcal{A}} \subseteq (F_{\underline{b}})_{\mathcal{A}}$ with equality holding by virtue of lemma 9.

Conversely, if there exists $x \in \mathcal{X}$ and $\mathcal{A} \in \mathcal{G}/\mathcal{R}$ such that $(F_x)_{\mathcal{A}} = (\mathcal{R})_{\mathcal{A}}$, then considering $\mathcal{X}\mathcal{A} = \mathcal{Y}$ we have that there exists $y^\star \in \mathcal{Y}$ such that $xa = y^\star$, $\forall a \in \mathcal{A}$. Let therefore y be any element in $\mathcal{Y}$; since there exists $a^\star\ (= y .\!\cdot x)$ such that $xa^\star \leqq y$, we have that $y^\star \leqq y$, whence $y^\star$ is minimum in $\mathcal{Y}$ and consequently minimal in $\mathcal{G}$.

We now prove the corresponding assertion concerning the equivalences of type B. Let $\underline{b}$ be minimal in the class $\mathcal{B}$ and consider $\mathcal{C}\mathcal{B} = \mathcal{A}$ with $\bar{c}$ maximum in $\mathcal{C}$ and $\underline{a}$ minimum in $\mathcal{A}$. By an argument similar to that in the first paragraph above, we have necessarily that $\bar{c}\,\underline{b} = \underline{a}$, whence $\bar{c}\,\underline{b} \leqq a$, $\forall a \in \mathcal{A}$, and consequently $\bar{c} = a \cdot\!.\, \underline{b}$, $\forall a \in \mathcal{A}$. It then follows that $(\mathcal{R})_{\mathcal{A}} = (B_{\underline{b}})_{\mathcal{A}}$.

Conversely, if there exists $x \in \mathcal{G}$ and $\mathcal{A} \in \mathcal{G}/\mathcal{R}$ such that $(B_x)_{\mathcal{A}} = (\mathcal{R})_{\mathcal{A}}$, then since each class modulo B_x contains a minimum element, it follows that each class modulo $\mathcal{R}$ in $\mathcal{A}$ contains a minimum element. The result then follows from the fact that $\mathcal{A}$ is itself a class modulo $\mathcal{R}$ and the proof is thus complete.

Introducing now the notation $\max \mathcal{G}$ ($\min \mathcal{G}$) to denote the set of maximal (minimal) elements in $\mathcal{G}$, we also have the following result :

THEOREM 5. — *If $\mathcal{G}$ contains a maximal element $\bar{x}$ then*

$$\max\mathcal{G} = \{\bar{x} \mathbin{.\!\cdot} a\}_{a\in\mathcal{G}} = \{\bar{x} \mathbin{\cdot\!.} a\}_{a\in\mathcal{G}};$$

correspondingly, if $\mathcal{G}$ contains a minimal element $\underline{y}$ then

$$\max\mathcal{G} = \{a \mathbin{.\!\cdot} \underline{y}\}_{a\in\mathcal{G}} = \{a \mathbin{\cdot\!.} \underline{y}\}_{a\in\mathcal{G}} \quad \textit{and} \quad \min\mathcal{G} = \{a\underline{y}\}_{a\in\mathcal{G}} = \{\underline{y}a\}_{a\in\mathcal{G}}.$$

The reader will have no difficulty in deducing these results as immediate corollaries to the proofs given in the first paragraph of lemma 10 and the first and third paragraphs of lemma 11, with analogous results for equivalences on the left.

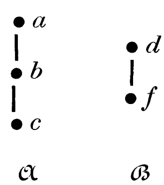
4. — *Semi-group case.*

THEOREM 6. — *If the ordered semi-group $\mathcal{S}$ is residuated, then $\mathcal{S}/\mathcal{R}$ is a group (homomorphic to $\mathcal{S}$).*

In fact, since $\mathcal{R}$ is compatible with multiplication (theorem 1), we have that $\mathcal{S}/\mathcal{R}$ is a semi-group. By virtue of theorem 3, $\mathcal{S}/\mathcal{R}$ is a quasi-group; $\mathcal{S}/\mathcal{R}$ is therefore a group.

