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Regularity theorems for fractional powers of a linear elliptic operator


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REGULARITY THEOREMS FOR FRACTIONAL POWERS
OF A LINEAR ELLIPTIC OPERATOR;

BY

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1. Introduction. — Let \( L \) be a linear elliptic operator with \( C^\infty \) coefficients in an open subset \( \Omega \) of \( \mathbb{R}^n \) (\( n \geq 2 \)). We suppose that \( L \) admits a (strictly) positive self-adjoint realisation \( \tilde{L} \) in \( L^2(\Omega) \). Let \( \{ E_\lambda \} \) be the spectral resolution of \( \tilde{L} \) so that

\[
\tilde{L} = \int \lambda \, dE_\lambda.
\]

We consider the family of operators \( \tilde{L}^s \), depending on a complex parameter \( s \), defined by

\[
\tilde{L}^s = \int \lambda^s \, dE_\lambda.
\]

The operators \( \tilde{L}^s \) may be viewed as "fractional powers" of \( L \). For \( s = -1, -2, \ldots \), we obtain the Green's operator and its iterates.

We study in this paper the regularity properties of the operators \( \tilde{L}^s \). For integral values of \( s \), it is known that the operators \( \tilde{L}^s \) define kernels which are "very regular" in the sense of SCHWARTZ ([17], chap. V, § 6) and that if further the coefficients of \( L \) are analytic the kernels of \( \tilde{L}^s \) are analytically very regular. For positive integral values of \( s \) the results are trivial, for negative integral values of \( s \) these follow from well-known regularity theorems for elliptic operators [11]. The question arises whether these results are true for all values of \( s \). We prove in this paper that this is in fact the case (Theorems 2 and 3). The case of elliptic operators with constant coefficients on a torus and on \( \mathbb{R}^n \) has already been dealt with respectively by S. BOCHNER [3] and L. SCHWARTZ ([16], chap. VII, § 10, ex. 7).

That the operators \( \tilde{L}^s \) possess kernels follows from regularity theorems for elliptic operators. In order to prove that the kernels are very regular,
we represent the kernels, for \( R(-s) \) sufficiently large, in terms of the Green's function \( G(t, x, y) \) of the associated parabolic operator. By using some results of G. Bergendal [1] and S. D. Eidelman [6] and showing that \( G(t, x, y) \) and its derivatives fall off exponentially as \( t \to \infty \), we then prove that the kernel \( L^s \) is very regular.

The proof of analytic regularity, when the coefficients are analytic, is more difficult. It involves in the first instance estimates for the norms \( \| A^k u \|_{L^2} \), where \( u \) is a function that is to be proved to be analytic and \( A \) a linear elliptic operator with analytic coefficients. Next we need to prove a general theorem (Theorem 1) to the effect that if \( A \) is a linear elliptic operator of order \( m \) with analytic coefficients in an open set \( \Omega' \) of \( \mathbb{R}^n \), and \( u \) is a function satisfying the inequalities

\[
\| A^k u \|_{L^2(\Omega')} \leq (km)! c^{k+1}
\]

for every integer \( k \geq 0 \), with a positive constant \( c \) independent of \( k \), then \( u \) is analytic in \( \Omega' \).

This theorem is a natural one in as much as the conditions

\[
\| A^k u \| \leq (km)! c^{k+1}
\]

on every compact set are necessary for \( u \) to be analytic. We notice also that this theorem contains the well-known result: if \( A \) is linear elliptic operator and has analytic coefficients, and if \( A u = f \) with \( f \) analytic, then \( u \) is analytic.

A weaker version of Theorem 1 has been proved by E. Nelson ([14], th. 7); he proves the analyticity of \( u \) under the stronger assumption

\[
\| A^k u \| \leq k! c^{k+1}.
\]

Theorem 1 is proved by suitably estimating the \( L^2 \)-norms of derivatives of order \( km \) of \( u \) in terms of \( L^2 \)-norms of \( u, Au, \ldots, A^k u \). The proof of this theorem uses some ideas of a paper of C. B. Morrey and L. Nirenberg [13].

The use of the parabolic equation in the proofs of Theorems 2 and 3 was suggested by a paper of S. Minakshisundaram [12].

For spaces of distributions we use the usual notation [17].

The results of this paper have been announced in [10].

