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ISOMORPHIC REFINEMENTS OF DECOMPOSITIONS OF A PRIMARY GROUP INTO CLOSED GROUPS ;

BY

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Introduction. — It is known that any two decompositions of a group into cyclic groups have isomorphic refinements. KOLETTIS [2] has shown that any two decompositions of a group into a closed group and cyclic groups have isomorphic refinements. The purpose of this paper is to show that any two decompositions of a group into closed groups have isomorphic refinements. Since cyclic groups are closed this will be an extension of Koletti's result.

Definitions. Preliminaries. — All groups will be assumed to be primary (relative to the same prime p) abelian groups without elements of infinite height $\left(\text{i. e. } \bigcap_n p^n G = 0\right)$. If $p^n G \neq 0$ for each n we say G has infinite length. We will use the notation $P(G)$ for the subgroup of G generated by elements of order p and $\text{height}_G(g)$ will denote the least non-negative integer n such that $g \in p^n G$, $g \neq 0$ and $\text{height}_G(0) = \infty$. FUCHS [1] calls a group closed if given a sequence of elements (g_i) of G such that the orders of the g_i are bounded and such that $g_i - g_{i+1} \in p^i G$ there is a $g \in G$ with $g - g_i \in p^i G$. We will

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use the equivalent definition which states that every Cauchy sequence having a bound on the orders of the elements converges when G is made into a topological group having the subgroups $p^n G$ as a fundamental system of neighborhoods of o.

We will need the fact that every pure closed subgroup of a group is a direct summand of the group and that any two decompositions of a closed group have isomorphic refinements (*see* [1], p. 116-117; corollary 34.5 and theorem 34.6). We further need that each group has a sequence of subgroups (B_i) and (H_i) such that $\sum_i B_i$ is a base, B_i is a direct sum of cyclic groups of order p^i,

$$G = \sum_{1 \leqq i \leqq j} B_i + H_j \quad \text{(direct)} \qquad \text{and} \qquad H_j \supset H_{j+1}$$

(*see* [1], p. 98).

The fact that a subgroup H of G is pure in G if $\text{height}_H(h) = \text{height}_G(h)$ for $h \in P(H)$ will also be used without stating the fact that it is being applied.

We firts prove several technical lemmas.

LEMMA 1. — *Let G be a group and suppose that*

$$G = G^1 + G^2 \quad (\textit{direct}) \qquad \textit{and} \qquad G = H^1 + H^2 \quad (\textit{direct})$$

where $P(G^1) = P(H^1)$. Then

$$G = G^1 + H^2 \quad (\textit{direct}) \qquad \textit{and} \qquad G = H^1 + G^2 \quad (\textit{direct}).$$

Proof. — We will show that $G = H^1 + G^2$ (direct). We have

$$P(H^1) \cap P(G^2) = P(G^1) \cap P(G^2) = 0 \qquad \text{thus} \quad H^1 \cap G^2 = 0.$$

Let $g \in G$ be of order p^k. If $k = 0, 1$, then

$$g \in P(G) = P(G^1) + P(G^2) = P(H^1) + P(G^2) \subset H^1 + G^2.$$

Now suppose that if the order of g is less than p^{k+1} then $g \in H^1 + G^2$ and let h be of order p^{k+1}. Then $p^k h \in H^1 + G^2$ by the induction hypothesis. If $p^k h = g_1 + g_2$ for $g_1 \in P(H^1) = P(G^1)$, $g_2 \in P(G^2)$, let $p^k h_1 = g_1$, $p^k h_2 = g_2$ for $h_1 \in H_1$, $h_2 \in G_2$. Then the order of $h - (h_1 + h_2)$ is less than p^{k+1} so

$$h = h - (h_1 + h_2) + (h_1 + h_2) \in H^1 + G^2.$$

LEMMA 2. — *Let G be the direct sum of its subgroups G^1 and G^2 and let π be the projection of G onto G^1 determined by this direct decomposition.*

If H is a subgroup of G such that $P(H) \subset P(G^1)$, then π restricted to H is an isomorphism. Furthermore, if H is pure in G, then $\pi(H)$ is pure in G.

