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VECTOR FIELDS AND INFINITESIMAL TRANSFORMATIONS ON RIEMANNIAN MANIFOLDS WITH BOUNDARY (1);

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Introduction.

In recent years many authors have made interesting and important contributions to the study of vector fields or infinitesimal transformations on compact orientable Riemannian manifolds without boundary. The purpose of this paper is to extend some of those contributions to Riemannian manifolds with boundary.

Paragraph 1 contains fundamental notations, and local operators and formulas for a Riemannian manifold.

In paragraph 2 fundamental formulas for Lie derivatives are given, and the infinitesimal transformations and their generating vector fields are defined in terms of Lie derivatives.

Paragraph 3 is devoted, for a compact orientable Riemannian manifold with boundary, to a discussion of local boundary geodesic coordinates and the derivation of some integral formulas, which will be needed in the remainder of this paper.

In the remainder of this introduction, M^n will always denote, unless stated otherwise, a compact orientable Riemannian manifold with boundary B^{n-1}. Paragraph 4 contains necessary and sufficient conditions for a vector field on a manifold M^n with zero tangential or normal component on the boundary B^{n-1} to be a Killing vector field. A boun-

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dary condition is also given for an infinitesimal affine collineation on the manifold M^n leaving the boundary B^{n-1} invariant to be a motion.

In paragraph 5 we obtain conditions for the nonexistence of a nonzero conformal Killing vector field on a manifold M^n with zero tangential or normal component on the boundary B^{n-1}, and necessary and sufficient conditions for a vector field on the manifold M^n with zero tangential or normal component on the bondary B^{n-1} to be a conformal Killing vector field. It is shown that if the manifold M^n has constant scalar curvature R and admits a certain special infinitesimal nonhomothetic conformal motion leaving the boundary B^{n-1} invariant, then $R > 0$. Moreover, on a compact orientable Einstein manifold M^n with boundary B^{n-1} and $R > 0$, those special infinitesimal nonhomothetic conformal motions leaving the boundary B^{n-1} invariant form a Lie algebra, and a decomposition of this algebra with interrelations between its subalgebras is also obtained.

Paragraph 6 contains conditions for the nonexistence of a projective Killing vector field on a manifold M^n with zero tangential or normal component on the boundary B^{n-1} and satisfying certain other boundary conditions. Finally, it is shown that on a compact orientable Einstein manifold M^n with boundary B^{n-1} and positive constant scalar curvature R a projective Killing vector field is the direct sum of a Killing vector field and an exact projective Klilling vector field in such a way that they all satisfy the same type of boundary conditions.

Throughout this paper, the dimensions of M^n and B^{n-1} are understood to be n ($\geqq 2$) and $n-1$ respectively, all Riemannian manifolds are of class C^3, and all differential forms and vector fields are of class C^2.

1. Notations and operators.

Let M^n be a Riemannian manifold of dimension n ($\geqq 2$), $\| g_{ij} \|$ with $g_{ij} = g_{ji}$ the matrix of the positive definite metric of the manifold M^n, and $\| g^{ij} \|$ the inverse matrix of $\| g_{ij} \|$. Throughout this paper all Latin indices take the values $1, \ldots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively; and that when a Latin letter appears in any term as a subscript and superscript, it is understood that this letter is summed for all the values $1, \ldots, n$. We shall also use v^i and v_i to denote the contravariant and covariant components of a vector field v respectively. Moreover, if we multiply, for example the components a_{ij} of a covariant tensor by the components b^{jk} of a contravariant tensor, it will always be understood that j is to be summed.

Let $\mathfrak{N}$ be the set $\{1, \ldots, n\}$ of positive integers less than or equal to n, and $I(p)$ denote an ordered subset $\{i_1, \ldots, i_p\}$ of the set $\mathfrak{N}$ for $p \leqq n$. If the elements $i_1, \ldots, i_p$ are in the natural order, that is, if $i_1 < \ldots < i_p$, then the ordered set $I(p)$ is denoted by $I_0(p)$. Furthermore, let $I(p; \hat{s} | j)$ be the ordered set $I(p)$ with the s-th element i_s replaced by another element j of $\mathfrak{N}$, which may or may not belong to $I(p)$. We shall use these notations for indices throughout this paper. When more than one set of indices is needed at one time, we may use other capital letters such as $J, K, \ldots$ in addition to I.

From the metric tensor g with components g_{ij} we have

$$g_{I(n),\ K(n)} = g_{i_1 j_1} \cdots g_{i_n j_n} \delta^{J(n)}_{K(n)} = \begin{vmatrix} g_{i_1 k_1} & \cdots & g_{i_1 k_n} \\ \cdots & \cdots & \cdots\cdot \\ g_{i_n k_1} & \cdots & g_{i_n k_n} \end{vmatrix},$$

where $\delta^{J(n)}_{K(n)}$ is zero when two or more j's or k's are the same, and is $+1$ or -1 according as the j's and k's differ from one another by an even or odd number of permutations. Thus the element of area of the manifold M^n at a point P with local coordinates $x^1, \ldots, x^n$ is

$$(1.1) \qquad dA_n = e_{1 \ldots n} dx^1 \wedge \ldots \wedge dx^n,$$

where d and the wedge $\wedge$ denote the exterior differentiation and multiplication respectively, and

$$(1.2) \qquad e_{1 \ldots n} = + \sqrt{g_{1 \ldots n,\ 1 \ldots n}}.$$

By using orthonormal local coordinates $x^1, \ldots, x^n$ and the relations

$$(1.3) \qquad e_{I(n)} = \delta^{1 \ldots n}_{I(n)} e_{1 \ldots n},$$

$$(1.4) \qquad \delta^{I(p) J(n-p)}_{1 \ldots n} \delta^{1 \ldots n}_{I(p) K(n-p)} = p!\, \delta^{J(n-p)}_{K(n-p)},$$

we can easily obtain

$$(1.5) \qquad e_{I(p) K(n-p)} e^{I(p) J(n-p)} = p!\, \delta^{J(n-p)}_{K(n-p)}.$$

From equations (1.3), (1.4), (1.5) it follows that

$$(1.6) \qquad e_{1 \ldots n} e^{1 \ldots n} = 1.$$

On the manifold M^n let $v_{(p)}$ be a differential form of degree p given by

$$(1.7) \qquad v_{(p)} = v_{I_0(p)} dx^{I_0(p)},$$

where we have placed

$$(1.8) \qquad dx^{I(p)} = dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

Then we have

$$(1.9) \qquad dv_{(p)} = (-1)^p \sum_{i_{p+1} > i_0} \left[\nabla_{i_{p+1}} v_{I_0(p)} - \sum_{s=1}^{p} \nabla_{i_s} v_{I_0(p; \hat{s} | i_{p+1})} \right] dx^{I_0(p+1)},$$

where ∇ denotes the covariant derivation with respect to the affine connection of the Riemannian metric g_{ij}, whose components in the local coordinates $x^1, \ldots, x^n$ are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih}(\partial g_{hj}/\partial x^k + \partial g_{hk}/\partial x^j - \partial g_{jk}/\partial x^h). \tag{1.10}$$

Moreover, the dual and codifferential operators $\star$ and δ are defined by (for this *see*, for instance, [9])

$$\star\, v_{(p)} = e_{I_0(p)J_0(n-p)} v^{I_0(p)} dx^{J_0(n-p)}, \tag{1.11}$$

$$\delta\, v_{(p)} = (-1)^{np+n+1} \star d \star v_{(p)}, \tag{1.12}$$

which imply immediately

$$\delta\, v_{(p)} = -p g^{ij} \nabla_j v_{iI_0(p-1)} dx^{I_0(p-1)}. \tag{1.13}$$

