P. Hill
C. Megibben

Minimal pure subgroups in primary groups


<http://www.numdam.org/item?id=BSMF_1964__92__251_0>
Throughout all groups are assumed to be primary abelian groups, and all topological references are to the \( p \)-adic topology. By a subsocle of a group we mean a subgroup of the socle. Thus \( S \) is a subsocle of \( G \) if \( S \) is a subgroup of \( G \) and if \( px = 0 \) for all \( x \) in \( S \). Let \( H \) be a subgroup of \( G \). If among the pure subgroups of \( G \) which contain \( H \) there exists a minimal one, we say that \( H \) is contained in, or is imbedded in, a minimal pure subgroup in \( G \). B. Charles studied minimal pure subgroups in [1]; he asserted that each of the conditions

1. \( H \) is a subsocle of \( G \)

and

2. There is a pure subgroup of \( G \) contained in \( H \) which is dense in \( H \)

is sufficient for the existence of a minimal pure subgroup for \( H \) in \( G \) provided \( G \) is without elements of infinite height. Head showed in [4] that condition (2) is not sufficient, and one of the authors showed in [6] that neither is condition (1).

In this paper we characterize the groups \( G \) in which each subgroup is imbedded in a minimal pure subgroup. The characterization is:

\( G \) is the sum of a divisible and a bounded group. We give a short proof of a theorem of Irwin and Walker [5] and give a solution to a new generalization of Fuchs' Problem 4. Some results are also given concerning minimal pure subgroups for subsocles.

It was shown in [3] that most groups have neat dense subgroups which do not contain basic subgroups. The following theorem shows, however, that if a neat subgroup has a dense subsocle, then it must contain a basic subgroup.
THEOREM 1. — Let $S$ be a dense subsocle of $G$, $S = G[p]$. If $H$ is maximal in $G$ with respect to $H[p] = S$, then $H$ is pure and dense in $G$.

PROOF. — Let $H$ be maximal in $G$ with respect to $H[p] = S$. Then $H$ is neat in $G$, that is, $H \cap pG = pH$. We need to show that $H \cap p^nG = p^nH$ for all natural numbers $n$; our proof is by induction. Assume that $H \cap p^nG = p^nH$ and suppose that $p^{n+1}x \in H$. Since $H$ is neat, there is an $h_0 \in H$ such that $p^{n+1}x = ph_0$. The element $p^n x - h_0$ is in $G[p]$. Since $S$ is dense in $G[p]$, there is an $s \in S$ such that $p^n x - h_0 - s$ is in $p^n G$. By the induction hypothesis, there is an $h_1 \in H$ such that $p^n x = ph_0 + s$. Thus $p^{n+1} h_1 = ph_0 + p^{n+1} x$ and $H$ is pure.

Since $H$ is pure, any element of order $p$ in $G/H$ can be represented by an element of order $p$ in $G$. Therefore, the density of $H[p] = S$ in $G[p]$ implies that each element of order $p$ in $G/H$ has infinite height. Hence $G/H$ is divisible, that is, $H$ is dense in $G$.

COROLLARY 1 (IRWIN and WALKER [5]). — Let $N$ be a subgroup of $G'$, the elements of infinite height in $G$. If $H$ is maximal in $G$ with respect to $H \cap N = o$, then $H$ is pure in $G$.


One may generalize problem 4 in [2] by replacing the subgroup $G'$ by an arbitrary fully invariant subgroup. The solution to the generalized problem is contained in the following corollary and a well known result of Szele.

COROLLARY 2. — Let $F$ be a fully invariant subgroup of $G$ and let $A$ be a subgroup of $G$ such that $A \cap F = o$. Then $A$ is contained in a pure subgroup $H$ of $G$ such that $H \cap F = o$.

PROOF. — If $F \subseteq G'$, the conclusion follows from the preceding corollary. Assume that $F$ is not contained in $G'$. Let $\sum B_n$ be the standard decomposition of a basic subgroup $B$ of $G$ into homogeneous groups $B_n$. Define $A_1 = G$ and $A_{n+1} = \{B_{n+1}, B_{n+2}, \ldots, p^n G\}$ for $n \geq 1$. Then $G = B_1 + B_2 + \ldots + B_n + A_{n+1}$. Since $F$ is fully invariant with elements of finite height in $G$, $F[p] = A_m[p]$ where $m$ is the smallest positive integer such that $F \cap B_m \neq o$.