Proof. — Let $g \in H$, $g \neq 0$. If p^N, $N \geqq 1$ is the order of g, then

$$p^{N-1}\pi(g) = \pi(p^{N-1}g) = p^{N-1}, \; g \neq 0 \text{ so } \pi(g) \neq 0.$$

Let $p^n x = g$ for

$$g \in P(\pi(H)) = \pi(P(H)) = P(H).$$

If H is pure in G there is a $y \in H$ such that $p^n y = g$ but then

$$g = \pi(g) = \pi(p^n y) = p^n \pi(y) \qquad \text{and} \qquad \pi(y) \in \pi(H).$$

LEMMA 3. — *Let $G = G^1 + G^2$ (direct) and suppose H is a pure closed subgroup of G such that $P(H) \subset G^1$. Then G^1 admits a direct decomposition $G^1 = H^1 + H^2$ such that $P(H^1) = P(H)$ and such that $H^1 \cong H$.*

Proof. — Let π be as in lemma 2. Then $\pi(H)$ is a pure closed subgroup of G hence a direct summand of G so let $H^1 = \pi(H)$ and H^2 be any supplement of H^1 in G^1.

The next two lemmas show us conditions that an isomorphism of a closed group into a group written as the direct sum of a family of its subgroups must satisfy. It closely resembles Kulikov's result that a closed group cannot be written as the direct sum of infinitely many subgroups of infinite length.

LEMMA 4. — *Let $G = \sum_{\nu \in N} G^\nu$ (direct) and let φ be an isomorphism of a closed group H into G. Then H admits a direct decomposition $H = H^* + H^{**}$ such that H^* is a direct sum of cyclic groups and such that $P(\varphi(H^{**}))$ is contained in the sum of a finite number of the G^ν.*

Proof. — Let (B_i) and (H_i) be sequences of subgroups of H such that B_i is a direct sum of cyclic groups of order p^i, $\sum B_i$ is a base of H and $H = B_1 + \ldots + B_n + H_n$ (direct). It suffices to show that for n sufficient large $P(\varphi(H_n))$ is contained in the sum of a finite number of the G^ν. If this is not the case, let π_ν be the projection of G onto G^ν associated with this decomposition.

For each non-negative integer n let

$$A_n = \{\nu \mid \nu \in N, \; \pi_\nu(P(\varphi(H_n))) \neq 0\}.$$

By assumption each A_n is infinite. Let $\nu_i \in A_i$ for each i and suppose $\nu_i \neq \nu_j$ for $i \neq j$ and let $h_i \in \varphi(P(H_i))$ be such that $\pi_{\nu_i}(h_i) \neq 0$. Then we know $\text{height}_H(h_i) \geqq i$ since $H = B_1 + \ldots + B_i + H_i$ (direct) and since $\sum B_n$ is a base of H. We choose a sequence of integers (n_i) by letting $n_1 = 1$ and in general

$$n_{i+1} > \max \{ \text{height}_G(\pi_\nu(h_{n_i})) \mid \nu \in N,\ \pi_\nu(h_{n_i}) \neq 0 \}.$$

Note that

$$(1) \qquad \text{height}_G(\pi_\nu(h_{n_{i+1}})) > \text{height}_G(\pi_\nu(h_{n_i}))$$

whenever $\pi_\nu(h_{n_i}) \neq 0$.

We form the Cauchy sequence $g_i = h_{n_1} + \ldots + h_{n_i}$. If g is the limit of this sequence let ν_{n_j} be such that $\pi_{\nu_{n_j}}(g) = 0$. Now $\pi_{\nu_{n_j}}(h_{n_j}) \neq 0$ so assume

$$(2) \qquad \text{height}_G(g - g_M) > \text{height}_G\left(\pi_{\nu_{n_j}}(h_{n_j})\right),$$

where $M \geqq j$. But

$$\text{height}_G\left(\pi_{\nu_{n_j}}(g - g_M)\right) = \text{height}_G\left(\pi_{\nu_{n_j}}(g_M)\right)$$

and

$$\text{height}_G\left(\pi_{\nu_{n_j}}(g_M)\right) = \min_{1 \leqq i \leqq M} \left\{ \text{height}_G\left(\pi_{\nu_{n_i}}(h_{n_i})\right) \right\}$$

by (1) above. This contradicts (2) above.