In particular, for a vector field v on the manifold M^n we obtain, from equations (1.9), (1.13),

$$(dv)_{ij} = \nabla_i v_j - \nabla_j v_i, \tag{1.14}$$

$$\delta\, v = -\nabla_i v^i, \tag{1.15}$$

$$(d\delta v)_i = -\nabla_i \nabla_j v^j, \tag{1.16}$$

$$(\delta d v)_i = \nabla^j \nabla_i v_j - \nabla^j \nabla_j v_i, \tag{1.17}$$

where $\nabla^j = g^{jk}\nabla_k$. A use of equations (1.16), (1.17) and the Ricci identity for the contravariant components v^i,

$$[\nabla_k, \nabla_j] v^i = v^l R^i{}_{ljk} \tag{1.18}$$

thus gives

$$(\Delta v)_i = -\nabla^j \nabla_j v_i + R_{ij} v^j, \tag{1.19}$$

where $[\nabla_k, \nabla_j] = \nabla_k \nabla_j - \nabla_j \nabla_k$, and Δ, $R^i{}_{jkl}$, R_{ij} are respectively the Laplace-Beltrami operator, the Riemann curvature tensor and the Ricci tensor defined by

$$\Delta = d\delta + \delta d, \tag{1.20}$$

$$R^i{}_{jkl} = \partial \Gamma^i_{jk}/\partial x^l - \partial \Gamma^i_{jl}/\partial x^k + \Gamma^s_{jk}\Gamma^i_{sl} - \Gamma^s_{jl}\Gamma^i_{sk}, \tag{1.21}$$

$$R_{ij} = R^k{}_{ijk}. \tag{1.22}$$

By contraction with respect to i and k, from equation (1.18) we have

$$v^j [\nabla_i, \nabla_j] v^i = R_{ij} v^i v^j. \tag{1.23}$$

Multiplication of equation (1.18) by g_{ih} gives the Ricci identity for the covariant components v_i,

$$[\nabla_k, \nabla_j] v_i = -v_l R^l{}_{ijk}, \tag{1.24}$$

which can also be written as

$$(1.25)\qquad -\nabla_k\nabla_i v_j + \nabla_k(\nabla_i v_j + \nabla_j v_i) - \nabla_j\nabla_k v_i = -v_l R^l{}_{ijk}.$$

Multiplying equation (1.25) by $g^{ik}g^{jh}$ and using equation (1.19) we thus obtain

$$(1.26)\qquad (Qv)^h - (\Delta v)^h = \nabla_i(\nabla^i v^h + \nabla^h v^i) - \nabla^h\nabla_i v^i,$$

where

$$(1.27)\qquad (Qv)^h = 2R_i^h v^i.$$

Let $u_{I(p)}$ and $v_{I(p)}$ be two tensor fields of the same order p on a compact orientable manifold M^n. Then the local and global scalar products $\langle u, v\rangle$ and (u, v) of the two tensor fields u and v are defined by

$$(1.28)\qquad \langle u, v\rangle = \frac{1}{p!}u^{I(p)}v_{I(p)},$$

$$(1.29)\qquad (u, v) = \int_{M^n}\langle u, v\rangle dA_n.$$

From equations (1.28), (1.29) it follows that (u, u) is non negative, and that $(u, u) = 0$ implies that $u = 0$ on the whole manifold M^n.

2. Lie derivatives and infinitesimal transformations.

Let v be a nonzero vector field on a Riemannian manifold M^n, and let i_v and L_v denote, respectively, the interior product and the Lie derivative with respect to the vector field v. Then for a covariant tensor a of order r, the interior product $i_v a$ is a tensor of order $r-1$ defined by

$$(2.1)\qquad (i_v a)_{I(r-1)} = v^j a_{jI(r-1)},$$

and according to H CARTAN [1] we have

$$(2.2)\qquad L_v = i_v d + di_v,$$

from which it follows

$$(2.3)\qquad L_v d = dL_v = di_v d.$$

For later developments, we shall use the following known formulas for Lie derivatives in terms of local coordinates $x^1, \ldots, x^n$ of the manifold M^n (for these formulas *see*, for instance, [8], [12]) :

$$(2.4)\qquad L_v u^{ij}{}_{kl} = v^s\nabla_s u^{ij}{}_{kl} - u^{sj}{}_{kl}\nabla_s v^i - u^{is}{}_{kl}\nabla_s v^j + u^{ij}{}_{sl}\nabla_k v^s + u^{ij}{}_{ks}\nabla_l v^s,$$

$$(2.5)\qquad L_v(\nabla_l u^i{}_{jk}) - \nabla_l(L_v u^i{}_{jk}) = (L_v\Gamma^i_{ls})u^s{}_{jk} - (L_v\Gamma^s_{lj})u^i{}_{sk} - (L_v\Gamma^s_{lk})u^i{}_{js},$$

$$(2.6)\qquad L_v(fu^i{}_{jk}) = (L_v f)u^i{}_{jk} + f(L_v u^i{}_{jk}),$$

(2.7) $$L_v(u^{ij}w_{ij}) = (L_v u^{ij})w_{ij} + u^{ij}L_v w_{ij},$$

(2.8) $$L_v g_{ij} = \nabla_i v_j + \nabla_j v_i,$$

(2.9) $$L_v \Gamma^i_{jk} = \nabla_k \nabla_j v^i + R^i{}_{jkl} v^l,$$

(2.10) $$L_v R^i{}_{jkl} = \nabla_l (L_v \Gamma^i_{jk}) - \nabla_k (L_v \Gamma^i_{jl}),$$

where f is a scalar, and $u^{ij}{}_{kl}$, $u^i{}_{jk}$, u^{ij}, w_{ij} are tensor fields of class least C^1 on the manifold M^n, the contravariant and covariant orders of each of which being the numbers of the superscripts and subscripts respectively. Applying equation (2.5) to g_{ij} and noticing that $\nabla g = 0$, we obtain

$$\nabla_l (L_v g_{jk}) = (L_v \Gamma^s_{lj}) g_{sk} + (L_v \Gamma^s_{lk}) g_{js}.$$

Subtracting this equation from the sum of two others obtained from it by interchanging l, j, k cyclically, and multiplying the resulting equation by g^{li} we are thus led to

(2.11) $$L_v \Gamma^i_{jk} = \frac{1}{2} g^{il} [\nabla_j (L_v g_{lk}) + \nabla_k (L_v g_{jl}) - \nabla_l (L_v g_{jk})].$$

The infinitesimal transformation on the manifold M^n generated by a nonzero vector field v is called an infinitesimal motion (or isometry), affine collineation, projective motion, or conformal motion, and the corresponding v a Killing, an affine Killing, a projective Killing, or a conformal Killing vector field according as

(2.12) $$L_v g_{ij} = 0,$$

(2.13) $$L_v \Gamma^i_{jk} = 0,$$

(2.14) $$L_v \Gamma^i_{jk} = p_j \delta^i_k + p_k \delta^i_j,$$

or

(2.15) $$L_v g_{ij} = 2\Phi\, g_{ij},$$

where

(2.16) $$p_i = \partial p / \partial x^i = \nabla_i p$$

is a gradient vector field on the manifold M^n, and Φ is a scalar. An infinitesimal conformal motion defined by equation (2.15) is called a homothetic motion, if Φ is constant.