2. Statement of the theorems. — Let \( \Omega \) be an open subset of \( \mathbb{R}^n \). Let \( \mathcal{C}(\Omega) \) be the space of complex-valued \( C^\infty \) functions with compact support in \( \Omega \). \( L^2(\Omega) \) is the Hilbert space of complex-valued square summable functions on \( \Omega \), with scalar product \( (\varphi, \psi) \) defined by

\[
(\varphi, \psi) = \int_{\Omega} \varphi \bar{\psi} \, dx
\]

for \( \varphi, \psi \in L^2(\Omega) \); \( \| \varphi \|_{L^2(\Omega)} \) means \( (\varphi, \varphi)^{1/2} \).
Let $A$ be a linear differential operator of order $m$,

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^x$$

with sufficiently differentiable complex-valued coefficients $a_\alpha(x)$ defined in $\Omega$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i$ being integer $\geq 0$ and we put:

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

$$D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \ldots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$ 

We say now $A$ is an elliptic operator in $\Omega$, if the homogeneous form of order $m$

$$\sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \neq 0$$

for every $x \in \Omega$ and for every non vanishing real vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$.

**Theorem 1.** — Let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $A$ be a linear elliptic operator of order $m$ with analytic coefficients in $\Omega$. Let $A^k$ be the $k$th iterate of $A$. Suppose that a function $u$ (of class $C^k$) satisfies the inequality

$$\|A^k u\|_{L^2(\Omega)} \leq (km)! c^{k+1}$$

for every integer $k \geq 0$ with a positive constant $c$ independent of $k$. Then the function $u$ is analytic in $\Omega$.

**Remark.** — The above theorem is also valid for elliptic systems; the demonstration is the same as for the scalar case.

As for the following theorems, we consider a linear elliptic operator $L$ defined on $\Omega$ such that

$$(L \varphi, \psi) = (\varphi, L \psi)$$

for every $\varphi, \psi \in \mathcal{D}(\Omega)$.

Suppose further that $L$ when defined on $\mathcal{D}(\Omega) (\subset L^2)$, where it is symmetric, has a strictly positive self-adjoint extension $\tilde{L}$.

Remark that these conditions entail that the form

$$L(x, \xi) = \sum_{|\alpha| \leq m} b_\alpha(x) \xi^\alpha$$

is real and definite for every $x \in \Omega$ and $\xi$ real vector, when $L = \sum_{|\alpha| \leq m} b_\alpha(x) D^x$ has sufficiently smooth coefficients.

Let $\{E_\lambda\}$ be the spectral resolution of $\tilde{L}$. By the hypothesis on $\tilde{L}$, we have $\lambda \geq c_0 > 0$ on the spectrum.
We can now define a family of operators $\tilde{L}^s$ depending on the complex parameter $s$, by

$$\tilde{L}^s = \int \lambda^s dE_\lambda.$$ 

As we shall see in section 5, $\tilde{L}^s$ thus defined is a continuous linear map of $\mathcal{D}(\Omega)$ into the space of distributions $\mathcal{D}'(\Omega)$ for every $s$, so that $\tilde{L}^s$ defines a kernel $L^s(x, y)$ ([17], [19]); the theorems to be proved concern the regularity of the kernel $L^s(x, y)$.

**Theorem 2.** — Let $L$ be a linear elliptic differential operator with $C^\infty$ coefficients in an open set $\Omega$ of $\mathbb{R}^n$. We suppose further that $L$ admits a strictly positive self-adjoint realisation

$$L = \int \lambda dE_\lambda.$$ 

in $L^2(\Omega)$. Let $s$ be a complex number. Then the operator

$$\tilde{L}^s = \int \lambda^s dE_\lambda,$$

defines a kernel which is very regular.

**Theorem 3.** — Let $L$ be a linear elliptic differential operator with analytic coefficients in an open set $\Omega$ of $\mathbb{R}^n$, admitting a strictly positive self-adjoint realisation $\tilde{L}$ in $L^2(\Omega)$. Then, for every complex number $s$, the kernel of the operator

$$\tilde{L}^s = \int \lambda^s dE_\lambda,$$

is analytically very regular.

For the definition of very regular kernels and analytically very regular kernels see ([17], chap. V, § 6).

As a consequence of the above theorems, $\tilde{L}^s(T)$ can be defined for $T$, a distribution with compact support and when $L$ has the $C^\infty$ (analytic) coefficients, $\tilde{L}^s(T)$ is an infinitely differentiable (resp. analytic) function in an open set of $\Omega$ where $T$ is an infinitely differentiable (resp. analytic) function.