It follows from [2] (theorem 22.2) that $A_m$ is an absolute direct summand of $G$. Hence if $H$ is maximal with respect to $H \cap F = o$, then $H$ is maximal with respect to $H \cap A_m = o$ and is a direct summand of $G$; in particular, $H$ is pure in $G$. 
The following theorem, which is of independent interest, (eventually) implies that most groups have subgroups which are not imbedded in minimal pure subgroups.

**Theorem 2.** — Let $L$ be a subgroup of $G$. If $H$ is a minimal pure subgroup of $G$ containing $L$, then $H = A + K$ where $A$ is bounded and $K[p] = L[p]$.

**Proof.** — There is no pure subgroup of $H$ properly between $L$ and $H$. It follows from theorem 1 that every subsocle of $H$ which contains $L[p]$ is closed in $H[p]$.

Define $S_n = L[p] \cap p^nH$ and let $S_n = Q_n + S_{n+1}$ for $n = 0, 1, 2, \ldots$. The height in $H$ of each nonzero element of $Q_n$ is exactly $n$. Moreover, if $C_n$ is (zero or) a direct sum of cyclic groups or order $p^n$ such that $C_n[p] = Q_{n-1}$, then $C = \sum C_n$ is pure in $H$. Extend $C$ to a basic subgroup $B = A + C$ of $H$.

Suppose that there is an element $x$ of order $p$ in $A \cap L$. Since $x$ is in $A$, it has finite height $t$ in $H$. The closure (in $H$) of $C[p]$ contains $L[p]$. Thus $x = p^{t+1}h + c$ where $c \in C[p]$ and $h \in H$. This implies that $x - c$ has height greater than $t$ in $H$ and, consequently, in $B$ since $B$ is pure in $H$. This is impossible since $B = A + C$, so $A \cap L = \emptyset$.

Assume that $A$ is unbounded. Then $A$ has a proper basic subgroup $A_i$. Since $B = A + C$ is basic in $H$, $B_i = A_i + C$ is basic in $H$. Thus

$$A_i[p] + L[p] \subseteq B_i[p] = H[p].$$

Since this contradicts the fact that $A_i[p] + L[p]$ is a proper closed subsocle of $H$, we conclude that $A$ is bounded.

Let $p^mA = \emptyset$. An argument similar to the one given above for the proof that $A \cap L = \emptyset$ shows that $A \cap (C, p^mH) = \emptyset$. Now we have that

$$H = \{ B, p^mH \} = \{ A + C, p^mH \} = A \cap (C, p^mH).$$

Define $K = \{ C, p^mH \}$. The purity of $C$ implies that

$$K[p] = \{ C[p], p^mH[p] \}.$$


**Proposition 1.** — If each subgroup of $G$ is contained in a minimal pure subgroup of $G$, then $G$ has a bounded basic subgroup.
PROOF. — Suppose that $B = \sum B_n$ is a basic subgroup of $G$ where $B_n \neq \emptyset$ for infinitely many $n$ and is a homogeneous group of degree $n$. Choose a sequence $n(i)$ of positive integers such that $n(i+1) - n(i) \geq 2$ and such that $B_{n(i)} \neq \emptyset$. Define

$$t(i) = n(2i+1) - n(2i) - 1$$

and let

$$L = \sum_{i=1}^{\infty} \{ b_{n(2i)} + p^{t(i)} b_{n(2i+1)} \}$$

where $\{ b_{n(i)} \}$ is a nonzero direct summand of $B_{n(i)}$. Suppose that $H$ is a minimal pure subgroup of $G$ containing $L$. By theorem 2, $H = A + K$ where $A$ is bounded and $K[p] = L[p]$.

Let $p^n A = \emptyset$. Then $p^n H[p] \subseteq L[p]$. Let $G = \{ b_{n(2i)} \} + \{ b_{n(2i+1)} \} + G_n$. Since $p^{n(2i+1)+1} b_{n(2i+1)}$ is in $L$, there is an element $h_0 = j b_{n(2i)} + b_{n(2i+1)} + g_o$ in $H$ where $j$ is an integer, $g_o \in G_o$, and

$$p^{n(2i+1)+1} h_0 = p^{n(2i+1)+1} b_{n(2i+1)}.$$

Now the element

$$h_i = (b_{n(2i)} + p^{t(i)} b_{n(2i+1)}) - p^{t(i)} h_0 = b_{n(2i)} - p^{t(i)} (j b_{n(2i)} + g_o)$$

is in $H$. Since $p^{n(2i)+1} h_i = p^{n(2i)+1} b_{n(2i)}$ and since $p^n H[p] \subseteq L[p]$, we conclude that $L$ contains $p^{n(2i)+1} b_{n(2i)}$ if $i \geq m$. However, it is immediate from the definition of $L$ that this is impossible, so $L$ is not contained in a minimal pure subgroup of $G$.