From equations (2.8), (2.12) it follows that v is a Killing vector field if

(2.17) $$\nabla_i v_j + \nabla_j v_i = 0,$$

which and equation (1.15) imply

(2.18) $$\delta v = 0.$$

From equations (2.9), (2.14) for any projective Killing vector field v we have

$$L_v \Gamma^i_{jk} = \nabla_k \nabla_j v^i + R^i{}_{jkl} v^l = p_j \delta^i_k + p_k \delta^i_j. \tag{2.19}$$

The contraction with respect to i and j in equation (2.19) and a use of the identity $R^i{}_{jkl} = 0$ give

$$p_j = \frac{1}{n+1} \nabla_j \nabla_i v^i. \tag{2.20}$$

By means of equations (1.16), (1.19), (1.27), (2.20) and the equation obtained by multiplying equation (2.19) by g^{jk} we thus have

$$\nabla v - \frac{2}{n+1} d\delta v = Qv. \tag{2.21}$$

Similarly, for a conformal Killing vector field v, from equations (2.9), (2.15), (2.11) we have

$$\nabla_i v_j + \nabla_j v_i = 2\Phi g_{ij}, \tag{2.22}$$

$$-\delta v = n\Phi, \tag{2.23}$$

$$L_v \Gamma^i_{jk} = \nabla_k \nabla_j v^i + R^i{}_{jkl} v^l = \Phi_j \delta^i_k + \Phi_k \delta^i_j - \Phi^i g_{jk}, \tag{2.24}$$

where we have placed

$$\Phi_j = \nabla_j \Phi = \partial\Phi/\partial x^j, \qquad \Phi^i = g^{ij} \Phi_j. \tag{2.25}$$

Multiplication of equation (2.24) by g^{jk} and substitution of equation (2.23) in the resulting equation yield immediately

$$\Delta v + \left(1 - \frac{2}{n}\right) d\delta v = Qv. \tag{2.26}$$

3. Local boundary geodesic coordinates and integral formulas.

Throughout this paper, by an $(n-1)$-dimensional boundary B^{n-1} of a compact n-dimensional submanifold M^n of an n-dimensional manifold $\mathfrak{M}^n (n \geq 2)$ we mean either an empty or a nonempty subdomain on the submanifold M^n satisfying the following condition. At every point P of the boundary B^{n-1} there is a full neighborhood $U(P)$ of the point P on the manifold $\mathfrak{M}^n$ and admissible local coordinates $x^1, \ldots, x^n$ such that $U(P) \cap M^n$ appears in the space of the x's as a hemisphere

$$\sum_{i=1}^{n} (x^i)^2 < \varepsilon^2, \qquad x^1 < 0, \tag{3.1}$$

the base $x^1 = 0$ of the hemisphere corresponding to the boundary B^{n-1}. For nonempty boundary B^{n-1} we shall choose the local coordinates $x^1, \ldots, x^n$ to be boundary geodesic coordinates so that at each point P of the boundary B^{n-1} the x^1-curve is a geodesic of the manifold M^n, with x^1 as its arc length measured from the boundary B^{n-1}, and is orthogonal to the x^i-curves, $i = 2, \ldots, n$. Thus on the boundary B^{n-1} we can easily obtain (for this see, for instance, [4], p. 57)

$$g_{11} = g^{11} = 1, \qquad g_{1i} = g^{1i} = 0 \qquad (i = 2, \ldots, n). \tag{3.2}$$

Moreover, by equation (3.1) the unit tangent vector N of the x^1-curve at every point P of the boundary B^{n-1} is the unit outer normal vector of the boundary B^{n-1} in the sense that x^1 is increasing along the x^1-curve in the direction of the vector N.

By using local boundary geodesic coordinates, from equations (1.10), (3.2) it is easily seen that on the boundary B^{n-1}:

$$\Gamma^1_{11} = 0, \qquad \Gamma^i_{11} = 0, \qquad \Gamma^1_{1i} = 0, \qquad 2\,\Gamma^i_{1j} = g^{ik}\partial g_{jk}/\partial x^1, \tag{3.3}$$

$$\begin{aligned} b_{ij} &= (\overline{\nabla}_j \overline{\nabla}_i x^h)\, g_{hk} N^k + g_{lr} N^r \Gamma^l_{hk} \nabla_i x^h \nabla_j x^k \\ &= \Gamma^1_{ij} = -\frac{1}{2}\,\partial g_{ij}/\partial x^1 \qquad (i, j = 2, \ldots, n), \end{aligned} \tag{3.4}$$

where b_{ij} are the coefficients of the second fundamental form of the boundary B^{n-1} relative to the outer normal vector N on the manifold M^n, and $\overline{\nabla}$ denotes the covariant derivation with respect to the metric tensor $g_{ij}\,(i, j = 2, \ldots, n)$ of the boundary B^{n-1} (for this *see*, for instance, [3], p. 147). Equations (3.3), (3.4) imply immediately

$$b^i_j \equiv g^{ik}\, b_{kj} = -\Gamma^i_{1j}. \tag{3.5}$$

The boundary B^{n-1} is said to be convex or concave on the manifold M^n according as the matrix $\| b_{ij} \|$ for $i, j = 2, \ldots, n$ is negative or positive definite. If $b_{ij} = 0$ for $i, j = 2, \ldots, n$, then all the geodesics of the boundary B^{n-1} are geodesics of the manifold M^n, and the boundary B^{n-1} is said to be totally geodesic on the manifold M^n. Moreover, in terms of local boundary geodesic coordinates the tangential and normal components of a vector v are respectively v_i, $i \neq 1$, and v_1.

Now consider a compact orientable Riemannian manifold M^n with boundary B^{n-1}, and let u be a vector field of class C^2 on the manifold M^n. Then on the manifold M^n we can construct the differential form

$$\omega = \star\, u_i dx^i. \tag{3.6}$$

By means of equations (1.11), (1.3) we can easily obtain

$$\omega = \delta^{1\ldots n}_{i\, j_0 (n-1)}\, e_{1\ldots n}\, u^i dx^{J_0 (n-1)}, \tag{3.7}$$

which becomes, on the boundary B^{n-1} in terms of local boundary geodesic coordinates,

$$\omega = u_1 dA_{n-1}, \tag{3.8}$$

where

$$dA_{n-1} = e_{1\ldots n} dx^2 \wedge \ldots \wedge dx^n \tag{3.9}$$

is the element of area of the boundary B^{n-1}. A use of equations (3.7), (1.1) gives immediately

$$d\omega = \delta^{1\,\ldots\,n}_{jJ_0(n-1)} e_{1\ldots n} \nabla_k u^j dx^k \wedge dx^{J_0(n-1)} = \nabla_j u^j dA_n. \tag{3.10}$$

By applying the Stokes' theorem we thus obtain the integral formula

$$\int_{M^n} \nabla_j u^j dA_n = \int_{B^{n-1}} u_1 dA_{n-1}. \tag{3.11}$$

For a vector field v on a compact orientable Riemannian manifold M^n with boundary B^{n-1}, replacement of the vector field u^j in equation (3.11) by the vectors $v^i \nabla^j v_i$, $v^i \nabla_i v^j$, $v^j \nabla_i v^i$ and use of equations (1.19), (1.27), (1.28), (1.29), (1.15) yield the integral formulas, respectively,

$$\left(\frac{1}{2} Qv - \Delta v, v\right) + 2(\nabla v, \nabla v) = \int_{B^{n-1}} v^i \nabla_1 v_i dA_{n-1}, \tag{3.12}$$

$$2(T\nabla v, \nabla v) + \int_{M^n} v^j \nabla_i \nabla_j v^i dA_n = \int_{B^{n-1}} v^i \nabla_i v_1 dA_{n-1}, \tag{3.13}$$

$$(\delta v, \delta v) + \int_{M^n} v^j \nabla_j \nabla_i v^i dA_n = \int_{B^{n-1}} v_1 \nabla_i v^i dA_{n-1}, \tag{3.14}$$

where for a covariant tensor field u_{ij},

$$(Tu)_{ij} = u_{ji}. \tag{3.15}$$

Subtraction of equation (3.14) from equation (3.13) and substitution of equations (1.23), (1.27) give immediately

$$\left(\frac{1}{2} Qv, v\right) + 2(T\nabla v, \nabla v) - (\delta v, \delta v) = \int_{B^{n-1}} (v^i \nabla_i v_1 - v_1 \nabla_i v^i) dA_{n-1}. \tag{3.16}$$