Restricting the multiplication in $\mathcal{G}$ to be associative does not materially alter the general form of $\mathcal{G}$ as a partially ordered set. This is manifested by the examples to follow. The general form of residuated semi-groups may therefore be enunciated as in theorem 4.

EXAMPLE $\mathcal{S}_1$.



		$\mathcal{A}$			$\mathcal{B}$	
		a	b	c	d	f
$\mathcal{A}$	a	a	b	c	f	f
	b	a	b	c	f	f
	c	c	c	c	f	f
$\mathcal{B}$	d	d	d	f	c	c
	f	f	f	f	c	c

This non-commutative ordered semi-group is residuated; the residual tables are as follows :

$\cdot\!.$	a	b	c	d	f
a	a	a	a	d	d
b	c	a	a	d	d
c	c	c	a	d	d
d	d	d	d	a	a
f	f	f	d	a	a

$.\!\cdot$	a	b	c	d	f
a	a	a	a	d	d
b	b	b	a	d	d
c	c	c	a	d	d
d	d	d	d	a	a
f	d	d	d	c	a

EXAMPLE $\mathcal{S}_2$. — The example $\mathcal{G}_2$ modified by taking $\mathcal{G}_2/\mathcal{R}$ isomorphic to any group of order 6. We have in this way a non-commutative residuated semi-group of type 2 with analogous formulae to those given in example $\mathcal{G}_2$.

EXAMPLE $\mathcal{S}_3$. — The example $\mathcal{G}_3$ modified by taking $\mathcal{G}_3/\mathcal{R}$ isomorphic to any group of order 6. Using the relation $b^{\star} + d^{\star} = (b + d^{\star})^{\star}$, it is easy to show that the multiplication becomes associative, so that we have a non-commutative residuated semi-group of type 3, again with analogous formulae to those of example $\mathcal{G}_3$.

[NOTE. — Example $\mathcal{S}_3$ is of especial importance in the theory of *nomal* semi-groups (*see* MOLINARO [3]). An A-nomal semi-group is one in which there exists an equivalence A_ε such that $\mathcal{S}/A_\varepsilon$ is a group, and an important property of such semi-groups in the commutative case is that the A-nomal equivalence A_ε is coarser than every equivalence of type A. Example $\mathcal{S}_3$ shows that this property does not hold in the non-commutative case; it can in fact be shown that ${}_xA \subseteq A_\varepsilon$, $\forall x \in \mathcal{S}_3$ but that $A_{(a,b)} \subseteq A_\varepsilon$ if, and only if, $b \equiv N - 1 \pmod N$.]

5. — *c-groupoid case.* — We recall that a c-groupoid is one in which the cancellation laws hold; i. e., $xy = xz$ and $yx = zx$ imply $y = z$. The following result is then immediate :

LEMMA 12. — *$\mathcal{G}$ is a c-groupoid if, and only if, every equivalence of type F reduces to equality.*

LEMMA 13. — *A residuated c-groupoid cannot contain a minimal element without the equivalence $\mathcal{R}$ reducing to equality.*

Suppose in fact that $\mathcal{G}_c$ contains a minimal element $\underline{x}$, and let $\mathcal{X}$ be the class of $\underline{x}$ modulo $\mathcal{R}$. Let $y \equiv \underline{x}\ (\mathcal{R})$; then since there exists $z \in \check{\underline{x}} \cap \check{y}$ (lemma 6), there exists $t\,(= \underline{x}\cdot.\,z)$ such that $t\underline{x} = ty\,(= \underline{x})$; consequently, by the cancellation law, $\underline{x} = y$. Since y was chosen arbitrarily in $\mathcal{X}$, we then have that $\mathcal{X} = \{\underline{x}\}$. It then follows that this is true for any class modulo $\mathcal{R}$ since each class modulo $\mathcal{R}$ contains a minimal element; consequently, $\mathcal{R}$ is equality.

We thus have the following general form of residuated c-groupoids :

THEOREM 7. — *If the c-groupoid $\mathcal{G}_c$ is residuated then either the equivalence $\mathcal{R}$ reduces to equality or each class modulo $\mathcal{R}$ contains a maximum element and no minimal elements, or no maximal elements and no minimal elements.*

EXAMPLE $(\mathcal{G}_c)_1$. — This type is trivial; the groupoid in this case is a totally unordered quasi-group.

