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Isomorphic refinements of decompositions of a primary group into closed groups

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ISOMORPHIC REFINEMENTS
OF DECOMPOSITIONS OF A PRIMARY GROUP
INTO CLOSED GROUPS;

BY

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Introduction. — It is known that any two decompositions of a group into cyclic groups have isomorphic refinements. KOLETTIS [2] has shown that any two decompositions of a group into a closed group and cyclic groups have isomorphic refinements. The purpose of this paper is to show that any two decompositions of a group into closed groups have isomorphic refinements. Since cyclic groups are closed this will be an extension of Koletti's result.

Definitions. Preliminaries. — All groups will be assumed to be primary (relative to the same prime $p$) abelian groups without elements of infinite height (i.e. $\bigcap_n p^n G = o$). If $p^n G \neq o$ for each $n$ we say $G$ has infinite length. We will use the notation $P(G)$ for the subgroup of $G$ generated by elements of order $p$ and $\text{height}_p(g)$ will denote the least non-negative integer $n$ such that $g \in p^n G$, $g \neq o$ and $\text{height}_p(o) = \infty$. FUCHS [1] calls a group closed if given a sequence of elements $(g_i)$ of $G$ such that the orders of the $g_i$ are bounded and such that $g_i - g_{i+1} \in p^i G$ there is a $g \in G$ with $g - g_i \in p^i G$. We will

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use the equivalent definition which states that every Cauchy sequence having a bound on the orders of the elements converges when \( G \) is made into a topological group having the subgroups \( p^n G \) as a fundamental system of neighborhoods of \( o \).

We will need the fact that every pure closed subgroup of a group is a direct summand of the group and that any two decompositions of a closed group have isomorphic refinements (see [1], p. 116-117; corollary 34.5 and theorem 34.6). We further need that each group has a sequence of subgroups \((B_i)\) and \((H_i)\) such that \( \sum B_i \) is a base, \( B_i \) is a direct sum of cyclic groups of order \( p^i \),

\[
G = \sum_{i \in I \subseteq J} B_i + H_j \quad \text{(direct)} \quad \text{and} \quad H_j \supset H_{j+1}
\]

(see [1], p. 98).

The fact that a subgroup \( H \) of \( G \) is pure in \( G \) if \( \text{height}_H(h) = \text{height}_G(h) \) for \( h \in P(H) \) will also be used without stating the fact that it is being applied.

We first prove several technical lemmas.

**Lemma 1.** — Let \( G \) be a group and suppose that \( G = G^1 + G^2 \) (direct) and \( G = H^1 + H^2 \) (direct) where \( P(G^i) = P(H^i) \). Then

\[
G = G^1 + H^2 \quad \text{(direct)} \quad \text{and} \quad G = H^1 + G^2 \quad \text{(direct)}.
\]

**Proof.** — We will show that \( G = H^1 + G^2 \) (direct). We have

\[
P(H^i) \cap P(G^j) = P(G^i) \cap P(G^j) = 0 \quad \text{thus} \quad H^i \cap G^j = 0.
\]

Let \( g \in G \) be of order \( p^k \). If \( k = 0, 1 \), then

\[
g \in P(G) = P(G^i) + P(G^j) = P(H^i) + P(G^j) \subset H^1 + G^2.
\]

Now suppose that if the order of \( g \) is less than \( p^{k+1} \) then \( g \in H^1 + G^2 \) and let \( h \) be of order \( p^{k+1} \). Then \( p^k h \in H^1 + G^2 \) by the induction hypothesis. If \( p^j h = g_1 + g_2 \) for \( g_1 \in P(H^i) = P(G^i), \quad g_2 \in P(G^j) \), let \( p^j h_1 = g_1, \quad p^j h_2 = g_2 \) for \( h_1 \in H^1, \quad h_2 \in G^2 \). Then the order of \( h - (h_1 + h_2) \) is less than \( p^{k+1} \) so

\[
h = h - (h_1 + h_2) + (h_1 + h_2) \in H^1 + G^2.
\]

**Lemma 2.** — Let \( G \) be the direct sum of its subgroups \( G^1 \) and \( G^2 \) and let \( \pi \) be the projection of \( G \) onto \( G^1 \) determined by this direct decomposition.
If $H$ is a subgroup of $G$ such that $P(H) \subseteq P(G')$, then $\pi$ restricted to $H$ is an isomorphism. Furthermore, if $H$ is pure in $G$, then $\pi(H)$ is pure in $G$.