LEMMA 5. — *If φ is a homomorphism of a closed group H into a direct sum of cyclic groups T, then H admits a direct decomposition $H = H^* + H^{**}$ such that $\varphi(P(H^{**})) = 0$ and H^* is a direct sum of cyclic groups.*

Proof. — Let (B_i) and (H_i) be as in the preceeding lemma and furthermore assume $H_i \supset H_{i+1}$. If $T = \sum T_i$ where T_i is a direct sum of cyclic groups of order p^i apply lemma 4 and suppose

$$\varphi(P(H_n)) \subset T_1 + \ldots + T_M.$$

Then

$$\varphi(P(H_{n+M})) \subset T_1 + \ldots + T_M.$$

But for $h \in P(H_{n+M})$, $\text{height}_H(h) \geqq M$ thus $\text{height}_{T_1 + \ldots + T_M}(\varphi(h)) \geqq M$ but clearly this implies $\varphi(h) = 0$.

Lemmas 4 and 5 will be used in lemma 6 to show that if a group is written as the direct sum of closed groups in two different ways, that the number of summands of infinite length, if infinite is an invariant of the group.

LEMMA 6. — *Suppose*

$$G = \sum_{\mu \in M} G^\mu + S \quad (\textit{direct})$$

and

$$G = \sum_{\nu \in N} H^{\nu} + T \quad (direct),$$

where each G^{μ} and H^{ν} are closed and of infinite length and where S and T are direct sums of cyclic groups. Then if either N or M is infinite, so is the other and both have the same cardinality.

Proof. — Assume M is infinite. By applying lemmas 4 and 5 for each $\mu \in M$ there is a direct decomposition $G^{\mu} = G^{\mu *} + G^{\mu **}$ and a finite subset N_{μ} of N such that $G^{\mu *}$ is a direct sum of cyclic groups and such that $P(G^{\mu **}) \subset \sum_{\mu \in N_{\mu}} H^{\mu}$.

If μ_i is a sequence of distinct elements of M we would like to show $J_{\mu_i} \neq J_{\mu_j}$ for some i and j. If this is not the case let $g_i \in P\left(G^{\mu_i^{**}}\right)$ be such that $g_i \neq 0$ and $\text{height}_G(g_i) > i$. Since each $g_i \in \sum_{\nu \in H_{\mu_1}} H^{\nu}$ which is closed then the series (h_n) where $h_n = g_1 + \ldots + g_n$ converges say to g. For some μ_j we would have $\pi_{\mu_j}(g) = 0$ where π_{μ_j} is the projection of G onto G_{μ_i} determined by the decomposition above. If $k = height_G(g_j)$ let m be a positive integer, $m > j$ such that $\text{height}_G(g - h_m) > k$. Then

$$k < \text{height}_G(\pi_{\mu_j}(g - h_m)) = \text{height}_G(\pi_{\mu_j}(g_j)) = k$$

which is a contradiction.

This shows that the cardinality of N is greater than or equal that of M. Similarly the cardinality of M is greater than or equal that of N; hence, the two are equal.

LEMMA 7. — *Let G be a group and*

$$G = \sum_{\gamma \in \Gamma} M^{\gamma}, \qquad G = \sum_{\delta \in \Delta} N^{\delta}$$

be two direct decompositions of G into closed groups and let $G = \sum_{\mu \in I} G^{\mu}$, $G = \sum_{\nu \in J} H^{\nu}$ be refinements of these decompositions respectively. Then if $\sum_{\gamma \in \Gamma} M^{\gamma}$ and $\sum_{\delta \in \Delta} N^{\delta}$ have isomorphic refinements, then so do $\sum_{\mu \in I} G^{\mu}$ and $\sum_{\nu \in J} H^{\nu}$.

Proof. — Consider the scheme

$$\begin{tikzcd}[column sep=small]
 & (M^{\gamma}) \arrow[dl] \arrow[dr] & & & (N^{\delta}) \arrow[dl] \arrow[dr] & \\
(G^{\mu}) \arrow[d] & & (P^{\lambda}) \arrow[d] \arrow[r, phantom, "\cong"] & (Q^{\lambda}) \arrow[d] & & (H^{\nu}) \arrow[d] \\
(\bar{R}^{\alpha}) \arrow[rr, phantom, "\cong"] & & (R^{\alpha}) \arrow[d] & (S^{\beta}) \arrow[d] \arrow[rr, phantom, "\cong"] & & (\bar{S}^{\beta}) \\
 & & (U^{\omega}) & (V^{\omega}) & &
\end{tikzcd}$$

where each family is a family of subgroups of G, giving a direct decomposition of G, where the arrows indicate refinements and the symbol $\cong$ denotes isomorphic decompositions, e. g. — P^{λ} is isomorphic to Q^{λ} for each λ. The existence of the isomorphic refinements $(\bar{R}^{\alpha})$ and (R^{α}), $(\bar{S}^{\beta})$ and (S^{β}) follows from the fact that each M^{γ} and N^{δ} are closed and from the fact that any two decompositions of a closed group have isomorphic refinements. Similarly the existence of the decompositions (U^{ω}) and (V^{ω}) follows from the fact that the P^{λ} and Q^{λ} are closed.