### 3. Preliminary lemmas.

We consider in this section some lemmas which are required in the proof of Theorem 1.

Let $\Omega'$ be any open subset of $\Omega$. Let $u$ be of class $C^\infty$ on the closure $\overline{\Omega'}$ of $\Omega'$. Let $k$ be an integer $\geq 0$. We define the $k$-norm of $u \in C^\infty(\overline{\Omega'})$ by

$$||u||_{k, \Omega'} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} ||D^\alpha u||_{L^1(\Omega')}.$$ 

where we put $x! = x_1! \cdot x_2! \cdots x_n!$ for $x = (x_1, x_2, \ldots, x_n)$. 
LEMMA 3.1. — Let \( k, k' \) be given integers \( \geq 0 \). Then we have

\[
\| u \|_{k+k', \Omega} = \sum_{|\alpha| = k} \frac{k!}{\alpha!} \| D^\alpha u \|_{k', \Omega},
\]

PROOF. — We have

\[
\frac{(k+k')!}{\gamma!} = \sum_{|\alpha| = k} \frac{k! k'}{\alpha! \beta!}.
\]

The lemma follows immediately from this equality.

The next lemma is a refined version of Friedrichs' inequality \[7\]. The proof is a modification of Friedrichs' proof as in \[13\].

We denote by \( \Omega_r \) the ball \( |x| < r \) of radius \( r \) in \( \mathbb{R}^n \).

LEMMA 3.2. — Let \( A \) be a linear elliptic operator of order \( m \) with \( \mathcal{C}^\infty \) coefficients in \( \Omega \). Let \( r, \delta \) be positive numbers such that \( \delta < r \) and \( \Omega_{r+\delta} \subset \Omega \). Then there exists a constant \( c > 0 \) independent of \( \delta \) such that for every \( u \in \mathcal{C}^\infty (\Omega) \) we have

\[
\| u \|_{m, \Omega_r} \leq c \left( \| A u \|_{0, \Omega_{r+\delta}} + \delta^{-m} \| u \|_{0, \Omega_{r+\delta}} \right).
\]

PROOF. — Let \( \zeta \in \mathcal{C}^\infty (\Omega) \) have its support in \( \Omega_{r+\delta} \) and be such that \( \zeta \equiv 1 \) on \( \Omega_r \) and satisfies

\[
\sup_{\Omega_{r+\delta}} |D^k \zeta (x)| \leq c_2 \delta^{-|\alpha|} \quad (\delta < r)
\]

with \( c_2 > 0 \) depending only on \( \alpha \).

For any \( u \in \mathcal{C}^\infty (\Omega) \), we shall consider \( \zeta^m u \), which is of class \( \mathcal{C}^\infty \) having its support in \( \Omega_{r+\delta} \). Since \( A \) is an elliptic operator with \( \mathcal{C}^\infty \) coefficients, we have the well-known inequality \[10\]

\[
\| \zeta^m u \|_{m, \Omega_{r+\delta}} \leq c \left( \| A (\zeta^m u) \|_{0, \Omega_{r+\delta}} + \| \zeta^m u \|_{0, \Omega_{r+\delta}} \right)
\]

with a constant \( c > 0 \) depending only on \( A \) and \( \Omega_{r+\delta} \).

By using the estimate (3.1), we obtain

\[
\| A (\zeta^m u) \|_{0, \Omega_{r+\delta}} \leq c \left\{ \| \zeta^m A u \|_{0, \Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k, \Omega_{r+\delta}} \right\},
\]

\[
\sum_{|\alpha| = m} \| \zeta^m D^\alpha u \|_{0, \Omega_{r+\delta}} \leq c \left\{ \| \zeta^m u \|_{m, \Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k, \Omega_{r+\delta}} \right\}.
\]
It follows then from (3.2),

$$\sum_{k=0}^{m} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \leq c \left\{ \| \zeta^m A u \|_{\Omega_{r+\delta}} + \sum_{k=0}^{m-1} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of $k$.

To complete the proof of the lemma, we need the following fact: for every $\varepsilon, \delta > 0$, there exists a constant $c$ independent of $\varepsilon, \delta$ and $u$ such that

$$\sum_{|\alpha|=k} \| \zeta^k D^\alpha u \|_{\alpha,\Omega} \leq \varepsilon \sum_{|\alpha|=k+1} \| \zeta^{k+1} D^\alpha u \|_{\alpha,\Omega} + c (\varepsilon^{-1} + \delta^{-1}) \sum_{|\alpha|=k-1} \| \zeta^{k-1} D^\alpha u \|_{\alpha,\Omega}$$

where $k \geq 1$.