**Proof.** — Let $g \in H$, $g \neq o$. If $p^N$, $N \geq 1$ is the order of $g$, then

$$p^{N-1} \pi(g) = \pi(p^{N-1} g) = p^{N-1} g, \ g \neq o \text{ so } \pi(g) \neq o.$$ 

Let $p^x g = g$ for

$$g \in P(\pi(H)) = \pi(P(H)) = P(H).$$ 

If $H$ is pure in $G$ there is a $y \in H$ such that $p^y g = g$ but then

$$g = \pi(g) = \pi(p^y g) = p^y \pi(y) \text{ and } \pi(y) \in \pi(H).$$

**Lemma 3.** — Let $G = G' + G^1$ (direct) and suppose $H$ is a pure closed subgroup of $G$ such that $P(H) \subseteq G'$. Then $G^1$ admits a direct decomposition $G^1 = H^1 + H^2$ such that $P(H^1) = P(H)$ and such that $H^1 \approx H$.

**Proof.** — Let $\pi$ be as in lemma 2. Then $\pi(H)$ is a pure closed subgroup of $G$ hence a direct summand of $G$ so let $H^1 = \pi(H)$ and $H^2$ be any supplement of $H^1$ in $G'$.

The next two lemmas show us conditions that an isomorphism of a closed group into a group written as the direct sum of a family of its subgroups must satisfy. It closely resembles Kulikov’s result that a closed group cannot be written as the direct sum of infinitely many subgroups of infinite length.

**Lemma 4.** — Let $G = \bigoplus_{\nu \in N} G^\nu$ (direct) and let $\varphi$ be an isomorphism of a closed group $H$ into $G$. Then $H$ admits a direct decomposition $H = H^* + H^{**}$ such that $H^*$ is a direct sum of cyclic groups and such that $P(\varphi(H^{**}))$ is contained in the sum of a finite number of the $G^\nu$.

**Proof.** — Let $(B_i)$ and $(H_i)$ be sequences of subgroups of $H$ such that $B_i$ is a direct sum of cyclic groups of order $p^i$, $\sum B_i$ is a base of $H$ and $H = B_1 + \ldots + B_n + H_n$ (direct). It suffices to show that for $n$ sufficient large $P(\varphi(H_n))$ is contained in the sum of a finite number of the $G^\nu$. If this is not the case, let $\pi_n$ be the projection of $G$ onto $G'$ associated with this decomposition.

For each non-negative integer $n$ let

$$A_n = \{ \nu | \nu \in N, \pi_n(P(\varphi(H_n))) \neq o \}.$$ 

By assumption each $A_n$ is infinite. Let $\nu_i \in A_i$ for each $i$ and suppose $\nu_i \neq \nu_j$ for $i \neq j$ and let $h_i \in \varphi(P(H_i))$ be such that $\pi_{\nu_i}(h_i) \neq o$. Then
we know \( \text{height}_{H}(h) \geq i \) since \( H = B_{1} + \ldots + B_{i} + H_{t} \) (direct) and since \( \sum B_{n} \) is a base of \( H \). We choose a sequence of integers \((n_{i})\) by letting \( n_{1} = i \) and in general

\[
n_{i+1} > \max \{ \text{height}_{G}(\pi_{\gamma}(h_{n_{i}})) \mid \gamma \in N, \pi_{\gamma}(h_{n_{i}}) \neq o \}.
\]

Note that

\[(1) \quad \text{height}_{G}(\pi_{\gamma}(h_{n_{i+1}})) > \text{height}_{G}(\pi_{\gamma}(h_{n_{i}}))\]

whenever \( \pi_{\gamma}(h_{n_{i}}) \neq o \).