EXAMPLE $(\mathcal{G}_c)_2$. — Consider the partially ordered set defined by

$$\begin{cases} \mathcal{T} = \{ a_i^\lambda ; \quad \lambda = 0, 1, \ldots, 5, \quad i = 1, 2, 3 \ldots \}, \\ a_i^\lambda \leqq a_j^\mu \quad \Leftrightarrow \quad \lambda = \mu, \quad i \geqq j. \end{cases}$$

Endow $\mathcal{T}$ with the following multiplication :

(a) $\mathcal{T}/\mathcal{R}$: the quasi-group (★) of example $\mathcal{G}_2$,

(b) $$a_i^\lambda . a_j^\mu = a_{i+j}^{[\lambda\mu]}.$$

It is easy to verify that this multiplication is isotone, non-commutative and non-associative; moreover, the cancellation laws hold. $(\mathcal{G}_c)_2$ so constructed is residuated :

$$a_i^\lambda .\cdot a_j^\mu = \begin{cases} a_1^{[\lambda .\cdot \mu]} & \text{if} \quad i < j + 1, \\ a_{i-j}^{[\lambda .\cdot \mu]} & \text{if} \quad i \geqq j + 1, \end{cases}$$

the formulae for left residuals being obtained on replacing $.\cdot$ by $\cdot .$ throughout.

EXAMPLE $(\mathcal{G}_c)_3$. — Consider the partially ordered set defined by

$$\begin{cases} \mathcal{T} = \{ a_{2^n, i}^\lambda ; \quad \lambda = 0, 1, \ldots, 5, \quad n = 0, \pm 1, \pm 2, \ldots, \quad i = 1, 2, 3, \ldots \}, \\ a_{2^n, i}^\lambda \leqq a_{2^m, j}^\mu \quad \Leftrightarrow \quad \lambda = \mu, \quad n \leqq m, \quad i \leqq j. \end{cases}$$

Endowing $\mathcal{T}$ with the following multiplication :

(a) $\mathcal{T}/\mathcal{R}$: the quasi-group of example $\mathcal{G}_2$,

(b) $$a_{2^n, i}^\lambda . a_{2^m, j}^\mu = a_{2^{n+m}, i+j}^{[\lambda\mu]},$$

it is easy to show that $(\mathcal{G}_c)_3$ so constructed is residuated

$$a_{2^n, i}^\lambda .\cdot a_{2^m, j}^\mu = \begin{cases} a_{2^{n-m}, i}^{[\lambda .\cdot \mu]} & \text{if} \quad i < j + 1, \\ a_{2^{n-m}, i-j}^{[\lambda .\cdot \mu]} & \text{if} \quad i \geqq j + 1, \end{cases}$$

the formulae for left residuals being obtained by replacing $.\cdot$ by $\cdot .$ throughout.

6. — *q-groupoid case.* — We recall that a q-groupoid is one in which quotients exist, i. e., for any $a, b \in \mathcal{G}_q$, there exist $x, y \in \mathcal{G}_q$ such that $ax = b$ and $ya = b$. The following result is immediate from the fact that for any choice of $a \in \mathcal{G}_q$ every element is minimum in its class modulo B_a and ${}_aB$:

LEMMA 14. — *$\mathcal{G}$ is a q-groupoid if, and only if, every equivalence of type B reduces to equality.*

LEMMA 15. — *A residuated q-groupoid cannot contain a maximal element without the equivalence $\mathcal{R}$ reducing to equality.*

Let $\bar{a}$ be maximal in $\mathcal{G}_q$; by lemma 14 we have, for any b, $b^\star \in \mathcal{G}_q$,

$$b(\bar{a} .\cdot b) = a = b^\star(\bar{a} .\cdot b^\star)$$

so it follows that $b \equiv b^\star\ (A_{\bar{a}}) \Rightarrow b \equiv b^\star\ ({}_cF)$ where $c = \bar{a} .\cdot b = \bar{a} .\cdot b^\star$. But from the proof of lemma 10 we have that $A_{\bar{a}} = \mathcal{R}$. Hence in the class $\mathcal{B}$ of b modulo $\mathcal{R}$ we have $\mathcal{R} \subseteq {}_cF$, where $c = \bar{a} .\cdot b$; in other words, $(\mathcal{R})_{\mathcal{B}} \subseteq ({}_cF)_{\mathcal{B}}$ whence we have equality by lemma 9. It then results from lemma 11 that $\mathcal{G}_q$ contains a minimal element, and consequently every class modulo $\mathcal{R}$ contains both a maximum and a minimum element.