In fact we have the equality

$$- (\zeta^k D^2 u, \zeta^k D^2 u) = (\zeta^{k-1} D^\alpha u, \zeta^{k+1} D, D^2 u) + 2 k (D_1 \zeta) \zeta^{k-1} D^\alpha u, \zeta^k D^2 u),$$

where $\alpha' = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$ (we suppose $\alpha_1 \geq 0$) and $D_1 = \partial/\partial x_1$.

Now we can obtain the inequality (3.4) by Schwarz's inequality and by taking into account the estimate (3.1) for $\zeta$.

In (3.4) we take $k = m - 1$ and choose $\varepsilon$ as $\varepsilon = \delta/2c$. Bringing the inequality thus obtained in the right side of (3.3), we have

$$\sum_{k=0}^{m} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \leq c \left\{ \| \zeta^m A u \|_{\Omega_{r+\delta}} + \sum_{k=0}^{m-2} \delta^{-m+k} \| \zeta^k u \|_{k,\Omega_{r+\delta}} \right\}$$

with $c > 0$ independent of $k$. Thus in the right side of (3.3), the terms corresponding to $k = m - 1$ can be absorbed in the left side. Repeating this procedure by using (3.4) with appropriate $\varepsilon$, we arrive finally at the desired inequality stated in the lemma.

**Lemma 3.3.** — Let $q$ be positive integer such that $q < m$. Let $r < r_0$, $r_0$ being fixed. Then there exists a constant $c_m > 0$ depending only on $m$ and $r_0$ such that for every $\varepsilon > 0$ and $u \in C^\infty(\Omega)$ one has

$$\| u \|_{q,\Omega_r} \leq \varepsilon \| u \|_{m,\Omega_r} + c_m \varepsilon^{-q/(m-q)} \| u \|_{r,\Omega_r}.$$ 

A proof of this lemma can be given by using Fourier transforms after extending the functions suitably to $\mathbb{R}^n$. Another proof can be found in [15] (Appendix).
REMARK. — Let $p$ be any integer $\ge 0$. By applying the above inequality to $D^2 u$ and by summing up the inequality thus obtained with respect to $z$ such that $|z| = mp$, we obtain from Lemma 3.1,

$$
\| u \|_{pm+q, \Omega} \le \varepsilon \| u \|_{(p+1)m, \Omega} + c_m z^{-q/(m-p)} \| u \|_{pm, \Omega},
$$

with the same constant $c_m$ as in the above lemma.

4. Proof of theorem 1. — In this section we shall prove Theorem 1. The proof is preceded by several lemmas which permit one to estimate suitably $\| u \|_{km}$ in terms of zero-norms of $u, Au, \ldots, A^k u$.

We suppose throughout this section that $A$ has analytic coefficients. In this section, $c(c_1, c_2, \ldots, \text{etc.})$ will denote a positive constant, always independent of $k$, which may vary from place to place.

The first lemma gives an estimate for the commutator of the operator $D^2$ and the operator of multiplication by an analytic function.

**Lemma 4.1.** — Let $a$ be an analytic function in $\overline{\Omega}$. We define the commutator $[a, D^2]$ by $[a, D^2] u = a \cdot D^2 u - D^2 (au)$, then we have for every integer $k > 0$.

$$(4.1) \quad \sum_{|\beta| = k} \frac{k!}{\alpha!} \| [a, D^2] u \|_{\alpha, \Omega} \le k! c^k \sum_{p=0}^{k-1} (p!)^{-1} c^{-p} \| u \|_{\alpha, \Omega}$$

with $c > 0$ independent of $k$.

**Proof.** — Since $a$ is analytic in $\overline{\Omega}$, we have

$$(4.2) \quad \sup_{\overline{\Omega}} |D^2 a| \le \alpha! c^{|\alpha| + 1}.$$

The Leibniz formula gives

$$D^2 (au) = \sum_{\beta \le \alpha} \frac{\alpha!}{\beta! (\alpha - \beta)!} (D^2 a)(D^{\alpha - \beta} u)$$

where $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ and $\beta \le \alpha$ means $\beta_i \le \alpha_i$ for each $i (i = 1, 2, \ldots, n)$.