We form the Cauchy sequence \( g_{i} = h_{n_{i}} + \ldots + h_{n_{i}} \). If \( g \) is the limit of this sequence let \( \nu_{n_{j}} \) be such that \( \pi_{\nu_{n_{j}}}(g) = o \). Now \( \pi_{\nu_{n_{j}}}(h_{n_{i}}) \neq o \) so assume

\[(2) \quad \text{height}_{G}(g - g_{u}) > \text{height}_{G}(\pi_{\nu_{n_{j}}}(h_{n_{i}})),\]

where \( M \geq j \). But

\[
\text{height}_{G}(\pi_{\nu_{n_{j}}}(g - g_{u})) = \text{height}_{G}(\pi_{\nu_{n_{j}}}(g_{u}))
\]

and

\[
\text{height}_{G}(\pi_{\nu_{n_{j}}}(g_{u})) = \min \{ \text{height}_{G}(\pi_{\nu_{n_{i}}}(h_{n_{i}})) \}
\]

by (1) above. This contradicts (2) above.

**Lemma 5.** — If \( \varphi \) is a homomorphism of a closed group \( H \) into a direct sum of cyclic groups \( T \), then \( H \) admits a direct decomposition \( H = H^{*} + H^{**} \) such that \( \varphi(P(H^{**})) = o \) and \( H^{*} \) is a direct sum of cyclic groups.

**Proof.** — Let \((B_{i})\) and \((H_{i})\) be as in the preceding lemma and furthermore assume \( H_{i} = H_{i+1} \). If \( T = \sum T_{i} \) where \( T_{i} \) is a direct sum of cyclic groups of order \( p^{i} \) apply lemma 4 and suppose

\[\varphi(P(H_{0})) \subset T_{1} + \ldots + T_{M}.\]

Then

\[\varphi(P(H_{n+M})) \subset T_{1} + \ldots + T_{M}.\]

But for \( h \in P(H_{n+M}) \), \( \text{height}_{H}(h) \geq M \) thus \( \text{height}_{T_{1} + \ldots + T_{M}}(\varphi(h)) \geq M \) but clearly this implies \( \varphi(h) = o \).

Lemmas 4 and 5 will be used in lemma 6 to show that if a group is written as the direct sum of closed groups in two different ways, that the number of summands of infinite length, if infinite is an invariant of the group.

**Lemma 6.** — Suppose

\[G = \sum_{\gamma \in M} G^{\gamma} + S \quad (\text{direct})\]
and
\[ G = \sum_{\gamma \in \Gamma} H^\gamma + T \quad \text{(direct)}, \]
where each \( G^\gamma \) and \( H^\gamma \) are closed and of infinite length and where \( S \) and \( T \) are direct sums of cyclic groups. Then if either \( N \) or \( M \) is infinite, so is the other and both have the same cardinality.

**Proof.** — Assume \( M \) is infinite. By applying lemmas 4 and 5 for each \( \mu \in M \) there is a direct decomposition \( G^\mu = G^{\mu*} + G^{\mu**} \) and a finite subset \( N_{\mu} \) of \( N \) such that \( G^{\mu*} \) is a direct sum of cyclic groups and such that \( P(G^{\mu*}) \subseteq \sum_{\nu \in N_{\mu}} H^\nu. \)

If \( \mu_i \) is a sequence of distinct elements of \( M \) we would like to show \( J_{\mu_i} \neq J_{\mu_j} \) for some \( i \) and \( j \). If this is not the case let \( g_i \in P(G^{\mu_i*}) \) be such that \( g_i \neq 0 \) and \( \text{height}_{\gamma_i}(g_i) > i. \) Since each \( g_i \in \sum_{\nu \in N_{\mu_i}} H^\nu \) which is closed then the series \( (h_n) \) where \( h_n = g_1 + \ldots + g_n \) converges say to \( g \). For some \( \mu_j \) we would have \( \pi_{\mu_j}(g) = 0 \) where \( \pi_{\mu_j} \) is the projection of \( G \) onto \( G_{\mu_j} \) determined by the decomposition above. If \( k = \text{height}_{\gamma_i}(g) \) let \( m \) be a positive integer, \( m > j \) such that \( \text{height}_{\gamma_i}(g - h_m) > k. \) Then
\[ k < \text{height}_{\gamma_i}(\pi_{\mu_i}(g - h_m)) = \text{height}_{\gamma_i}(\pi_{\mu_j}(g)) = k \]
which is a contradiction.

