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ON Σ -GROUPS

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Recently, IRWIN and WALKER [2] introduced the notion of Σ -groups. One question left open is whether or not the property of being a Σ -group is hereditary, (i. e.) whether every subgroup of a Σ -group is again a Σ -group. We will show that this is not always the case by constructing an easy example. Further, it is asked in [3] whether every high subgroup of a Σ -group G is an endomorphic image of G . We will give an affirmative answer to this question. Lastly, we investigate when a cotorsion group is a Σ -group.

All groups considered in this note are abelian. If G is any group, then G^1 denotes the subgroup of elements of infinite height in G , that is,

$$G^1 = \bigcap_{n=1}^{\infty} nG. \text{ We will sometimes refer to it as the } \textit{Radical} \text{ of } G.$$

A subgroup maximally disjoint with G^1 is known as a high subgroup of G . A group is called a Σ -group, if all its high subgroups are direct sums of cyclic groups. It is known [3] that if G is a Σ -group, then all its high subgroups are isomorphic. A group G is called cotorsion if it is reduced and $\text{Ext}(Q, G) = 0$, where Q is the additive group of rational numbers. For general properties of high subgroups we refer to [2]. If G is any group, then G_t denotes the torsion part of G .

LEMMA 1.

- (i) Every torsion group G contains a Σ -group R such that $G^1 = R^1$, R is pure in G and G/R divisible.
- (ii) Every torsion-free group G contains a Σ -group R such that $G^1 = R^1$ and R is pure in G .

PROOF. — Assertion (i) is the content of theorem 10 of [2], while (ii) follows on noting that for the torsion free group G , the radical G^1 is divisible and so is itself a Σ -group.

LEMMA 2. — *Every reduced torsion group G is isomorphic to the radical of some reduced torsion Σ -group R .*

PROOF. — Appealing to the assertion (i) of lemma 1, we see that it is enough if we show that G is a subgroup of a torsion group H , with $H^1 = G$. To each $a \in G$, associate a sequence of elements $(a_2, a_3, \dots, a_n, \dots)$ with the conditions that $na_n = a$. Let H be the group obtained by adjoining to G all such sequences of elements with the stated conditions. It is then immediately checked up that $H^1 = G$ and that H is torsion.

THEOREM 1 — *Not every subgroup of a Σ -group need again be a Σ -group* ⁽¹⁾.

PROOF. — Since every divisible group is a Σ -group, the assertion follows trivially, when we consider a divisible group containing a group G which is not a direct sum of cyclic groups and for which $G^1 = 0$.

We could even give an example of a reduced group which is a Σ -group and not all subgroups of which are Σ -groups.

Let G be an unbounded closed p -group. Then it follows from [1] (p. 116) that G is not a Σ -group. But, by lemma 2, there is a reduced Σ -group R which contains G (even as its radical). This R is the required counter example.

The following theorem answers a question raised in [3].

THEOREM 2. — *Let G be a Σ -group. Then every high subgroup of G is an endomorphic image of G .*

PROOF. — We assume, without loss in generality, that G is reduced. If G is torsion, then every high subgroup of G is a basic subgroup of G and hence is an endomorphic image of G . If G is torsion free, since G is reduced, $G^1 = 0$ and so, G is its own high subgroup, so that it clearly has the required property. Let now G be a mixed group. Then we distinguish two cases :

Case 1. — Let G/G_t be reduced. Now every high subgroup H of G is a direct sum of cyclic groups and hence H splits. Then, by theorem 2 of [3], G also splits; $G = G_t + F$. Since G_t and F are Σ -groups, all their high subgroups are endomorphic images. Then we readily check up that every high subgroup of G is an endomorphic image of G .

⁽¹⁾ The author thanks Prof. E. A. WALKER for offering comments towards the simplification of the example. He also thanks him for having given the benefit of papers [2] and [3] long before their publication.

Case 2. — Let G/G_i be not reduced. Let M/G_i be the maximal divisible subgroup of G/G_i . Let $H = H_i + H_f$ be a high subgroup of G . Then by theorem 1 of [3], $G = M + H_f$.