From (4.2) and the definition of $[a, D^2]$, it follows immediately

$$\sum_{|\beta| = k} \frac{k!}{\alpha!} \| [a, D^2] u \|_{\alpha, \Omega} \le \sum_{|\beta| = k} \sum_{|\gamma| \le k} \frac{k!}{\gamma!} c^{k-|\gamma|} \| D^\gamma u \|_{\alpha, \Omega}.$$

Now the number of $\alpha$'s such that $\alpha \ge \gamma$ for fixed $\gamma$ is at most of order $n^{|\gamma|}$, so that the right side is majorised by

$$\sum_{p=0}^{k} \frac{k!}{p!} (nc)^{k-p} \sum_{|\gamma| = p} \frac{p!}{\gamma!} \| D^\gamma u \|_{\alpha, \Omega};$$

this proves the lemma.
LEMMA 4.2. — Let \( r, \delta \) be as in the lemma 3.2. Let \( z \) be any positive number. Then there exist constants \( c, c_1 \) (\( c \) depending only on \( A \) and \( c_1 \) depending on \( A \) and \( z \)) such that one has for every \( k \) and \( u \in C^\alpha(\Omega)\),

\[
(4.3) \quad \| u \|_{mk^+1m, \Omega_+3} \leq c \left\{ \left\| Au \right\|_{mk, \Omega_+3} + \delta^{-m} \left\| u \right\|_{mk, \Omega_+3} + z \left\| u \right\|_{mk^+1m, \Omega_+3} + \left( (k+1)m! c_1^{k+1} \sum_{p=0}^{k} (pm)!^{-1} c_1^{-p} \| u \|_{pm, \Omega_+3} \right) \right\}
\]

PROOF. — From Lemma 3.1 and the Friedrichs' inequality (Lemma 3.2), we have

\[
(4.4) \quad \| u \|_{mk^+1m, \Omega_+3} \leq \sum_{|\alpha| = km} \frac{(km)!}{\alpha!} \| D^\alpha u \|_{m, \Omega_+3} \leq c \left\{ \left\| Au \right\|_{mk, \Omega_+3} + \delta^{-m} \left\| u \right\|_{mk, \Omega_+3} \right. \\
\left. + \sum_{|\alpha| = km} \frac{(km)!}{\alpha!} \left\| [A, D^\alpha] u \right\|_{0, \Omega_+3} \right\}.
\]

Now, writing \( A \) explicitly as \( A = \sum_{|\beta| \leq m} a_\beta D^\beta \) with analytic coefficients \( a_\beta \) and applying the Lemma 4.1 for \([A, D^\alpha] u = \sum_{|\beta| \leq m} [a, D^\beta] D^\alpha u\), we obtain

\[
(4.5) \quad \sum_{|\alpha| = km} \frac{(km)!}{\alpha!} \left\| [A, D^\alpha] u \right\|_{0, \Omega_+3} \leq \sum_{p=0}^{km-1} \sum_{q=0}^{m} \frac{(km)!}{p!} \left( k^{m-p} \left\| u \right\|_{p+q, \Omega_+3} \right) \left( k^{m-1} - (\frac{1}{\alpha})^m \right)
\]

Since we may suppose \( c_1 > 1 \) in (4.5), it follows immediately that there exists a constant \( c_2 > 0 \) independent of \( k \) such that

\[
(4.6) \quad \sum_{|\alpha| = km} \frac{(km)!}{\alpha!} \left\| [A, D^\alpha] u \right\|_{0, \Omega_+3} \leq \sum_{s=0}^{(k+1)m-1} \frac{(k+1)m!}{s!} \left( k^{m-s} \left\| u \right\|_{s, \Omega_+3} \right) \left( k^{m-1} - (\frac{1}{\alpha})^m \right)
\]

We wish now to majorize the right side of (4.6), containing terms \( \left\| u \right\|_s \), for \( s = 0, 1, \ldots, (k+1)m-1 \), by an expression which contains only \( \| u \|_{pm} \), for \( p = 0, 1, \ldots, (k+1) \).