This shows that the cardinality of \( N \) is greater than or equal that of \( M. \) Similarly the cardinality of \( M \) is greater than or equal that of \( N; \) hence, the two are equal.

**Lemma 7.** — Let \( G \) be a group and
\[ G = \sum_{\gamma \in \Gamma} M^\gamma, \quad G = \sum_{\gamma \in \Delta} N^\gamma \]
be two direct decompositions of \( G \) into closed groups and let \( G = \sum_{\mu \in I} G^\mu, \)
\[ G = \sum_{\nu \in \mathcal{J}} H^\nu \]
be refinements of these decompositions respectively. Then if \( \sum_{\gamma \in \Gamma} M^\gamma \) and \( \sum_{\gamma \in \Delta} N^\gamma \) have isomorphic refinements, then so do \( \sum_{\mu \in I} G^\mu \) and \( \sum_{\nu \in \mathcal{J}} H^\nu. \)
Proof. — Consider the scheme
\[
\begin{array}{cccc}
& (M^\gamma) & \downarrow (G^\omega) & \downarrow (P^\lambda) & \downarrow (Q^\mu) & \downarrow (H^\nu) \\
& \downarrow (\bar{R}^\iota) & \downarrow (R^\kappa) & \downarrow (S^\sigma) & \downarrow (\bar{S}^\tau) \\
& (U^\omega) & (V^\nu) \\
\end{array}
\]
where each family is a family of subgroups of \( G \), giving a direct decom-
position of \( G \), where the arrows indicate refinements and the symbol \( \cong \)
denotes isomorphic decompositions, e.g. \( P^\lambda \) is isomorphic to \( Q^\mu \) for
each \( \lambda \). The existence of the isomorphic refinements \( (\bar{R}^\iota) \) and \( (R^\kappa) \),
\( (\bar{S}^\tau) \) and \( (S^\sigma) \) follows from the fact that each \( M^\gamma \) and \( N^\delta \) are closed and
from the fact that any two decompositions of a closed group have iso-
morphic refinements. Similarly the existence of the decompositions
\( (U^\omega) \) and \( (V^\nu) \) follows from the fact that the \( P^\lambda \) and \( Q^\mu \) are closed.

Lemma 7 gives us an indication of how we may proceed to find
isomorphic refinements of two decompositions of a group \( G \) into closed
groups. To be more specific, suppose

\[
G = \sum_{\mu \in I} G^\mu \quad \text{(direct)},
\]
\[
G = \sum_{\nu \in J} H^\nu \quad \text{(direct)}
\]
and let \( I^* (J^*) \) be the subset of \( I \) (of \( J \)) consisting of all \( \mu \in I \) (\( \nu \in J \))
such that \( G^\mu \) (\( H^\nu \)) is of infinite length and let \( I^{**} (J^{**}) \) be the complement
of \( I^* \) (of \( J^* \)) in \( I \) (in \( J \)). If \( I^* \) or \( J^* \) were finite, KOLETTIS and Kulikov’s
results could be applied to find isomorphic refinements; hence, suppose
they are both infinite. Then by lemma 6 we know they have the same
cardinality.