Let therefore $\underline{x}$ be any minimal element of $\mathcal{G}_q$. Then by theorem 5 the minimal elements of $\mathcal{G}_q$ are simply the multiples of $\underline{x}$. But since every equivalence of type B reduces to equality, every element of $\mathcal{G}_q$ is a multiple of $\underline{x}$. It follows that $\mathcal{R}$ is equality.

We are thus led to the following general form of residuated q-groupoids :

THEOREM 8. — *If the q-groupoid $\mathcal{G}_q$ is residuated then either the equivalence $\mathcal{R}$ reduces to equality or each class modulo $\mathcal{R}$ contains no maximal elements and no minimal elements.*

EXAMPLE $\mathcal{G}_q$. — The q-groupoid defined as follows :

$$\begin{cases} \mathcal{R} = \{ a_i^\lambda;\quad \lambda = 0, 1, \ldots, 5,\quad i = 0, \pm 1, \pm 2, \ldots \}, \\ \qquad a_i^\lambda \leqq a_j^\mu \iff \lambda = \mu,\quad i \leqq j; \end{cases}$$

$$\begin{cases} (a) \quad \mathcal{T}/\mathcal{R} : \text{ the quasi-group } (\bigstar) \text{ of example } \mathcal{G}_2, \\ (b) \quad \begin{cases} a_{2n}^\lambda . a_{2m-1}^\mu = a_{2n-1}^\lambda . a_{2m-1}^\mu = a_{2n-1}^\lambda . a_{2m}^\mu = a_{2n-2}^\lambda . a_{2m}^\mu, \\ a_i^\lambda . a_i^\mu = a_i^{[\lambda\mu]}. \end{cases} \end{cases}$$

This q-groupoid is residuated; we have

$$a_i^\lambda .\cdot a_j^\mu = a_{2i-j}^{[\lambda .\cdot \mu]} \qquad \text{and} \qquad a_i^\lambda \cdot. a_j^\mu = a_{2i-j}^{[\lambda \cdot. \mu]}.$$

7. — *c-semi-group case.* — The general form of residuated c-semi-groups is deduced from that of residuated semi-groups and that of residuated c-groupoids; its enunciation is as in theorem 7. The following are examples of non-trivial residuated c-semi-groups :

EXAMPLE $(\mathcal{S}_c)_2$. — Example $(\mathcal{G}_c)_2$ modified by taking for $\mathcal{T}/\mathcal{R}$ the dihedral group of order 6 (the smallest non-commutative group).

EXAMPLE $(\mathcal{S}_c)_3$. — Example $(\mathcal{G}_c)_3$ modified in the same way.

8. — *Quasi-group case.* — The general form of residuated quasi-groups is deduced from that of residuated c-groupoids and residuated q-groupoids; its enunciation is as in theorem 8. An example of a non-trivial residuated quasi-group is the following :

EXAMPLE $\mathcal{Q}$.

$$\begin{cases} \mathcal{P} = \{ a^{\lambda}_{2^n,i};\quad \lambda = 0, 1, \ldots, 5,\quad i, n = 0, \pm 1, \pm 2, \ldots \}, \\ \qquad a^{\lambda}_{2^n,i} \leqq a^{\mu}_{2^m,j} \iff \lambda = \mu,\quad n \leqq m,\quad i \leqq j; \end{cases}$$

$$\begin{cases} (a) \quad \mathcal{P}/\mathcal{R} : \text{ the quasi-group } (\bigstar) \text{ of example } \mathcal{G}_2, \\ (b) \qquad\qquad a^{\lambda}_{2^n,i} \cdot a^{\mu}_{2^m,j} = a^{[\lambda\mu]}_{2^{n+m},\,i+j}. \end{cases}$$

This quasi-group is residuated; we have the following formulae

$$a^{\lambda}_{2^n,i} \mathbin{.\cdot} a^{\mu}_{2^m,j} = a^{[\lambda\, .\cdot\, \mu]}_{2^{n-m},\,i-j}; \qquad a^{\lambda}_{2^n,i} \mathbin{\cdot.} a^{\mu}_{2^m,j} = a^{[\lambda\, \cdot.\, \mu]}_{2^{n-m},\,i-j}.$$

We also have the following results concerning residuated quasi-groups, of which the first is immediate.