By subtracting equation (3.16) from equation (3.12) we obtain, in consequence of equation (1.14),

$$\begin{aligned} &-(\Delta v, v) + (dv, dv) + (\delta v, \delta v) \\ &= \int_{B^{n-1}} [v^i(\nabla_1 v_i - \nabla_i v_1) + v_1 \nabla_i v^i]\, dA_{n-1}. \end{aligned} \tag{3.17}$$

Similarly, addition of equations (3.12), (3.16) and use of equation (2.8) yield

$$(3.18)\qquad (Qv - \Delta v, v) + (L_v g, L_v g) - (\delta v, \delta v) = \int_{B^{n-1}} [v^i(\nabla_1 v_i + \nabla_i v_1) - v_1 \nabla_i v^i]\, dA_{n-1}.$$

The integral formulas (3.11), ..., (3.14), (3.16), (3.17), (3.18) were first derived by the author in a previous paper [4] by means of general local boundary coordinates $x^1, \ldots, x^n$, with $x^n = 0$ corresponding to the boundary B^{n-1}, and the combined operator of the exterior product $\wedge$ of differentials on the manifold M^n and the vector product of $n + m - 1$ vectors in a Euclidean space E^{n+m} of dimension $n + m$ for any $m > 0$, provided that the manifold M^n is isometrically imbedded in the space E^{n+m}.

It is easily seen that equation (3.16) can be written in the following three forms

$$(3.19)\qquad \left(\frac{1}{2} Qv, v\right) + 2(\nabla v, \nabla v) - (dv, dv) - (\delta v, \delta v) = \int_{B^{n-1}} (v^i \nabla_i v_1 - v_1 \nabla_i v^i)\, dA_{n-1},$$

$$(3.20)\qquad \left(\frac{1}{2} Qv, v\right) - 2(\nabla v, \nabla v) + (L_v g, \nabla_v g) - (\delta v, \delta v) = \int_{B^{n-1}} (v^i \nabla_i v_1 - v_1 \nabla_i v^i)\, dA_{n-1},$$

$$(3.21)\qquad \left(\frac{1}{2} Qv, v\right) - 2(\nabla v, \nabla v) - \frac{n-2}{n}(\delta v, \delta v) + (tv, tv) = \int_{B^{n-1}} (v^i \nabla_i v_1 - v_1 \nabla_i v^i)\, dA_{n-1},$$

where

$$(3.22)\qquad tv = L_v g + 2\frac{\delta v}{n} g.$$

LEMMA 3.1. — *Let f be a scalar field of class C^2 on a compact orientable Riemannian manifold M^n with boundary B^{n-1}. Suppose that the normal component $\nabla_1 f$ of the gradient vector ∇f on the boundary B^{n-1} vanishes, and that f satisfies*

$$(3.23)\qquad -\Delta f \equiv \nabla^i \nabla_i f = \lambda f$$

with constant λ. Then $f = 0$ or constant on the manifold M^n according as $\lambda > 0$ or $\lambda = 0$.

The proof of lemma 3.1 follows immediately from the integral formula (3.11) with the vector field u replaced by $\nabla(f^2)$.

4. Killing vector fields and infinitesimal motions.

From equations (2.17), (2.18), (1.26) it follows immediately that a Killing vector field v on any Riemannian manifold M^n satisfies

$$\Delta v - Qv = 0. \tag{4.1}$$

For the converse, suppose that on a compact orientable Riemannian manifold M^n with boundary B^{n-1} a vector field v has zero tangential component on the boundary B^{n-1} and satisfies equations (2.18), (4.1). If $\nabla_1 v_1 = 0$ on the boundary B^{n-1} in local boundary geodesic coordinates, then by using equations (2.18), (4.1), from equation (3.18) it follows that the vector field v satisfies equation (2.12), and therefore is a Killing vector field on the manifold M^n. Hence we obtain

THEOREM 4.1 T. — *On a compact orientable Riemannian manifold M^n with boundary B^{n-1}, a necessary and sufficient condition for a vector field v with zero tangential component on the boundary B^{n-1} to be a Killing vector field is that it satisfy equations* (2.18), (4.1) *on the manifold M^n and*

$$\nabla_1 v_1 = 0 \quad \textit{on } B^{n-1} \tag{4.2}$$

in local boundary geodesic coordinates.

It should be noted that if the boundary B^{n-1} is totally geodesic, and the vector field v on the manifold M^n with zero tangential component on the boundary B^{n-1} satisfies equation (2.18), the condition (4.2) in local boundary geodesic coordinates is automatically satisfied, as on the boundary B^{n-1} in local boundary geodesic coordinates we have, in consequence of equations (2.18), (3.5),

$$\nabla_1 v^1 = -\sum_{i=2}^{n} \nabla_i v^i = -v^1 \sum_{i=2}^{n} b_i^i,$$

which vanishes for a totally geodesic boundary B^{n-1}.

Similarly, from equation (3.18) we have

THEOREM 4.1 N. — *On a compact orientable Riemannian manifold M^n with boundary B^{n-1}, a necessary and sufficient condition for a vector field v with zero normal component on the boundary B^{n-1} to be a Killing vector field is that it satisfy equations* (2.18), (4.1) *on the manifold M^n and*

$$\sum_{i=2}^{n} v^i(\nabla_1 v_i + \nabla_i v_1) = 0 \quad \textit{on } B^{n-1} \tag{4.3}$$

in local boundary geodesic coordinates.

It should be noted that the letters T and N in theorems 4.1 T and 4.1 N are used to denote similar theorems on vector fields with zero tangential and normal components on the boundary B^{n-1} of the manifold M^n respectively; for convenience we shall use this notation throughout this paper.

From equations (2.11), (2.12), (2.13) it follows that on any Riemannian manifold M^n an infinitesimal motion is an infinitesimal affine collineation. For the converse, suppose that on a compact orientable Riemannian manifold M^n with boundary B^{n-1} an affine Killing vector field v has zero normal component on the boundary B^{n-1}. From equations (2.9), (2.13) it follows

$$\nabla_k \nabla_j v^i + R^i{}_{jkl} v^l = 0, \tag{4.4}$$

which implies equation (4.1) by multiplication by g^{jk}. By putting $i = j = a$ in equation (4.4), summing for a and making use of the identity $R^a{}_{akl} = 0$, we obtain $\nabla_k \nabla_a v^a = 0$ and therefore $\nabla_a v^a = \mathrm{Cte}$, which and equation (3.11) yield equation (2.18) because of the vanishing of the integrand on the right side of equation (3.11). On the other hand, from the definition of infinitesimal transformations it is readily seen that a necessary and sufficient condition for an infinitesimal transformation generated by a vector field v on a compact Riemannian manifold M^n with boundary B^{n-1} to leave the boundary B^{n-1} invariant is that the vector field v has zero normal component on the boundary B^{n-1}. An application of theorem 4.1 N thus gives

THEOREM 4.2 N. — *On a compact orientable Riemannian manifold M^n with boundary B^{n-1}, an infinitesimal affine collineation leaving the boundary B^{n-1} invariant is a motion, if its generating vector field v has zero normal component and satisfies equation* (4.3) *on the boundary B^{n-1}.*

5. **Conformal Killing vector fields and infinitesimal conformal motions.**

At first we suppose that a Riemannian manifold M^n be an Einstein manifold so that

$$R_{ij} = R g_{ij}/n, \tag{5.1}$$

from which follows immediately

$$R = g^{ij} R_{ij}. \tag{5.2}$$

On contracting with respect to i and m from the Bianchi identity

$$\nabla_m R^i{}_{jkl} + \nabla_k R^i{}_{jlm} + \nabla_l R^i{}_{jmk} = 0, \tag{5.3}$$

and multiplying the resulting equation by g^{jk} we can easily obtain

$$2\nabla_s R^s{}_l = \nabla_l R. \tag{5.4}$$

On the other hand, multiplying equation (5.1) by g^{ik} we have

$$R^k{}_j = R\delta^k_j/n. \tag{5.5}$$

Substitution of equation (5.5) in equation (5.4) shows immediately that for $n > 2$, R is constant.

