T.J. Head

Remarks on a problem in primary abelian groups


<http://www.numdam.org/item?id=BSMF_1963__91__109_0>
1. All groups considered in this note are assumed to be $p$-primary abelian groups. If $A$ is a subgroup of $G$ then $\overline{A}$ will denote the closure of $A$ in the usual topology of $G$ ([2], page 114). The closure of a subgroup is a subgroup, but the closure of a pure subgroup need not be pure. It is a consequence of lemma 20 of [3] that if $G$ is a closed $p$-group (for definition see [2], page 114) then the closure of each pure subgroup of $G$ is pure.

**Problem.** — If $G$ is a primary abelian group without elements of infinite height in which the closure of each pure subgroup is pure does it follow that $G$ is a closed $p$-group?

We do not know the answer to this question, but we can give an affirmative answer in the case of direct sums of cyclic groups:

**Theorem.** — If $G$ is a direct sum of cyclic $p$-groups and the closure of each pure subgroup of $G$ is pure in $G$ then $G$ is a bounded $p$-group.

An outline of the proof of this theorem is given in paragraph 3 below.

2. **The relation of the problem to minimal pure embeddings.** — Following B. Charles [1], when a subgroup $S$ of a group $G$ is contained in a pure subgroup $P$ of $G$ which has the property that no proper pure subgroup of $P$ contains $S$ we say that $P$ is *minimal pure containing* $S$. 
When such a $P$ exists we say that $S$ has a minimal pure embedding in $G$. We will denote the subgroup of elements of infinite height in a group $G$ by $G'$.

In the proofs below we use the following two observations:

If $A$ is a subgroup of $G$ then $\overline{A}$ is that subgroup of $G$ containing $A$ for which $\overline{A}/A = (G/A)'$. If a subgroup $S$ of a group $G$ is contained in $G'$ and if $P$ is minimal pure containing $S$ then $P$ is divisible.

The latter observation follows from the fact that if $P$ were not divisible $P$ would contain a finite cyclic direct summand $\langle x \rangle$ and if $P = \langle x \rangle \oplus C$ then $S$ would be contained in $C$ where $C$, being a direct summand of $P$, would be pure in $G$.

For a subgroup $S$ of $G$ we denote by $S'$ the subgroup of $G$ containing $S$ for which $S'/S$ is the maximal divisible subgroup of $G/S$.

**Proposition.** — Let $P$ be a pure subgroup of a primary abelian group $G$. Let $H$ be a subgroup of $G$ for which $P \subset H \subset \overline{P}$. Then $H$ has a minimal pure embedding in $G$ if and only if $H \subset P'$.

**Proof.** — Suppose $P_1$ is minimal pure containing $H$. Then $P_1/P$ is divisible. Then $P_1 \subset P'$ and $H \subset P'$. Conversely, if $H \subset P'$ then $H/P$ is contained in the maximal divisible subgroup of $G/P$. Then there exists a subgroup $P_1$ of $G$ containing $P$ such that $P_1/P$ is minimal divisible containing $H/P$ in $G/P$. Then $P_1$ is minimal pure containing $H$ in $G$.

It has been suggested ([1], page 224) that if $G$ is a primary abelian group without elements of infinite height and $S$ is a subgroup of $G$ which is the union of an ascending chain of discrete subgroups of $G$ then $S$ has a minimal pure embedding in $G$. The proposition and theorem above are sufficient to show that this is not true even if the discrete subgroups are finite:

Let $G$ be a countable unbounded direct sum of cyclic $p$-groups. Let $P$ be a pure subgroup of $G$ for which $\overline{P}$ is not pure. $\overline{P}$ is the union of an ascending chain of finite (hence discrete) subgroups of $G$. Since $P'$ is pure, $\overline{P} \neq P'$. Consequently $\overline{P}$ is not contained in any subgroup of $G$ which is minimal pure containing $P$.

This same example is a counter-example to part 2 of theorem 6 of [1] because $P$ is a pure subgroup of $G$ which is dense in $\overline{P}$ and yet $\overline{P}$ has no minimal pure embedding in $G$. Along this line we have:
COROLLARY. — For a primary abelian group $G$ the following two conditions are equivalent:

1. Each subgroup $H$ of $G$ that contains a subgroup $P$ which is pure in $G$ and dense in $H$ (relative to the topology of $G$) has a minimal pure embedding in $G$.

2. For each pure subgroup $P$ of $G$, $\overline{P}$ is pure in $G$.

Proof. — Assume (1) and let $P$ be pure in $G$. Then $\overline{P}$ has a minimal pure embedding in $G$. By the proposition $P = P'$ and $\overline{P}$ is pure in $G$.

Assume (2) and let $H$ be a subgroup of $G$ which contains a subgroup $P$ which is pure in $G$ and dense in $H$. We have $P \subset H \subset \overline{P}$. Since $\overline{P}$ is pure in $G$ it follows from the proposition that $\overline{P} = P'$. The proposition then gives the conclusion that $H$ has a minimal pure embedding in $G$.

3. Outline of the proof of the theorem stated in paragraph 1. —

It is sufficient to show that if $G = \sum \limits_{n=1}^{\infty} Z(p^{i(n)})$ where $i(n)$ is a strictly increasing sequence of positive integers, $i(1) \geq 2$, and $Z(p^{i(n)})$ is a cyclic group of order $p^{i(n)}$ then $G$ contains a pure subgroup $P$ for which $\overline{P}$ is not pure. For each positive integer $n$ let $g(n)$ be a generator of $Z(p^{i(n)})$. Then it may be verified that the following sequence of elements of $G$ is a linearly independent set and that the subgroup, $P$, generated by this set is pure in $G$:

$$s(n) = g(2n - 1) + p^{(2n - 1)} g(2n) + p^{(2n + 1)} g(2n + 1),$$

$$1 \leq n < \infty.$$ Let

$$x = p^{(i(1) - 1)} g(1).$$

Then $x \in \overline{P}$ since modulo $P$ we have:

$$x = p^{(i(1) - 1)} g(1) = -p^{(i(3) - 1)} g(3) = \ldots = (-1)^n p^{(i(2n + 1) - 1)} g(2n + 1) = \ldots.$$ Let $y$ be any element of $G$ for which $p^{(i(1) - 1)} y = x$. There is an integer $N$ such that the component of $y$ in $Z(p^{(i(N)})$ is different from 0 and the component of $y$ in $Z(p^{(i(n)})$ is 0 for each $n > N$. By proceeding from the fact the component of $y$ in $Z(p^{(i(1)})$ is the unique component of $y$ which is not annihilated by $p^{(i(1) - 1)}$, it can be verified that the neighborhood $y + p^{v+2} G$ of $y$ is disjoint from $P$. Then $y \notin \overline{P}$ and the equation $p^{(i(1) - 1)} z = x$, which has the solution $z = g(1)$ in $G$, is not solvable for $z$ in $\overline{P}$. Thus $\overline{P}$ is not pure in $G$. 
REFERENCES.


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Thomas J. Head,
Iowa State University,
Ames, Iowa (États-Unis).