Lemma 7 gives us an indication of how we may proceed to find isomorphic refinements of two decompositions of a group G into closed groups. To be more specific, suppose

$$G = \sum_{\mu \in I} G^{\mu} \quad \text{(direct)},$$

$$G = \sum_{\nu \in J} H^{\nu} \quad \text{(direct)}$$

and let I^* (J^*) be the subset of I (of J) consisting of all $\mu \in I$ $(\nu \in J)$ such that G^{μ} (H_{ν}) is of infinite length and let I^{**} (J^{**}) be the complement of I^* (of J^*) in I (in J). If I^* or J^* were finite, Kolettis and Kulikov's results could be applied to find isomorphic refinements; hence, suppose they are both infinite. Then by lemma 6 we know they have the same cardinality.

Suppose that β is the first ordinal having the same cardinality as I^*. We will show that we can write

$$I^* = \bigcup_{\sigma < \beta} I_{\sigma},$$

$$J^* = \bigcup_{\tau < \beta} J_{\tau},$$

where each I_{σ}, J_{τ} are finite and $I_{\sigma} \cap I_{\sigma'} = \emptyset = J_{\tau} \cap J_{\tau'}$ if $\sigma \neq \sigma'$ and $\tau = \tau'$

such that if we let

$$C^{\sigma} = \sum_{\mu \in I_{\sigma}} G^{\mu},$$

$$D^{\tau} = \sum_{\nu \in J_{\tau}} H^{\nu},$$

we can find decompositions

$$C^{\sigma} = C^{\sigma_1} + C^{\sigma_2} + C^{\sigma_3} \quad \text{(direct)},$$
$$D^{\tau} = D^{\tau_1} + D^{\tau_2} + D^{\tau_3} \quad \text{(direct)}$$

such that :

(1) $D^{\sigma_1} \cong C^{\sigma_1}$ for each $\sigma < \beta$;

(2) $D^{\sigma_2} \cong C^{(\sigma+1)_2}$ for each $\sigma < \beta$;

(3) C^{σ_3} and D^{σ_3}

are direct sums of cyclic groups for each $\sigma < \beta$;

(4) $P\left(\sum_{\sigma<\beta} C^{\sigma_1} + C^{\sigma_2}\right) = P\left(\sum_{\tau<\beta} D^{\tau_1} + D^{\tau_2}\right)$;

(5) $C^{\sigma_2} = 0$ if $\sigma = 0$

or a limit ordinal $< \beta$. If this is possible, then lemma 1 coupled with (4) above gives

$$G = \sum_{\sigma<\beta} (C^{\sigma_1} + C^{\sigma_2}) + \sum_{\tau<\beta} D^{\tau_3} + \sum_{\nu \in J^{**}} H^{\nu}.$$

But

$$G = \sum_{\sigma<\beta} (C^{\sigma_1} + C^{\sigma_2}) + \sum_{\sigma<\beta} C^{\sigma_3} + \sum_{\mu \in J^{**}} G^{\mu},$$

thus $\sum_{\sigma<\beta} C^{\sigma_3} + \sum_{\mu \in I^{**}} G^{\mu} \cong \sum_{\tau<\beta} D^{\tau_3} + \sum_{\nu \in J^{**}} H^{\nu}$,
but any two direct decompositions of a group into cyclic groups have isomorphic refinements. Using this fact, (1) and (2) above, and lemma 7, we get that

$$\sum_{\mu \in I} G^{\mu} \quad \text{and} \quad \sum_{\nu \in J} H^{\nu}$$

have isomorphic refinements.

To complete the proof that the two decompositions have isomorphic refinements the task remains to determine the sets I_{σ}, J_{τ} and the required decompositions of the groups C^{σ} and D^{τ}.

The next lemma will give us a method of choosing the sets I_{σ} and J_{τ}.