Now let v be a conformal Killing vector field on a compact orientable manifold M^n with boundary B^{n-1}. Then equation (3.16) becomes, in consequence of equations (2.22), (2.23),

$$\begin{aligned}\int_{M^n} [R_{ij}v^i v^j - 2\langle\nabla v, \nabla v\rangle - n(n-2)\Phi^2]dA^n \\ = \int_{B^{n-1}} (v^i\nabla_i v_1 - v_1\nabla_i v^i)dA_{n-1}.\end{aligned} \tag{5.6}$$

If the vector field v has zero tangential component on the boundary B^{n-1}, then by means of local boundary geodesic coordinates and equation (3.5) we have, on the boundary B^{n-1},

$$v^i\nabla_i v_1 - v_1\nabla_i v^i = -v_1\sum_{i=2}^{n}\nabla_i v^i = v_1 v^1\sum_{i=2}^{n} b^i_i. \tag{5.7}$$

Similarly, if the vector field v has zero normal component on the boundary B^{n-1}, then on the boundary B^{n-1},

$$v^i\nabla_i v_1 - v_1\nabla_i v^i = \sum_{i=2}^{n} v^i\nabla_i v_1 = \sum_{i,j=2}^{n} b_{ij}v^i v^j. \tag{5.8}$$

By making use of equation (5.6) and the fact that the right side of both equations (5.7), (5.8) is nonnegative for a totally geodesic or concave boundary B^{n-1}, we can easily obtain

THEOREM 5.1. — *On a compact orientable Riemannian manifold M^n with a totally geodesic or concave boundary B^{n-1} and negative definite Ricci curvature everywhere, there exists no nonzero conformal Killing vector field, with zero tangential or normal component on the boundary B^{n-1}, and therefore no infinitesimal conformal motion, other than the identity, leaving the boundary B^{n-1} invariant.*

Theorems 4.1 N, 4.2 N, 5.1 were obtained by YANO [14], and the author [4] with some boundary conditions missing due to a minor mistake, for which see [5].

In particular, if M^n is an Einstein manifold, then from equation (5.1) negative constant scalar curvature R implies negative definite Ricci curvature $R_{ij}v^iv^j$. Thus we have

COROLLARY 5.1. — *On a compact orientable Einstein manifold M^n with a totally geodesic or concave boundary B^{n-1} and negative constant scalar curvatur R, there exists no nonzero conformal Killing vector field, with zero tangential or normal component on the boundary B^{n-1}, and therefore no infinitesimal conformal motion, other than the identity, leaving the boundary B^{n-1} invariant.*

For any vector field v on a compact orientable Riemannian manifold M^n with boundary B^{n-1}, we obtain, by adding equations (3.12), (3.21) and making use of equation (3.13),

$$(5.9)\qquad (tv, tv) - (\Delta v + (1 - 2/n)d\,\delta v - Qv, v) = \int_{B^{n-1}} [v^i(\nabla_1 v_i + \nabla_i v_1) - 2v_1\nabla_i v^i/n]\, dA_{n-1}.$$

If the vector field v has zero tangential component on the boundary B^{n-1}, then by using local boundary geodesic coordinates and equations (2.22), (2.23) we have, on the boundary B^{n-1},

$$(5.10)\qquad n\nabla_1 v_1 - \nabla_i v^i = 0.$$

From equation (3.5) it follows immediately that on the boundary B^{n-1},

$$(5.11)\qquad \sum_{j=2}^{n} \nabla_j v^j = -v^1 \sum_{j=2}^{n} b^j_j,$$

and therefore equation (5.10) becomes

$$(5.12)\qquad \frac{n-1}{n}\nabla_i v^i + v^1 \sum_{j=2}^{n} b^j_j = 0 \quad \text{on } B^{n-1}.$$

For the converse, suppose that on a compact orientable Riemannian manifold M^n with boundary B^{n-1} a vector field v with zero tangential component on the boundary B^{n-1} satisfies equations (2.26), (5.12). Then the integrand on the right side of equation (5.9) vanishes due to equation (5.10), and from equations (5.9), (2.26), (3.22), (2.23) follows immediately equation (2.15), which shows that v is a conformal Killing vector field. Thus we arrive at

THEOREM 5.2 T. — *Let M^n be a compact orientable Riemannian manifold with boundary B^{n-1}. Then equations* (2.26), (5.12) *are necessary and* *sufficient conditions for a vector field v on the manifold M^n with zero tangential component on the boundary B^{n-1} to be a conformal Killing vector field.*

Theorem 5.2 T is due to YANO [14].

Similarly, if the vector field v has zero normal component on the boundary B^{n-1}, then by using local boundary geodesic coordinates on the boundary B^{n-1} from equation (2.22) it follows that the vector field v satisfies equation (4.3). For the converse, suppose that on a compact orientable Riemannian manifold M^n with boundary B^{n-1} a vector field v with zero normal component on the boundary B^{n-1} satisfies equations (2.26), (4.3). Then the integrand on the right side of equation (5.9) vanishes, and v is a conformal Killing vector field. Hence we have

THEOREM 5.2 N. — *Let M^n be a compact orientable Riemannian manifold with boundary B^{n-1}. Then equations* (2.26), (4.3) *are necessary and sufficient conditions for a vector field v on the manifold M^n with zero normal component on the boundary B^{n-1} to be a conformal Killing vector field and therefore to generate an infinitesimal conformal motion on the manifold M^n leaving the boundary B^{n-1} invariant.*

Theorem 5.2 N is due to YANO [14], and to LICHNEROWICZ (*see* [7]; [8], p. 129; [10] or [13]) for the case of empty B^{n-1}.