Suppose that \( \beta \) is the first ordinal having the same cardinality as \( I^* \). We
will show that we can write
\[
I^* = \bigcup_{\sigma < \beta} I_\sigma,
\]
\[
J^* = \bigcup_{\tau < \beta} J_\tau,
\]
where each \( I_\sigma, J_\tau \) are finite and \( I_\sigma \cap I_{\sigma'} = \emptyset = J_\tau \cap J_{\tau'} \) if \( \sigma \neq \sigma' \) and
\( \tau = \tau' \).
such that if we let

\[ C^\alpha = \sum_{\mu \in I_{\alpha}} G^\mu, \]
\[ D^\gamma = \sum_{\nu \in J_{\gamma}} H^\nu, \]

we can find decompositions

\[ C^\alpha = C_{\alpha_1} + C_{\alpha_2} + C_{\alpha_3} \quad \text{(direct)}, \]
\[ D^\gamma = D_{\gamma_1} + D_{\gamma_2} + D_{\gamma_3} \quad \text{(direct)} \]

such that:

1. \( D_{\gamma_1} \cong C_{\alpha_1} \) for each \( \sigma < \beta \);
2. \( D_{\gamma_2} \cong C_{(\sigma+1)_1} \) for each \( \sigma < \beta \);
3. \( C_{\alpha} \) and \( D_{\gamma} \) are direct sums of cyclic groups for each \( \sigma < \beta \);
4. \( P \left( \sum_{\sigma < \beta} C_{\sigma} + C_{\alpha_1} \right) = P \left( \sum_{\tau < \beta} D_{\tau_1} + D_{\tau_2} \right) \);
5. \( C_{\alpha_0} = 0 \) if \( \sigma = 0 \) or a limit ordinal \( < \beta \). If this is possible, then lemma 1 coupled with (4) above gives

\[ G = \sum_{\sigma < \beta} (C_{\sigma_1} + C_{\sigma_0}) + \sum_{\tau < \beta} D_{\tau_1} + \sum_{\nu \in J_{**}} H^\nu. \]

But

\[ G = \sum_{\sigma < \beta} (C_{\sigma_2} + C_{\sigma_0}) + \sum_{\tau < \beta} C_{\tau_1} + \sum_{\mu \in I_{**}} G^\mu, \]

thus \( \sum_{\sigma < \beta} C_{\sigma_1} + \sum_{\mu \in I_{**}} G^\mu \cong \sum_{\tau < \beta} D_{\tau_1} + \sum_{\nu \in J_{**}} H^\nu, \)

but any two direct decompositions of a group into cyclic groups have isomorphic refinements. Using this fact, (1) and (2) above, and lemma 7, we get that

\[ \sum_{\mu \in I} G^\mu \quad \text{and} \quad \sum_{\nu \in J} H^\nu \]

have isomorphic refinements.

To complete the proof that the two decompositions have isomorphic refinements the task remains to determine the sets \( I_{\sigma}, J_{\gamma} \) and the required decompositions of the groups \( C^\alpha \) and \( D^\gamma \).

The next lemma will give us a method of choosing the sets \( I_{\sigma} \) and \( J_{\gamma} \).
Lemma 8. — Let

\[ G = G^* + \sum_{\nu \in J} G^\nu \]  
\text{(direct)}

and

\[ H = \sum_{\nu \in J} H^\nu \]  
\text{(direct)}

where \( G^* \) and each \( G^\nu \) and \( H^\nu \) is a closed group. Then if \( \varphi \) is an isomorphism of \( G \) onto \( H \) there exists finite subsets \( J^* \) of \( J \), \( I^* \) of \( I \) and direct decompositions

\[ G^* = C^1 + C^3 \]
\[ \sum_{\nu \in I^*} H^\nu = D^1 + D^2 + D^3 \]
\[ \sum_{\nu \in I^*} G^\nu = C^2 + C^* \]

such that:

1. \( C^1 \cong D^1 \);
2. \( C^3 \cong D^3 \);
3. \( C^1 \) and \( D^3 \) are direct sums of cyclic groups;
4. \( P(\varphi(C^1 + C^2)) = P(D^1 + D^3) \).