LEMMA 8. — *Let*

$$G = G^* + \sum_{\mu \in I} G^\mu \quad (\textit{direct})$$

and

$$H = \sum_{\nu \in J} H^\nu \quad (\textit{direct})$$

where G^* *and each* G^μ *and* H^ν *is a closed group. Then if* φ *is an isomorphism of* G *onto* H *there exists finite subsets* J^* *of* J, I^* *of* I *and direct decompositions*

$$G^* = C^1 + C^3,$$

$$\sum_{\nu \in J^*} H^\nu = D^1 + D^2 + D^3,$$

$$\sum_{\mu \in I^*} G^\mu = C^2 + C^*$$

such that :

(1) $C^1 \cong D^1$;

(2) $C^2 \cong D^2$;

(3) C^3 *and* D^3 *are direct sums of cyclic groups*;

(4) $P(\varphi(C^1 + C^2)) = P(D^1 + D^2)$.

Proof. — It is not difficult to see that there is no loss in generality in assuming, as we shall, that $G = H$ and that φ is the identity map. By lemma 4 there exists a direct decomposition $G^* = C^1 + C^3$ where C^3 is a group of bounded order and where $P(C^1) \subset \sum_{\nu \in J^*} H^\nu$ for some finite subset J^* of J. Note that we can assume J^* contains any one element of J; which element will be specified when the lemma is applied. Furthermore, C^1 is closed and pure in G; hence by lemma 3 there exists a decomposition

$$\sum_{\nu \in J^*} H^\nu = D^1 + \bar{D} \quad (\text{direct}),$$

where $P(D^1) = P(C^1)$ and $D^1 \cong C^1$. Since D^1 and C^1 are direct summands of G by lemma 1 we get

$$G = C^1 + \bar{D} + \sum_{\nu \notin J^*} H^\nu \quad (\text{direct}),$$

but

$$G = C^1 + C^3 + \sum_{\mu \in I} G^\mu \quad \text{(direct).}$$

Thus, let ψ be the isomorphism of

$$\overline{D} + \sum_{\nu \in J^{**}} H^\nu \qquad \text{onto} \quad C^3 + \sum_{\mu \in I} G^\mu,$$

where $\psi(x) = y$ if and only if $x - y \in C^1$. Then apply lemma 4 and let

$$\overline{D} = D^2 + D^3 \quad \text{(direct)}$$

be such that D^3 is a direct sum of cyclic groups and

$$P(\psi(D^2)) \subset \sum_{\mu \in I^*} G^\mu$$

for some finite subset I^* of I which can be assumed to contain any specified element of I. Then by lemma 3,

$$\sum_{\mu \in I^*} G^\mu = C^2 + C^*,$$

where

$$C^2 \cong \psi(D^2) \cong D^2$$

and where

$$P(C^2) = P(\psi(D^2)).$$

Now $\psi(D^2) + C^1 = D^2 + C^1$ by the definition of ψ; hence

$$P(C^1 + C^2) = P(C^1) + P(C^2) = P(C^1) + P(\psi(D^2)) = P(C^1) + P(D^2),$$

but

$$P(C^1) = P(D^1), \qquad \text{thus} \quad P(C^1 + C^2) = P(D^1 + D^2)$$

and our proof is complete. Remark that C^* as a direct summand of a closed group is closed. We now come to the main result of this paper.

THEOREM. — *Any two decompositions of a group into closed groups have isomorphic refinements.*

Proof. — Let

$$G = \sum_{\mu \in I} G^\mu \quad \text{(direct),}$$

$$G = \sum_{\nu \in J} H^\nu \quad \text{(direct),}$$

where each G^{μ} and H^{ν} is closed; and assume I^* consists of all $\mu \in I$ such that G^{μ} is of infinite length. Define J^* in a similar manner. As we remarked before, we may assume I^* is infinite. Then by lemma 6, I^* and J^* have the same cardinality. Let β be the least ordinal having the same cardinality as I. Let I and J be well ordered so that both have well order type β. For any ordinal $\sigma < \beta$ let i_σ and j_σ denote the σ — th element of I and J respectively. If $\eta < \beta$ suppose that for each ordinal $\sigma \leqq \eta$ we have chosen a finite non-empty subset I_σ of I such that $I_\sigma \cap I_{\sigma'} = \emptyset$ if $\sigma \neq \sigma'$ and such that $i_\varepsilon \in \bigcup_{\sigma \leqq \eta} I_\sigma$ for all $\varepsilon \leqq \eta$ and that for all $\tau < \eta$ we have chosen finite non-empty subsets J_τ of J such that $J_\tau \cap J_{\tau'} = \emptyset$ if $\tau \neq \tau'$ and such that

$$j_\varepsilon \in \bigcup_{\tau < \eta} J_\tau \text{ for all } \varepsilon < \eta.$$