PROPOSITION 2. — If $G$ is a bounded group, each subgroup of $G$ is contained in a minimal pure subgroup of $G$.

PROOF. — Our proof is by induction on $n$ where $p^n G = \emptyset$. If $p G = \emptyset$, every subgroup is pure. Suppose that $L$ is a subgroup of $G$ and that $p^{n+1} G = \emptyset$. Since a homogeneous subgroup of $G$ of degree $n + 1$ is an absolute direct summand, we may assume that $p^n G \leq L$.

Let

$$L = L_{n+1} + G_n, \quad p G \cap C_n = L_n + C_{n-1}, \quad \ldots \ldots \ldots \ldots, \quad p^n G \cap C_i = L_i,$$

where $L_i$ is a homogeneous group of degree $i$ with $L_{n+1}$ chosen maximal in $L$ and $L_i$ chosen maximal in $p^{n+1-i} G \cap C_i$ for $i = n, n-1, \ldots, 1$. Observe that there are homogeneous subgroups $B_i$ of $G$ of degree $n + 1$.
such that $B_i \supseteq L_i$ and $B_i[p] = L_i[p]$. Define $B = \sum B_i$. Then $B[p] = p^n G \cap L = p^n G$.

Since $B$ is an absolute direct summand of $G$, there are decompositions

$$|L, B| = K + B$$

and

$$G = H + B$$

such that $H \supseteq K$. Since $p^n G = B[p]$, $p^n H = 0$. By the induction hypothesis, $K$ is contained in a minimal pure subgroup $A$ of $H$. We prove that $A + B$ is a minimal pure subgroup of $G$ containing $L$.

Suppose that $S$ is a pure subgroup of $A + B$ containing $L$. We wish to show that $S = A + B$. Proceeding by induction, assume that $p^i A \subseteq S$ and that $p^{i+1} B \subseteq S$. From these two conditions it follows that $p^i B \subseteq S$, and it remains to show that $p^{i-1} A \subseteq S$. Routine considerations show that it suffices to prove that $p^{i-1} A[p] \subseteq S$.

Let $T = S \cap p^{i-1} A[p]$ and let $p^{i-1} A[p] = T + R$. Assume that $R \neq 0$. Choose a pure subgroup $R^*$ of $A$ such that $R^*[p] = R$. Observe that $R^*$ is homogeneous of degree $i$. From the construction of $B$, it can be shown that $p^i A \cap K \subseteq \{L, p^i B\}$. From this fact it follows that $R^* \cap \{p^i A, K\} = 0$. Choose a subgroup $F \supseteq \{p^i A, K\}$ and maximal in $A$ with respect to $F \cap R^* = 0$. Since $A$ is minimal pure for $K$ in $H$, $F$ cannot be pure in $A$. Hence $R^* + F$ is a proper subgroup of $A$. Choose an element $a \in A$ such that $a \notin R^* + F$ and such that $pa \in R^* + F$. Letting $pa = r^* + f$ where $r^* \in R^*$ and $f \in F$, we obtain contradictory statements: $r^*$ has height zero in $R^*$; and $p^{i-1} r^* = 0$.

We conclude that $R = 0$, that is, $p^{i-1} A[p] \subseteq S$.

**Corollary 3.** — Let $L$ be a subgroup of $G$. If the heights (computed in $G$) of the elements of $L$ are bounded, then $L$ is contained in a minimal pure subgroup (direct summand) of $G$.

**Proof.** — There is a positive integer $n$ such that $L \cap p^n G = 0$. The group $p^n G$ is a fully invariant subgroup of $G$. Apply corollary 2 and proposition 2.

Now consider the case where $G$ is the sum of a divisible group $D$ and a bounded group $B$, $G = D + B$. Let $L$ be a subgroup of $G$. In order to show that $L$ is contained in a minimal pure subgroup of $G$, we may assume that $D[p] \subseteq L$ since a divisible subgroup is an absolute direct summand. In this case, $H$ is minimal pure for $L$ if $H/D$ is minimal pure for $|L, D|/D$ in $G/D$, a bounded group. This completes the proof of

**Theorem 3.** — Each subgroup of $G$ is contained in a minimal pure subgroup of $G$ if and only if $G$ is the sum of a divisible group and a bounded group.
We now turn our attention to the question of the existence of minimal pure subgroups for subsocles. Theorem 2 shows that if a subsocle $S$ is imbedded in a minimal pure subgroup in $G$, then $S$ supports a pure subgroup, that is, there is a pure subgroup $H$ of $G$ such that $H[p] = S$. Thus the question of whether or not a subsocle is imbedded in a minimal pure subgroup is just the question of whether or not that subsocle supports a pure subgroup. It is well known that every subsocle of a bounded group supports a pure subgroup.