Proof. — It is not difficult to see that there is no loss in generality in assuming, as we shall, that \( G = H \) and that \( \varphi \) is the identity map. By lemma 4 there exists a direct decomposition \( G^* = C^1 + C^3 \) where \( C^3 \) is a group of bounded order and where \( P(C^1) \subset \sum_{\nu \in J^*} H^\nu \) for some finite subset \( J^* \) of \( J \). Note that we can assume \( J^* \) contains any one element of \( J \); which element will be specified when the lemma is applied. Furthermore, \( C^1 \) is closed and pure in \( G \); hence by lemma 3 there exists a decomposition

\[ \sum_{\nu \in I^*} H^\nu = D^1 + D^2 \]  
\text{(direct),}

where \( P(D^1) = P(C^1) \) and \( D^1 \cong C^1 \). Since \( D^1 \) and \( C^1 \) are direct summands of \( G \) by lemma 1 we get

\[ G = C^1 + D^1 + \sum_{\nu \in I^*} H^\nu \]  
\text{(direct),}
but
\[ G = C' + C^1 + \sum_{\mu \in I} G^\mu \quad \text{(direct)}. \]

Thus, let \( \psi \) be the isomorphism of
\[ \bar{D} + \sum_{\nu \in J^*} H^\nu \quad \text{onto} \quad C^1 + \sum_{\mu \in I} G^\mu, \]
where \( \psi(x) = y \) if and only if \( x - y \in C' \). Then apply lemma 4 and let
\[ \bar{D} = D^3 + D^1 \quad \text{(direct)} \]
be such that \( D^3 \) is a direct sum of cyclic groups and
\[ P(\psi(D^3)) \subset \sum_{\nu \in J^*} G^\nu \]
for some finite subset \( J^* \) of \( I \) which can be assumed to contain any specified element of \( I \). Then by lemma 3,
\[ \sum_{\nu \in J^*} G^\nu = C^1 + C^*, \]
where
\[ C^1 \cong \psi(D^1) \cong D^3 \]
and where
\[ P(C^2) = P(\psi(D^3)). \]
Now \( \psi(D^3) + C^1 = D^3 + C^1 \) by the definition of \( \psi \); hence
\[ P(C^1 + C^1) = P(C^1) + P(C^1) = P(C^1) + P(\psi(D^3)) = P(C^1) + P(D^3), \]
but
\[ P(C^1) = P(D^1), \quad \text{thus} \quad P(C^1 + C^1) = P(D^1 + D^3) \]
and our proof is complete. Remark that \( C^* \) as a direct summand of a closed group is closed. We now come to the main result of this paper.

**Theorem.** — Any two decompositions of a group into closed groups have isomorphic refinements.

**Proof.** — Let
\[ G = \sum_{\mu \in I} G^\mu \quad \text{(direct),} \]
\[ G = \sum_{\nu \in J} H^\nu \quad \text{(direct),} \]
where each $G^\mu$ and $H^\nu$ is closed; and assume $I^*$ consists of all $\mu \in I$ such that $G^\mu$ is of infinite length. Define $J^*$ in a similar manner. As we remarked before, we may assume $I^*$ is infinite. Then by lemma 6, $I^*$ and $J^*$ have the same cardinality. Let $\beta$ be the least ordinal having the same cardinality as $I$. Let $I$ and $J$ be well ordered so that both have well order type $\beta$. For any ordinal $\sigma < \beta$ let $i_\sigma$ and $j_\sigma$ denote the $\sigma$-th element of $I$ and $J$ respectively. If $\eta < \beta$ suppose that for each ordinal $\sigma \leq \eta$ we have chosen a finite non-empty subset $I_\sigma$ of $I$ such that $I_\sigma \cap I_\sigma' = \emptyset$ if $\sigma \neq \sigma'$ and such that $i_\sigma \in \bigcup \limits_{\sigma \leq \eta} I_\sigma$ for all $\varepsilon \leq \eta$
and that for all $\tau < \eta$ we have chosen finite non-empty subsets $J_\tau$ of $J$ such that $J_\tau \cap J_\tau' = \emptyset$ if $\tau \neq \tau'$ and such that