Setting

$$C^\sigma = \sum_{\mu \in I_\sigma} G^\mu, \qquad D^\tau = \sum_{\nu \in J_\tau} H^\nu$$

for $\sigma \leqq \eta$, $\tau < \eta$ suppose we have decompositions

$$C^\sigma = C^{\sigma_1} + C^{\sigma_2} + C^{\sigma_3} \quad \text{(direct)},$$
$$D^\tau = D^{\tau_1} + D^{\tau_2} + D^{\tau_3} \quad \text{(direct)}$$

and a direct summand C^{η_2} of C^η. We will let C^{η^*} denote a supplement of C^{η_2} in C^η.

Then we consider the following conditions :

(A^η) (1) $C^{\varepsilon_1} \cong D^{\varepsilon_1}$, $\varepsilon < \eta$;

(2) $C^{(\varepsilon+1)_2} \cong D^{\varepsilon_2}$, $\varepsilon < \eta$;

(3) C^{ε_3} and D^{ε_3} are direct sums of cyclic groups, $\varepsilon \leqq \eta$;

(4) $P\left(\sum_{\varepsilon < \eta} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\eta_2}\right) = P\left(\sum_{\varepsilon < \eta} (D^{\varepsilon_1} + D^{\varepsilon_2})\right)$;

(5) $C^{\varepsilon_2} = 0$ for $\varepsilon = 0$ and for all limit ordinals $\varepsilon \leqq \eta$.

We can easily see that if we let $I_0 = \{ i_0 \}$, $G^{(i_0)_2} = 0$ and $G^{i_0^*} = G^{i_0}$ then (A^0) holds. Now suppose we have chosen our subsets of I and J and the decompositions of C^σ and D^τ in such a manner that (A^η) holds for all $\eta < \rho$ where $\rho \leqq \beta$. If $\rho = \beta$, then by the remarks after lemma 7, we see that the theorem is proved. Hence, suppose $\rho < \beta$. Then we must choose a direct decomposition $C^{\rho^*} = C^{\rho_1} + C^{\rho_3}$, subsets

$I_{\rho+1}$ and I_ρ of I and J respectively, a direct summand $C^{(\rho+1)_2}$ of C^ρ and a direct decomposition

$$D^\rho = D^{\rho_1} + D^{\rho_2} + D^{\rho_3}$$

so that (A^ρ) holds. First, suppose ρ is a limit ordinal. In this case the sets I_σ and J_τ for σ, $\tau < \rho$ have already been chosen. We let I_ρ be any finite non-empty subset of the complement of $\bigcup_{\sigma<\rho} J_\sigma$ in I subject to the condition that

$$i_\rho \in I_\rho \text{ if } i_\rho \notin \bigcup_{\sigma<\rho} I_\sigma.$$

Now let

$$C^\rho = \sum_{\mu \in I_\rho} G^\mu.$$

Let $C^\rho = C^{\rho_2} + C^{\rho^*}$ where $C^{\rho_2} = 0$ and $C^{\rho^*} = C^\rho$. Then it is easy to see that (A^ρ) holds. In case ρ is not a limit ordinal let $\lambda + 1 = \rho$. By (4) of (A^λ) we have

$$P\Big(\sum_{\varepsilon<\lambda}(C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\lambda_2}\Big) = P\Big(\sum_{\varepsilon<\lambda}(D^{\varepsilon_1} + D^{\varepsilon_2})\Big).$$

Now both

$$\sum_{\varepsilon<\lambda}(C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\lambda_2} \qquad \text{and} \qquad \sum_{\varepsilon<\lambda}(D^{\varepsilon_1} + D^{\varepsilon_2})$$

are direct summands of G so by lemma 1

$$G = \sum_{\varepsilon<\lambda}(C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\lambda_2} + \sum_{\substack{\nu \notin \bigcup J_\varepsilon \\ \varepsilon<\lambda}} H^\nu + \sum_{\varepsilon<\lambda} D^{\varepsilon_3} \quad \text{(direct).}$$

But also

$$G = \sum_{\varepsilon<\lambda}(C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\lambda_2} + C^{\lambda^*} + \sum_{\substack{\mu \notin \bigcup I_\varepsilon \\ \varepsilon<\lambda}} G^\mu + \sum_{\varepsilon<\lambda} C^{\varepsilon_3}.$$