An infinitesimal transformation generated by a vector field v on a Riemannian manifold M^n is called an infinitesimal conformal collineation, if the vector field v satisfies equation (2.24). Multiplying equation (2.24) by g^{jk} and using equations (2.23), (2.25) we readily obtain equation (2.26). An application of theorems 5.2 T and 5.2 N thus gives

THEOREM 5.3 T. — *On a compact orientable Riemannian manifold M^n with boundary B^{n-1} a vector field v with zero tangential component on the boundary B^{n-1} and satisfying equations* (2.24), (5.12) *is a conformal Killing vector field.*

THEOREM 5.3 N. — *On a compact orientable Riemannian manifold M^n with boundary B^{n-1} an infinitesimal conformal collineation leaving the boundary B^{n-1} invariant, generated by a vector field v with zero normal component on the boundary B^{n-1} and satisfying equation* (4.3), *is an infinitesimal conformal motion.*

Now let v be a conformal Killing vector field on a Riemannian manifold M^n so that it satisfies equation (2.24). Substituting equation (2.24) in equation (2.10) and noticing that $\nabla_l \Phi_j = \nabla_j \Phi_l$, we obtain

$$(5.13) \qquad L_v R^i{}_{jkl} = -\delta^i_l \nabla_k \Phi_j + \varphi^i_k \nabla_l \Phi_j - g_{jk} \nabla_l \Phi^i + g_{jl} \nabla_k \Phi^i,$$

which is reduced to, by contraction with respect to i and l,

$$(5.14) \qquad L_v R_{ij} = -g_{ij} \nabla_h \Phi^h - (n-2) \nabla_j \Phi_i.$$

On the other hand, from equation (2.4) follows immediately

$$(5.15) \qquad L_v g^{ij} = -(\nabla^i v^j + \nabla^j v^i).$$

In virtue of equations (2.7), (5.2), (5.14), (5.15) we are therefore led to

$$(5.16) \qquad L_v R = -(\nabla^i v^j + \nabla^j v^i) R_{ij} - 2(n-1) \nabla_i \Phi^i.$$

By putting

$$(5.17) \qquad K_{ij} = -R_{ij} + \frac{R}{2(n-1)} g_{ij}$$

and using equations (2.), (2.15), (5.2), (2.25), (5.14), (5.16) we have

$$(5.18) \qquad \nabla_j \nabla_i \Phi = \frac{1}{n-2} L_v K_{ij}.$$

Multiplication of equation (5.18) by g^{ij} and use of equations (2.7), (2.22), (5.15), (5.17) give

$$(5.19) \qquad \nabla^i \nabla_i \Phi = -\frac{1}{2(n-1)} (L_v R + 2R\Phi).$$

In particular, if R is constant, then equation (5.19) becomes

$$(5.20) \qquad \nabla^i \nabla_i \Phi = -R\Phi/(n-1).$$

Now we further suppose that the gradient vector $\nabla\Phi$ has zero normal component on the boundary B^{n-1}. Then equation (5.20) and lemma 3.1 imply that if $R < 0$, then $\Phi = 0$, and therefore from equations (2.12), (2.15) the infinitesimal conformal motion generated by the vector field v is a motion.

Similarly, if $R = 0$, then Φ is constant, and the vector field v is homothetic by definition and an affine Killing vector field by equation (2.24). If the vector field v further has zero normal component on the boundary B^{n-1}, then it generates a motion leaving the boundary B^{n-1} invariant by theorem 4.2 N with the condition (4.3) automatically satisfied.

On a compact Riemannian manifold M^n with boundary B^{n-1} an infinitesimal conformal motion generated by a vector field v satisfying equation (2.15) is called an infinitesimal boundary conformal motion, if the gradient vector $\nabla\Phi$ has zero normal component on the boundary B^{n-1}; and v is called a boundary conformal Killing vector field on the manifold M^n. Thus we obtain

THEOREM 5.4 N. — *On a compact orientable Riemannian manifold M^n of constant nonpositive scalar curvature R with boundary B^{n-1}, an infinitesimal boundary conformal motion leaving the boundary B^{n-1} invariant is a motion.*

COROLLARY 5.4 N. — *If a compact orientable Riemannian manifold M^n of constant scalar curvature R with boundary B^{n-1} admits an infinitesimal nonhomothetic boundary conformal motion leaving the boundary B^{n-1} invariant, then $R > 0$.*

For the case of empty boundary B^{n-1}, theorem 5.4 N and corollary 5.4 N are due to YANO [13].

Now let v be a conformal Killing vector field on an Einstein manifold M^n. Making use of equations (5.17), (5.18), (5.1), (2.15) and the fact that R is constant, we then obtain

$$\nabla_j \nabla_i \Phi = \lambda \Phi g_{ij}, \tag{5.21}$$

and therefore, in consequence of the first equation of (2.25),

$$\nabla_i \Phi_j + \nabla_j \Phi_i = 2\lambda \Phi g_{ij}, \tag{5.22}$$

where

$$\lambda = -\frac{R}{n(n-1)}. \tag{5.23}$$

From equations (2.21), (5.22) follows immediately

$$\nabla_i w_j + \nabla_j w_i = 0, \tag{5.24}$$

so that w_i is a Killing vector field on the manifold M^n, where we have placed

$$w_i = v_i - \Phi_i/\lambda. \tag{5.25}$$

If v_i is a boundary conformal Killing vector field with zero normal component on the boundary B^{n-1}, then Φ_i is also due to equations (5.22), (5.23). By applying corollary 5.4 N we thus obtain

THEOREM 5.5 N. — *If a compact orientable Einstein manifold M^n with boundary B^{n-1} and $R > 0$ admits an infinitesimal nonhomothetic boundary conformal motion leaving the boundary B^{n-1} invariant, generated by a vector field v with zero normal component on the boundary B^{n-1}, then the vector field v can be decomposed into*

$$v^i = w^i + \Phi^i/\lambda, \tag{5.26}$$

where $\lambda = -R/n(n-1) < 0$, w^i is a Killing vector field with zero normal component on the boundary B^{n-1}, and $\Phi_i = \nabla_i \Phi$ is a boundary conformal Killing vector field with zero normal component on the boundary B^{n-1}.

For the case of empty boundary B^{n-1}, theorem 5.5 N was obtained by LICHNEROWICZ ([7] or [8], p. 136) by using de Rham's decomposition of a vector field on a compact orientable manifold.

Let $[u, u^\star]$ be the Lie product of two vector fields u and $u^\star$ on a Riemannian manifold M^n, so that

$$(5.27) \qquad [u, u^\star] = uu^\star - u^\star u.$$

Since in terms of the local coordinates $x^1, \ldots, x^n$ we can express u and $u^\star$ as

$$(5.28) \qquad u = u^i \frac{\partial}{\partial x^i}, \qquad u^\star = u^{\star i} \frac{\partial}{\partial x^i},$$

from equation (5.27) it follows that

$$[u, u^\star] = u^i \frac{\partial}{\partial x^i}\left(u^{\star j} \frac{\partial}{\partial x^j}\right) - u^{\star i} \frac{\partial}{\partial x^i}\left(u^j \frac{\partial}{\partial x^j}\right)$$
$$= \left(u^i \frac{\partial u^{\star j}}{\partial x^i} - u^{\star i} \frac{\partial u^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

Thus the contravariant and covariant components of the vector $[u, u^\star]$ are given by

$$(5.29) \qquad [u, u^\star]^j = u^i \nabla_i u^{\star j} - u^{\star i} \nabla_i u^j,$$

$$(5.30) \qquad [u, u^\star]_j = u^i \nabla_i u^\star_j - u^{\star i} \nabla_i u_j.$$

From equations (2.4), (5.29) it follows that

$$(5.31) \qquad [u, u^\star]^j = L_u u^{\star j}.$$

If u is a conformal Killing vector field satisfying equation (2.15), by equations (2.4), (5.30) we obtain

$$(5.32) \qquad [u, u^\star]_j = L_u u^\star_j - 2\Phi u^\star_j.$$

Now suppose that on an Einstein manifold M^n with boundary B^{n-1} there exist two infinitesimal nonhomothetic boundary conformal motions leaving the boundary B^{n-1} invariant and generated by two vector fields v and $v^\star$ respectively. Then we have equations (2.15), (5.21), (5.22), (5.24), (5.26) and similar equations for the vector field $v^\star$, which will be denoted by the same numbers with a star. By means of equations (2.8), (5.32), (2.5), (2.15)*, (2.24) we obtain

$$(5.33) \qquad L_{[v, v^\star]}\, g_{ij} = \nabla_i (L_v v^\star_j - 2\Phi v^\star_j) + \nabla_j (L_v v^\star_i - 2\Phi v^\star_i)$$
$$= 2(v^k \Phi^\star_k - v^\star_k \Phi^k) g_{ij}.$$