$$j_\varepsilon \in \bigcup \limits_{\tau < \eta} J_\tau \text{ for all } \varepsilon \leq \eta.$$ Setting

$$C^\sigma = \sum \limits_{\mu \in I_\sigma} G^\mu, \quad D^\tau = \sum \limits_{\nu \in J_\tau} H^\nu$$

for $\sigma \leq \eta$, $\tau < \eta$ suppose we have decompositions

$$C^\sigma = C^{\tau_1} + C^{\tau_2} + C^{\tau_3}, \quad \text{(direct),}$$

$$D^\tau = D^{\tau_1} + D^{\tau_2} + D^{\tau_3}, \quad \text{(direct)}$$

and a direct summand $C'^\varepsilon$ of $C^\varepsilon$. We will let $C'^\varepsilon$ denote a supplement of $C'^\varepsilon$ in $C^\varepsilon$.

Then we consider the following conditions:

(1) $C^{\varepsilon_1} \cong D^{\varepsilon_1}, \quad \varepsilon < \eta$;
(2) $C^{\varepsilon_1 + \varepsilon_2} \cong D^{\varepsilon_2}, \quad \varepsilon < \eta$;
(3) $C^{\varepsilon}$ and $D^{\varepsilon}$ are direct sums of cyclic groups, $\varepsilon \leq \eta$;
(4) $P \left( \sum \limits_{\varepsilon < \eta} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\varepsilon_3} \right) = P \left( \sum \limits_{\varepsilon < \eta} (D^{\varepsilon_1} + D^{\varepsilon_2}) \right)$;
(5) $C^{\varepsilon} = 0$ for $\varepsilon = 0$ and for all limit ordinals $\varepsilon \leq \eta$.

We can easily see that if we let $I_\varepsilon = \{ i_\varepsilon \}$, $G^{i_\varepsilon} = \circ$ and $G^i = G^\circ$ then $(A^\circ)$ holds. Now suppose we have chosen our subsets of $I$ and $J$ and the decompositions of $C^\varepsilon$ and $D^\varepsilon$ in such a manner that $(A^\varepsilon)$ holds for all $\eta < \rho$ where $\rho \leq \beta$. If $\rho = \beta$, then by the remarks after lemma 7, we see that the theorem is proved. Hence, suppose $\rho < \beta$. Then
we must choose a direct decomposition $C^{\rho'} = C^{\rho_1} + C^{\rho_2}$, subsets
$I_{\rho+1}$ and $I_{\rho}$ of $I$ and $J$ respectively, a direct summand $C^{\rho+1}$ of $C^\rho$ and a direct decomposition

$$D^\rho = D^{\rho_1} + D^{\rho_2} + D^{\rho_3}$$

so that $(A^\rho)$ holds. First, suppose $\rho$ is a limit ordinal. In this case the sets $I_\sigma$ and $J_\tau$ for $\sigma, \tau < \rho$ have already been chosen. We let $I_\rho$ be any finite non-empty subset of the complement of $\bigcup_{\sigma < \rho} J_\sigma$ in $I$ subject to the condition that

$$i_\rho \in I_\rho \text{ if } i_\rho \not\in \bigcup_{\sigma < \rho} I_\sigma.$$ 

Now let

$$C^\rho = \sum_{\nu \in I_\lambda} G^{\nu \rho}.$$ 

Let $C^\rho = C^{\rho_1} + C^{\rho_2}$ where $C^{\rho_1} = 0$ and $C^{\rho_2} = C^\rho$. Then it is easy to see that $(A^\rho)$ holds. In case $\rho$ is not a limit ordinal let $\lambda + 1 = \rho$. By $(\ell)$ of $(A^\rho)$ we have

$$P\left( \sum_{\varepsilon < \lambda} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\varepsilon_3} \right) = P\left( \sum_{\varepsilon < \lambda} (D^{\varepsilon_1} + D^{\varepsilon_2}) \right).$$