Let ψ be the isomorphism of

$$C^{\lambda^*} + \sum_{\substack{\mu \notin \bigcup I_\varepsilon \\ \varepsilon<\lambda}} G^\mu + \sum_{\varepsilon<\lambda} C^{\varepsilon_3}.$$

onto

$$\sum_{\substack{\nu \notin \bigcup J_\varepsilon \\ \varepsilon<\lambda}} H^\nu + \sum_{\varepsilon<\lambda} D^\varepsilon$$

where $\psi(x) = y$ if

$$x - y \in \sum_{\varepsilon<\lambda} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\lambda_2}.$$

We have that

$$\sum_{\varepsilon<\lambda} C^{\varepsilon_3} \quad \text{and} \quad \sum_{\varepsilon<\lambda} D^{\varepsilon_3}$$

are direct sums of cyclic groups. C^{λ^*} is a direct summand of a closed group $\sum_{\mu\in J_\lambda} G^\mu$ hence is closed. Letting C^{λ^*} correspond to the G^* in lemma 8, we can find finite non-empty subsets $I_\rho = I_{\lambda+1}$ and J_λ in the complements of $\bigcup_{\varepsilon\leq\lambda} J_\varepsilon$ and $\bigcup_{\varepsilon<\lambda} J_\varepsilon$ in I and J respectively and the following direct decompositions :

$$C^{\lambda^*} = C^{\lambda_1} + C^{\lambda_3},$$
$$D^\lambda = D^{\lambda_1} + D^{\lambda_2} + D^{\lambda_3},$$
$$C^\rho = C^{\rho_2} + C^{\rho^*},$$

where we have :

(*a*) $C^{\lambda_1} \cong D^{\lambda_1}$;
(*b*) $C^{\rho_2} \cong D^{\lambda_2}$;
(*c*) C^{λ_3} and D^{λ_3} are direct sums of cyclic groups;
(*d*) $P(\psi(C^{\lambda_2} + C^{\rho_2})) = P(D^{\lambda_1} + D^{\lambda_2})$.

We can suppose, as remarked in the proof of lemma 8, that

$$i_\rho \in I_\rho \qquad \text{if} \quad i_\rho \notin \bigcup_{\eta>\rho} I_\eta$$

and likewise require that

$$j_\lambda \in J_\lambda \qquad \text{if} \quad j_\lambda \notin \bigcup_{\eta<\lambda} J_\eta.$$

Then (1), (2) and (3) of (A^ρ) obviously hold for $\varepsilon < \lambda$ since (A^λ) is true. For $\varepsilon = \lambda$, (1), (2) and (3) of (A^ρ) hold since (*a*), (*b*) and (*c*) above hold. (5) of (A^ρ) obviously holds so we only need show (4) of (A^ρ) holds.

By (*d*) above we have that

$$P(\psi(C^{\lambda_1} + C^{\rho_2})) = P(D^{\lambda_1} + D^{\lambda_2}).$$

By the definition of ψ we get that

$$P(C^{\lambda_1} + C^{\rho_2} + \sum_{\varepsilon<\lambda} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\lambda_2}) = P(\psi(C^{\lambda_1} + C^{\rho_2}) + \sum_{\varepsilon<\lambda} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\lambda_2}).$$

Now since

$$P(\psi(C^{\lambda_1}+C^{\rho_2}))=P(D^{\lambda_1}+D^{\lambda_2})$$

by (*d*) above and since

$$P\left(\sum_{\varepsilon<\lambda}(C^{\varepsilon_1}+C^{\varepsilon_2})+C^{\lambda_2}\right)=P\left(\sum_{\varepsilon<\lambda}(D^{\varepsilon_1}+D^{\varepsilon_2})\right)$$

by (4) of (A^{λ}) we get that (4) of (A^{ρ}) holds.

Thus by transfinite induction we can suppose (A^{η}) holds for all $\eta<\rho$ and hence that the two decompositions of G have isomorphic refinements.

REFERENCES.

[1] FUCHS (L.). — *Abelian groups*. New York, Oxford, London, Paris, Pergamon Press, 1960.

2] KOLETTIS (George, Jr.). — Semi-complete primary abelian groups, *Proc. Amer. math. Soc.*, t. 11, 1960, p. 200-205.

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