On the other hand, from equations (2.8), (5.24), (5.24)* it follows that

$$(5.34) \qquad L_{tv}\, g = 0, \qquad L_{tv^\star}\, g = 0,$$

and therefore

(5.35) $$[L_w, L_{w^\star}]\, g \equiv (L_w L_{w^\star} - L_{w^\star} L_w)\, g = 0.$$

Since w is a Killing vector field, we have, in consequence of equations (2.11), (2.12) for w,

(5.36) $$L_w \Gamma^i_{jk} = 0,$$

which and equations (5.32) for $\Phi = 0$, (2.8), (2.5), (5.24)* lead immediately to

(5.37) $$\begin{aligned} L_{[w, w^\star]}\, g_{ij} &= \nabla_i (L_w w^\star_j) + \nabla_j (L_w w^\star_i) \\ &= L_w (\nabla_i w^\star_j + \nabla_j w^\star_i) + 2\, w^\star_s L_w \Gamma^s_{ij} = 0. \end{aligned}$$

From equations (2.4), (2.7), (5.32) and $\nabla_k \Phi_i = \nabla_i \Phi_k$, it follows that

(5.38) $$\nabla_i (L_w \Phi) = \nabla_i (w^j \Phi_j) = L_w \Phi_i = [w, \nabla \Phi]_i.$$

Similarly, making use of equations (2.5), (5.36), (5.21), (5.30), (5.21)* we can easily obtain

(5.39) $$\nabla_j (L_w \Phi_i) = L_w (\nabla_j \Phi_i) = L_w (\lambda \Phi\, g_{ij}) = \lambda (L_w \Phi)\, g_{ij},$$

(5.40) $$[\Phi, \Phi^\star]_k = \lambda (\Phi_k \Phi^\star - \Phi^\star_k \Phi),$$

(5.41) $$L_{[\Phi, \Phi^\star]}\, g_{ij} = \nabla_i [\Phi, \Phi^\star]_j + \nabla_j [\Phi, \Phi^\star]_i = 0.$$

Now we observe that if v and $v^\star$ are two vector fields on the Einstein manifold M^n with zero normal components on the boundary B^{n-1}, then by using local boundary geodesic coordinates and equations (5.30), (2.15), (2.15)*, (3.5) we can easily see that on the boundary B^{n-1},

(5.42) $$[v, v^\star]_1 = 0.$$

Similarly, if v and $v^\star$ are two boundary conformal Killing vector fields on the Einstein manifold M^n with zero normal components on the boundary B^{n-1}, then by noticing the relations $\nabla_i \Phi_k = \nabla_k \Phi_i$, $\nabla_i \Phi^\star_k = \nabla_k \Phi^\star_i$ we obtain, on the boundary B^{n-1},

(5.43) $$\nabla_1 (v^k \Phi^\star_k - v^\star_k \Phi^k) = 0.$$

Furthermore, equations (5.38), (5.39), (5.21) imply that $[w, \nabla \Phi]$ can be considered as $\nabla \overline{\Phi}$, where $\overline{\Phi}$ is related to a nonhomothetic boundary conformal Killing vector field $\bar{v}$ on the manifold M^n by the equation $L_{\bar{v}}\, g = 2\, \overline{\Phi}\, g$.

Combining the above results and applying theorem 5.5 N we thus arrive at

THEOREM 5.6 N. — *Let M^n be a compact orientable Einstein manifold with boundary B^{n-1} and $R > 0$. Then the infinitesimal nonhomothetic* *boundary conformal motions on the manifold M^n leaving the boundary B^{n-1} invariant form a Lie algebra L, which can be decomposed into the direct sum*

$$L = L_1 + L_2 \tag{5.44}$$

with the relations

$$[L_1, L_1] \subset L_1, \qquad [L_1, L_2] \subset L_2, \qquad [L_2, L_2] \subset L_1, \tag{5.45}$$

$$\dim L_1 \geqq \dim L_2 - 1, \qquad \dim L_1 \geqq \frac{1}{2}(\dim L - 1), \tag{5.46}$$

where L is the subalgebra of L defined by the infinitesimal motions on the manifold M^n leaving the boundary B^{n-1} invariant, and L_2 the vector space of $\nabla\Phi$, Φ being given in equation (2.15) defining all the nonhomothetic infinitesimal boundary conformal motions on the manifold M^n.

Equations (5.46) are obtained from the fact that if $\omega_0, \omega_1, \ldots, \omega_q$ form a basis of L_2, then the q elements $[\omega_0, \omega_i]$ $(i = 1, \ldots, q)$ of L_1 are linearly independent. For the case of empty boundary B^{n-1}, theorem 5.6 N is due to LICHNEROWICZ ([7] or [8], p. 138), and was proved again by YANO [13] by a different method, which is extended in this paper.

6. **Projective Killing vector fields and infinitesimal projective motions.**

On a Riemannian manifold M^n with boundary B^{n-1} let a vector field v generate an infinitesimal projective motion so that equation (2.14) holds. Substituting equation (2.14) in equation (2.10), contracting with respect to i and l, changing k to i and noticing that

$$\nabla_i p_j = \nabla_j p_i, \tag{6.1}$$

we can easily obtain

$$L_v R_{ij} = (1-n)\nabla_j p_i, \tag{6.2}$$

which and equations (2.5), (2.10) yield immediately

$$L_v(\nabla_k R_{ij}) = (1-n)\nabla_k\nabla_j p_i - 2p_k R_{ij} - p_i R_{jk} - p_j R_{ki}. \tag{6.3}$$

If M^n is an Einstein manifold, then $\nabla_k R_{ij} = 0$ and equation (6.3) can be reduced by equation (5.1) to

$$(1-n)\nabla_k\nabla_j p_i = c(2p_k g_{ij} + p_i g_{jk} + p_j g_{ki}), \tag{6.4}$$

where c is a constant defined by

$$c = R/n. \tag{6.5}$$

From equation (6.4) and the Ricci identity (1.24) for p_i, it follows

$$(6.6) \qquad -p_l R^l{}_{ijk} = \frac{c}{1-n}(p_k g_{ij} - p_j g_{ik}).$$

Since $p_k = \nabla_k p$, by multiplying equation (6.4) by g^{ij} we obtain

$$(6.7) \qquad \nabla_k[(1-n)\nabla^i\nabla_i p - 2(n+1)cp] = 0,$$

which implies

$$(6.8) \qquad (1-n)\nabla^i\nabla_i(p-p_0) = 2(n+1)c(p-p_0),$$

where p_0 is constant. If $\frac{c}{1-n} > 0$ or $R < 0$ and in local boundary geodesic coordinates $\nabla_1 p = 0$ on the boundary B^{n-1}, then by lemma 3.1, $p - p_0 = 0$ or $p_i = 0$ on the manifold M^n.

Thus from equation (2.14) we have $L_v \Gamma^i_{jk} = 0$, that is, the infinitesimal projective motion generated by the vector field v is an infinitesimal affine collineation, which is a motion by theorem 4.2 N if on the boundary B^{n-1} the vector field v has zero normal component and satisfies equation (4.3). An application of corollary 5.1 thus gives

THEOREM 6.1 N. — *On a compact orientable Einstein manifold M^n with a totally geodesic or concave boundary B^{n-1} and negative constant scalar curvature R, there exists no infinitesimal projective motion, other than the identity, leaving the boundary B^{n-1} invariant such that on the boundary B^{n-1} its generating vector field v has zero normal component and satisfies $\nabla_1 p = 0$ and equation* (4.3) *in local boundary geodesic coordinates.*

For the case of empty boundary B^{n-1}, theorem 6.1 N is due to YANO and NAGANO [11].