**Proposition 3.** — Let $S = \bigcup S_i$ be the union of an ascending sequence of subsocles $S_i$ of $G$. If $S_i \cap p^i G = 0$ for $i = 1, 2, \ldots$, then $S$ supports a pure subgroup. Indeed, $S$ supports a direct summand of a basic subgroup.

**Proof.** — Since $S_i$ is contained in a bounded direct summand of $G$, it supports a pure subgroup $H_i$ of $G$. But $\{H_i, S_{i+1} \cap p^{i+1} G = 0\}$; hence $\{H_i, S_{i+1}\}$ is contained in a bounded direct summand $B_{i+1}$ of $G$. Since $H_i$ is bounded and pure in $B_{i+1}$, it is a direct summand of $B_{i+1}$; let $B_{i+1} = H_i + A_{i+1}$. Then

$$S_{i+1} = H_i[p] + (A_{i+1} \cap S_{i+1}).$$

But $A_{i+1} \cap S_{i+1}$ supports a pure subgroup $C_{i+1}$ in $A_{i+1}$ since $A_{i+1}$ is bounded. Let $H_{i+1} = H_i + C_{i+1}$. The union $H$ of the ascending sequence of pure subgroups $H_i$ of $G$ is a pure subgroup of $G$ with $H[p] = S$. Kulikov’s criteria shows that $H$ is a direct sum of cyclic groups (and therefore a direct summand of a basic subgroup of $G$).

**Corollary 4.** — If $G$ is a direct sum of cyclic groups, then each subsocle $S$ supports a pure subgroup.

**Proof.** — Let $G = \sum B_i$ where $B_i$ is (zero or) a homogeneous group of degree $i$ and let $S_i = (B_i + B_{i+1} + \ldots + B_j) \cap S$. The conditions of proposition 3 are satisfied.

Following established terminology, we say that $G$ is a closed group if it is the primary part of a complete direct sum of cyclic groups [2].

**Proposition 4.** — Each subsocle of a closed group supports a pure subgroup.

**Proof.** — Let $S$ be a subsocle of a closed group $G$. Choose $S_i$ such that $S \cap p^i G = S_i + (p^{i+1} G \cap S)$ for $i = 0, 1, \ldots$. Let $T_0 = S$, $T_i = S_i + S_{i+1} + \ldots + S_{i-1}$ if $i \geq 1$, and let $T = \bigcup T_i$. By proposition 3, $T$ supports a direct summand $B_i$ of a basic subgroup $B$ of $G$, $B = B_i + B$. Since $G$ is a closed group, $G = B_i + B$. Since $T$ is
dense in \( S \), \( S \subseteq T \). But \( \bar{T} = \overline{B_i[p]} = \overline{B_i} \). Thus \( S \) is a dense subsocle of \( B_i \), a direct summand of \( G \). The proof is completed by theorem 1.

**Theorem 4.** — If \( G = A + B \) where \( A \) is a direct sum of cyclic groups and \( B \) is a closed group, then each subsocle of \( G \) supports a pure subgroup.

**Proof.** — By theorem 1, it suffices to prove that each closed subsocle of \( G \) supports a pure subgroup. Let \( S \) be a closed subsocle of \( G \) and let \( S' = S \cap B \). Then \( S \) is a closed subsocle of \( B \). By proposition 4, \( S' \) supports a pure subgroup \( C \) of \( B \). Since \( S \) is closed, \( C \) is closed in \( B \) (and therefore is a closed group). Hence \( C \) is a direct summand of \( B \); let \( B = C + K \). Then \( S = S \cap (A + K) + S' \). Notice that \( S \cap K = \emptyset \).

Define \( S_i = (A_1 + A_2 + \ldots + A_i + K) \cap S \) where \( A = \sum A_i \) is the standard decomposition of \( A \). Then \( S \cap (A + K) = \bigcup S_i \) and \( S_i \cap p'(A + K) = \emptyset \). Thus by proposition 3, there is a pure subgroup of \( A + K \) with \( S \cap (A + K) \) as its socle, and the theorem is proved.

**References.**


(Manuscrit reçu en juin 1964.)

Paul Hill and Charles Megibben,
Department of Mathematics,
Auburn University,
Auburn, Alabama (États-Unis).