LEMMA 16. — *$\mathcal{G}$ is a quasi-group if, and only if, every equivalence of types B and F reduce to equality.*

LEMMA 17. — *If $\mathcal{G}$ is a quasi-group then every equivalence of type A reduces to equality.*

Given any $a, c \in \mathcal{G}$, there exists a unique $b \in \mathcal{G}$ such that $ab = c$. Consider the set of elements $x \in \mathcal{G}$ satisfying $ax \leqq c$; we have $ax \leqq ab$, whence $x \leqq ab \mathbin{.\cdot} a = b$ since F_a is equality. It follows from this that $b = c \mathbin{.\cdot} a$. In a similar way, we have that $a = c \mathbin{\cdot.} b$, so that $a = c \mathbin{\cdot.} (c \mathbin{.\cdot} a)$. The elements a and c being arbitrary, it follows that every equivalence of type A reduces to equality.

DEFINITION. — An ordered multiplicative structure $\mathfrak{M}$ is said to be $\cup$*-semi-reticulated* if it is a $\cup$-semi-lattice with respect to its partial ordering and the following equalities satisfied :

$$a(b \cup c) = ab \cup ac, \qquad (b \cup c)\, a = ba \cup ca, \qquad \forall\, a, b, c \in \mathfrak{M}.$$

$\mathfrak{M}$ is said to be *reticulated* if it is $\cup$-semi-reticulated and is a lattice.

THEOREM 9. — *Every semi-reticulated quasi-group is residuated.*

Given any $a, b \in \mathcal{Q}$, there exists a unique $c \in \mathcal{Q}$ such that $ac = b$; let us show that c is the greatest of the elements $x \in \mathcal{Q}$ satisfying $ax \leqq b$. Let x be such that $ax \leqq b$ and consider the element $z = c \cup x$; we have

$$az = a(c \cup x) = ac \cup ax = b \cup ax = b = ac$$

whence $z = c$ by the cancellation law, and so $x \leqq c$.

COROLLARY. — *Every semi-reticulated quasi-group is reticulated.*

Given any element $a \in \mathcal{Q}$, consider the mapping $\psi_a : \mathcal{Q} \to \mathcal{Q}$ defined by setting $\psi_a(x) = a \mathbin{.\!\cdot} x$. Since the equivalence A_a is equality, ψ_a is clearly an injection; also, since the equivalence ${}_aA$ is equality, every element of $\mathcal{Q}$ is a right residual of a and so ψ_a is a surjection. ψ_a is therefore a bijection, and since it satisfies

$$x \leqq y \iff \psi_a(y) \leqq \psi_a(x),$$

it follows that $\mathcal{Q}$ is isomorphic to its dual and so is a lattice.

9. — *Group case.* — Every ordered group is residuated; for, in such a group, the relation $ax \leqq b$ implies that $x = a^{-1}ax \leqq a^{-1}b$ so that, since $a(a^{-1}b) = b$, we have that $b \mathbin{.\!\cdot} a$ exists and is equal to $a^{-1}b$. Similarly, $b \mathbin{\cdot\!.} a$ exists and is equal to ba^{-1}.

The general form of residuated groups may be enunciated as in theorem 8.

10. — In this section we give a generalisation of a recent result due to McFADDEN [2].

DEFINITION. — By a *proper fundamental equivalence* in a residuated groupoid, we shall mean an equivalence of type A, B or F which is distinct from $\mathcal{R}$ and from equality.