Now let us consider a projective Killing vector field v on a Riemannian manifold M^n with boundary B^{n-1}. Then by subtracting equation (3.14) multiplied by $2/(n+1)$ from equation (3.17) and making use of equation (2.21) we can immediately obtain

$$(6.9) \qquad -(Qv, v) + \frac{n-1}{n+1}(\delta v, \delta v) + (dv, dv)$$
$$= \int_{B^{n-1}} [v^i(\nabla_1 v_i - \nabla_i v_1) + \frac{n-1}{n+1} v_1 \nabla_i v^i]\, dA_{n-1}.$$

Thus, from equations (6.9), (1.20), (2.21), (4.1), and theorem 4.1 T, follows

THEOREM 6.2 T. — *On a compact orientable Riemannian manifold M^n with boundary B^{n-1}, if a projective Killing vector field v satisfies $R_{ij} v^i v^j \leqq 0$,* *and on the boundary B^{n-1} has zero tangential component and satisfies $\delta v = 0$, then it must satisfy $R_{ij} v^i v^j = 0$, and is a parallel vector field, that is, $\nabla v = 0$ on the manifold M^n, for a totally geodesic boundary B^{n-1}. In particular, on the manifold M^n if the Ricci curvature $R_{ij} v^i v_j$ is negative definite definite everywhere, then there exists no such nonzero projective Killing vector field v.*

Similarly, we have, in consequence of theorem 4.1 N,

THEOREM 6.2 N. — *On a compact orientable Riemannian manifold M^n with boundary B^{n-1}, if a projective Killing vector field v with zero normal component on the boundary B^{n-1} satisfies $R_{ij} v^i v^j \leqq 0$ and*

$$\sum_{i=2}^{n} v^i (\nabla_1 v_i - \nabla_i v_1) = 0 \quad \text{on } B^{n-1} \tag{6.10}$$

in local boundary geodesic coordinates, then it must satisfy $R_{ij} v^i v^j = 0$, and is a parallel vector field for a totally geodesic boundary B^{n-1}. In particular, on the manifold M^n if the Ricci curvature $R_{ij} v^i v^j$ is negative definite everywhere, then there exists no such nonzero projective Killing vector field v.

For the case of empty boundary B^{n-1}, theorems 6.2 T and 6.2 N are due to COUTY [2]. For the corresponding theorems on the nonexistence of a Killing vector field on a Riemannian manifold M^n with boundary B^{n-1}, *see* [6].

Now let v be a projective Killing vector field on an Einstein manifold M^n. Then, from equations (1.27), (5.1), it follows immediately

$$Qv = \frac{2R}{n} v. \tag{6.11}$$

By means of equations (2.21), (1.20), (6.11) we thus obtain

$$v = \frac{n}{2R} \delta dv + \frac{n(n-1)}{2R(n+1)} d\delta v. \tag{6.12}$$

On the other hand, in consequence of equations (1.16), (2.20), and (6.2), (5.1) we have, respectively,

$$(d\delta v)_i = -(n+1) p_i, \tag{6.13}$$

$$L_v g_{ij} = -\frac{n(n-1)}{R} \nabla_j p_i. \tag{6.14}$$

From equations (6.12), (6.13), (6.14) it is readily seen that

$$L_v (\delta dv) = 0, \tag{6.15}$$

so that δdv is a Killing vector field. Since a Killing vector field is a special projective Killing vector field, equation (6.12) implies that $d\delta v$ is a projective Killing vector field. For later use we need the formula

$$L_{\nabla p}\Gamma^{i}_{jk} = -\frac{2R}{n(n-1)}(p_j \delta^i_k + p_k \delta^i_j), \tag{6.16}$$

which can easily be obtained from equations (2.9), (6.4), (6.6), and also shows by definition that $d\delta v$ is a projective Killing vector field.

An application of theorems 6.1 N and 6.2 T together with a use of equations (6.1), (6.14), (6.16) thus gives

THEOREM 6.3. — *If a compact orientable Einstein manifold M^n with boundary B^{n-1} and positive constant scalar curvature R admits a projective Killing vector field v such that on the boundary B^{n-1} it satisfies one of the following two sets of conditions in local boundary geodesic coordinates :*

$$v_1 = 0, \qquad \nabla_1 p = 0, \qquad \sum_{i=2}^{n} (\nabla_1 v_i + \nabla_i v_1) = 0, \tag{6.17}$$

$$v_i = 0, \qquad \nabla_i p = 0, \qquad \delta v = 0 \qquad (i \neq 1), \tag{6.18}$$

then the vector field v has a decomposition given by equation (6.12), *where δdv is a Killing vector field, δdv is an exact projective Killing vector field, and on the boundary B^{n-1} they both satisfy the conditions* (6.17) *or* (6.18) *at the same time as the vector field v.*

For the case of empty boundary B^{n-1}, theorem 6.3 is due to YANO and NAGANO [11].

REFERENCES.

[1] CARTAN (H.). — Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie, *Colloque de Topologie* [1950, Bruxelles], p. 15-27. — Liège, G. Thone; Paris, Masson, 1951.

[2] COUTY (R.). — Transformations infinitésimales projectives, *C. R. Acad. Sc., Paris*, t. 247, 1958, p. 804-806.

[3] EISENHART (L. P.). — *Riemannian geometry.* — Princeton, Princeton University Press, 1949.

[4] HSIUNG (C.-C.). — Curvature and Betti numbers of compact Riemannian manifolds with boundary, *Rend. Sem. Mat., Univ. e Politec. Torino*, t. 17, 1957-1958, p. 95-131.

[5] HSIUNG (C.-C.). — A note of correction, *Rend. Sem. Mat., Univ. e Politec. Torino*, t. 21, 1961-1962, p. 127-129.

[6] HSIUNG (C.-C.). — Curvature and homology of Riemannian manifolds with boundary, *Math. Z.*, t. 82, 1963, p. 67-81.

[7] LICHNEROWICZ (A.). — Transformations infinitésimales conformes de certaines variétés riemanniennes compactes, *C. R. Acad. Sc. Paris*, t. 241, 1955, p. 726-729.

[8] Lichnerowicz (A.). — *Géométrie des groupes de transformations.* — Paris, Dunod, 1958 (Travaux et Recherches mathématiques, 3).
[9] de Rham (G.) and Kodaira (K.). — *Harmonic integrals.* — Princeton, Institute for Advanced Study, 1950 (Mimeographed notes).
[10] Sato (L.). — On conformal Killing tensor fields, *Bull. Yamagata University* (*Nat. Sc.*), t. 3, 1956, p. 175-180.
[11] Yano (K.) and Nagano (T.). — Some theorems on projective and conformal transformations, *Koninkl. Nederl. Akad. Wetensch., Proc.*, Serie A, t. 60 (*Indag. Math.*, t. 19), 1957, p. 451-458.
[12] Yano (K.). — *The theory of Lie derivatives and its applications.* — Amsterdam, North Holland publishing Company, 1957 (Bibliotheca mathematica, 3).
[13] Yano (K.). — Some integral formulas and their applications, *Michigan Math. J.*, t. 5, 1958, p. 63-73.
[14] Yano (K.). — Harmonic and Killing vector fields in compact orientable Riemannian spaces with boundary, *Annals of Math.*, t. 69, 1959, p. 588-596.

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