Now both

$$\sum_{\varepsilon < \lambda} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\varepsilon_3} \text{ and } \sum_{\varepsilon < \lambda} (D^{\varepsilon_1} + D^{\varepsilon_2})$$

are direct summands of $G$ so by lemma 1

$$G = \sum_{\varepsilon < \lambda} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\varepsilon_3} + \sum_{\nu \in J_\varepsilon} H^\nu + \sum_{\varepsilon < \lambda} D^{\varepsilon_3} \text{ (direct).}$$

But also

$$G = \sum_{\varepsilon < \lambda} (C^{\varepsilon_1} + C^{\varepsilon_2}) + C^{\varepsilon_3} + \sum_{\nu \in J_\varepsilon} C^{\nu \varepsilon} + \sum_{\varepsilon < \lambda} G^{\varepsilon_3}.$$ 

Let $\psi$ be the isomorphism of

$$C^{\varepsilon_3} + \sum_{\nu \in J_\varepsilon} G^{\nu \varepsilon} + \sum_{\varepsilon < \lambda} C^{\varepsilon_3}$$

onto

$$\sum_{\nu \in J_\varepsilon} H^\nu + \sum_{\varepsilon < \lambda} D^{\varepsilon_3}.$$
where \( \psi(x) = y \) if
\[
x - y \in \sum_{z < \lambda} (C^z + C^z) + C^z.
\]
We have that
\[
\sum_{z < \lambda} C^z \quad \text{and} \quad \sum_{z < \lambda} D^z
\]
are direct sums of cyclic groups. \( C^z \) is a direct summand of a closed group \( \sum_{z \in I^*} G^z \) hence is closed. Letting \( C^z \) correspond to the \( G^z \) in lemma 8, we can find finite non-empty subsets \( I_p = I_{k+1} \) and \( J_k \) in the complements of \( \bigcup_{z \leq \lambda} J_z \) and \( \bigcup_{z < \lambda} J_z \) in \( I \) and \( J \) respectively and the following direct decompositions:
\[
C^z = C^z + C^z, \quad D^z = D^z + D^z, \quad C^p = C^p + C^p,
\]
where we have:
(a) \( C^z \cong D^z \);
(b) \( C^z \cong D^z \);
(c) \( C^z \) and \( D^z \) are direct sums of cyclic groups;
(d) \( P(\psi(C^z + C^z)) = P(D^z + D^z) \).

We can suppose, as remarked in the proof of lemma 8, that
\[
i_p \in I_p \quad \text{if} \quad i_p \notin \bigcup_{\gamma > \rho} I_{\gamma}
\]
and likewise require that
\[
j_k \in J_k \quad \text{if} \quad j_k \notin \bigcup_{\gamma < k} J_{\gamma}.
\]
Then (1), (2) and (3) of \((A^\rho)\) obviously hold for \( z < \lambda \) since \((A^\rho)\) is true. For \( z = \lambda \), (1), (2) and (3) of \((A^\rho)\) hold since (a), (b) and (c) above hold. (5) of \((A^\rho)\) obviously holds so we only need show (4) of \((A^\rho)\) holds.

By (d) above we have that
\[
P(\psi(C^z + C^z)) = P(D^z + D^z).
\]
By the definition of \( \psi \) we get that
\[
P(C^z + C^z + \sum_{z < \lambda} (C^z + C^z) + C^z) = P(\psi(C^z + C^z) + \sum_{z < \lambda} (C^z + C^z) + C^z) = 0.
\]
Now since
\[ P(\psi(C^{i_1} + C^{i_2})) = P(D^{i_1} + D^{i_2}) \]
by (d) above and since
\[ P\left( \sum_{\zeta < \lambda} (C^{j_1} + C^{j_2}) + C^{j_3} \right) = P\left( \sum_{\zeta < \lambda} (D^{i_1} + D^{i_2}) \right) \]
by (4) of (A7) we get that (4) of (A7) holds.

Thus by transfinite induction we can suppose (A8) holds for all \( \eta < \rho \)
and hence that the two decompositions of \( G \) have isomorphic refinements.

REFERENCES.


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