Residuated unitary groupoids with no proper fundamental equivalences are completely characterised by the following result :

THEOREM 10. — *Let $\mathcal{G}$ be a residuated groupoid with identity e. If $\mathcal{G}$ has no proper fundamental equivalences, there are but two possibilities :*

1. *the classes modulo $\mathcal{R}$ have at most two elements and the class of e is isomorphic to the Boolean algebra* $\{0, 1\}$,

or 2. *$\mathcal{G}$ is a loop; and if the class of e consists only of e itself, the loop ordering is the trivial ordering* $a \not\parallel b \Leftrightarrow a = b$.

First of all, if every fundamental equivalence reduces to equality then $\mathcal{G}$ is a loop by virtue of lemma 16; and if the class of e consists only of e itself then e is minimal in $\mathcal{G}$ and so, by virtue of theorem 5, every element of $\mathcal{G}$ is minimal and consequently $\mathcal{R}$ is equality.

Suppose therefore that not all fundamental equivalences reduce to equality; then by lemma 9, $\mathcal{R}$ is not equality and there exists (by hypothesis) at least one equivalence of type A, B or F which coincides with $\mathcal{R}$. By either of lemmas 10 and 11 we then have that each class modulo $\mathcal{R}$ contains a maximum element. Let therefore $\bar{a}$ be the maximum element in the unit class modulo $\mathcal{R}$; then from $e \leqq \bar{a}$ we have $\bar{a} = \bar{a}e \leqq \bar{a}\,\bar{a}$, whence $\bar{a} = \bar{a}\,\bar{a}$ and consequently $e \equiv \bar{a}(F_{\bar{a}})$. Now if $F_{\bar{a}} = \mathcal{R}$, we have

$\bar{a}x = \bar{a}\,\bar{a}$, $\forall x \equiv \bar{a}(\mathcal{R})$, so that $\bar{a}$ is also minimum in its class modulo $\mathcal{R}$; for since $\bar{a}\,\bar{a} = \bar{a}$, we have $\bar{a}x = \bar{a}$, $\forall x \equiv \bar{a}(\mathcal{R})$ and if there existed $y \equiv \bar{a}(\mathcal{R})$ such that $y < \bar{a}$ then there would not exist $x \in \mathcal{G}$ such that $\bar{a}x \leq y$, contrary to the hypothesis that $\mathcal{G}$ is residuated. It follows from this that, if $F_{\bar{a}} = \mathcal{R}$, then the unit class modulo $\mathcal{R}$ consists of only the identity element e; and as we have seen above, this implies that $\mathcal{R}$ is equality, contrary to the hypothesis. Hence we cannot have $F_{\bar{a}} = \mathcal{R}$ and so we must have that $F_{\bar{a}}$ is equality. It then follows that $e = \bar{a}$; in other words, e is maximum in its class modulo $\mathcal{R}$. Since we always have $e \leq x \,.\!\cdot\, x$, $\forall x \in \mathcal{G}$, it then follows that $e = x \,.\!\cdot\, x$, $\forall x \in \mathcal{G}$. Consider now any class $\mathcal{X}$ modulo $\mathcal{R}$; let $\bar{x}$ be the maximum element in $\mathcal{X}$ and consider $x_1 \leq x_2 < \bar{x}$. By the antitone property of residuals, we have $x_2 \,.\!\cdot\, x_1 \geq x_2 \,.\!\cdot\, x_2 = e$, whence we have equality and so $x_1 \equiv x_2\,(A_{x_2})$. But since $x_2 \,.\!\cdot\, x_2 = e$, we also have that $x_2 \cdot\!.\,(x_2 \,.\!\cdot\, x_2) = x_2 \cdot\!.\, e = x_2$ so that x_2 is maximum in its class modulo A_{x_2} and so $x_2 \not\equiv \bar{x}(A_{x_2})$. It follows that $A_{x_2} \neq \mathcal{R}$ and so we must have that A_{x_2} is equality, whence $x_1 = x_2$. Consequently, any element covered by $\bar{x}$ is minimal in $\mathcal{X}$ and such an element must be minimum in $\mathcal{X}$ by virtue of lemma 7. The proof is completed by remarking that when the unit class modulo $\mathcal{R}$ contains two elements, it must be isomorphic to the Boolean algebra $\{0, 1\}$ in order for it to be a residuated sub-semi-group of $\mathcal{G}$.

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