Now H_i is high in G_i and $G_i \subseteq M$. We show that H_i is high in M . Clearly, $H_i \cap M' = 0$. If H_i is not high in M , let $K \supseteq H_i$ be high in M . Then, $K + H_f$ includes $H_i + H_f = H$ and is high in G , which contradicts the maximality of H . Hence $K = H_i$ so that H_i is high in M .

Now M is a mixed group such that it has a high subgroup which is torsion. On the other hand, the group M , being a direct summand of G , is a Σ -group. Thus all high subgroups of M are isomorphic and hence torsion. Now if $y \in M$ and $o(y) = \infty$, then $\{y\} \cap M' \neq 0$ since otherwise, $\{y\}$ can be expanded to a high subgroup of M which will no longer be torsion. Hence there exists an integer k such that $ky \in M'$. This implies, in particular, that $M^* = M/M'$ is a torsion group. Let H^* be the image of H_i in M^* . Then $H^* \cong H_i$ and hence is a direct sum of cyclic groups.

We show that H^* is pure in M^* . Let $nx^* = a^* \in H^*$, where $x^* \in M^*$. Then $nx = a + b$, where $x \in M$, $a \in H$, $b \in M'$. Since b is in M' , $b = ny$, $y \in M$. Therefore,

$$\begin{aligned} n(x - y) &= a \in H_i \\ &= nz, \quad z \in H_i, \quad \text{since } H_i \text{ is pure.} \end{aligned}$$

$$\begin{aligned} \text{Hence } nx &= nz + b \\ \text{(i. e.) } nx^* &= nz^*, \quad z^* \in H^*. \end{aligned}$$

This proves that H^* is pure in M^* .

Thus H^* is a direct sum of cyclic groups and is pure in the torsion group M^* . Then H^* is a direct summand of a basic subgroup B^* of M^* and so H^* is an endomorphic image of M^* . Since M^* is an epimorphic image of M , it follows that H^* is an epimorphic image of M . But H^* is isomorphic to H_i so that we are assured of an epimorphism of M on H_i . That is, H_i is an endomorphic image of M . Let this mapping be θ .

Now $G = M + H_f$. Let π and π' be the corresponding projections, $\pi(G) = M$, $\pi'(G) = H_f$. Then define a mapping δ of G in to itself as $\delta = \theta\pi + \pi'$. Now we can readily check up that δ is an endomorphism and $\delta(G) = H$. Thus H is an endomorphic image of G and this completes the proof.

REMARK. — It is worth noting that the subgroup M is fully invariant in G . This follows from the fact that M is the unique complementary summand of H_f in G which together with theorem 22.3 of [1] implies that M is fully invariant.

We now investigate when a cotorsion group is a Σ -group. Before that we consider two lemmas which are of independent interest.

LEMMA 3. — *A high subgroup H of a cotorsion group G is an endomorphic image of G if and only if $G = H$.*

PROOF. — We need only to prove the necessary part. If H is an endomorphic image of G , then clearly it should be cotorsion. Since H is high in G , G/H is divisible and so, the exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0$$

gives the exact sequence,

$\text{Hom}(Q, G) = 0 \rightarrow \text{Hom}(Q, G/H) \rightarrow \text{Ext}(Q, H) \rightarrow \text{Ext}(Q, G) = 0$, where the first and the last terms are zero since G is cotorsion. Since G/H is divisible, $\text{Hom}(Q, G/H) \neq 0$. This means that $\text{Ext}(Q, H) \neq 0$ so that H is not cotorsion, which is a contradiction unless $G/H = 0$, (i. e.) $G = H$.

LEMMA 4. — *A cotorsion group is a direct sum of cyclic groups if and only if it is bounded.*

The proof follows from the observations :

- (i) A direct summand of a cotorsion group is again cotorsion.
- (ii) An infinite cyclic group S is cotorsion if and only if $S = 0$.
- (iii) A torsion cotorsion group is bounded.

THEOREM 3. — *A cotorsion group is a Σ -group if and only if it is bounded.*

PROOF. — If G is a Σ -group, by theorem 2, H (high in G) is an endomorphic image of G . But then, by lemma 3, G coincides with H . Now H is a direct sum of cyclic groups and so lemma 4 settles that G is bounded.

On the other hand, if G is bounded, then clearly it is a Σ -group.

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