For this purpose, we write \( s \) as \( s = pm + q \) with \( 0 \leq p \leq k \), and \( 0 \leq q < m \). Then the remark of Lemma 3.3 gives

\[
(4.7) \quad \| u \|_{pm+q, \Omega_+3} \leq \varepsilon' \| u \|_{(p+1)m, \Omega_+3} + c_m \varepsilon'^{-q/m-q} \| u \|_{pm, \Omega_+3}
\]

with \( c_m \) independent of \( \varepsilon' \) and \( \delta \).
In (4.7), we choose \( \varepsilon' \) as
\[
\varepsilon' = \varepsilon \frac{(pm + q)!}{(p + 1)!} c_{z}^{(m-q)}
\]
where \( \varepsilon (0 < \varepsilon < 1) \) is given.
Then we have
\[
\frac{\varepsilon'^{q(m-q)}}{s} \| u \|_{\Omega_{+,\delta}} \leq \varepsilon \frac{c_{z}^{(p+1)m}}{(p + 1)!} \| u \|_{(p+1)m, \Omega_{+,\delta}}
+ c_{m} \frac{m}{\varepsilon} c_{z}^{-(p+1)m} \| u \|_{pm, \Omega_{+,\delta}}
\]
so that we obtain for \( s = pm + q \),
\[
(4.8) \quad \sum_{[x] = km} \frac{(km)!}{x!} \| [A, D^{x}] u \|_{0, \Omega_{+,\delta}} \leq m \varepsilon \| u \|_{(k+1)m, \Omega_{+,\delta}} + c' \varepsilon \sum_{p = 0}^{k} \frac{((k+1)m)!}{(pm)!} c_{z}^{(k+1)m-pm} \| u \|_{pm, \Omega_{+,\delta}}
\]
where we put \( c' \varepsilon = 1 + m \varepsilon + \left( \frac{m}{\varepsilon} \right) c_{m} \). We take now in (4.9) the constant \( c_{z} \) large enough to absorb the constant \( c' \varepsilon \) which is independent of \( k \). Then, from (4.4), the desired inequality follows.

**Definition (see [13]).** — Let \( \lambda \) be a positive number. For each integer \( k \geq 0 \), we define
\[
\sigma^{k}(u, \lambda, R) = \left( (km)! \right)^{-1} \lambda^{-k} (R - r)^{km} \sup_{R/2 \leq r < R} \| u \|_{km, \Omega_{r}}.
\]

**Lemma 4.3.** — Let \( R < 1 \). There exists a constant \( \lambda \) depending only on \( A \) and \( R \) such that for every \( k \) and \( u \in C^{0}(\Omega) \) we have
\[
(4.10) \quad \sigma^{k+1}(u, \lambda, R) \leq \left[ (k+1)^{m} \right] \sigma^{k}(A u, \lambda, R)
+ \sum_{p = 0}^{k} \sigma^{p}(u, \lambda, R).
\]

**Proof.** — Multiplied by \( \left[ (k+1)^{m} \right] \) on both sides of the inequality of Lemma 4.2 and taking the supremum for \( R/2 \leq r < R \), we obtain
\[
(4.11) \quad \sigma^{k+1}(u, \lambda, R) \leq \sup_{R/2 \leq r < R} (I_{1} + \varepsilon I_{2} + I_{3} + I_{4})
\]
where
\[
\begin{align*}
I_1 &= c \left[ \left( (k+1) m \right)! \right]^{-1} \lambda^{-\left( k+1 \right)} \left( R - r \right)^{k+1} \left( ku \right) \left[ k, \Omega_{r+\delta} \right], \\
I_2 &= c \left[ \left( (k+1) m \right)! \right]^{-1} \lambda^{-\left( k+1 \right)} \left( R - r \right)^{k+1} \left( ku \right) \left[ k+1, \Omega_{r+\delta} \right], \\
I_3 &= c \left[ \left( (k+1) m \right)! \right]^{-1} \lambda^{-\left( k+1 \right)} \left( R - r \right)^{k+1} \left( ku \right) \left[ k+1, \Omega_{r+\delta} \right], \\
I_4 &= c \lambda^{-\left( k+1 \right)} \left( R - r \right)^{k+1} \sum_{\rho=0}^{k} \frac{c_{k+1-p}^{\rho}}{(pm)!} \left( ku \right) \left[ pm, \Omega_{r+\delta} \right].
\end{align*}
\]

We choose in what follows \( \delta = \frac{R - r}{k + 1} \); then we have
\[
\left( \frac{R - r}{R - r - \delta} \right)^{km} = \left( 1 - \frac{1}{k + 1} \right)^{-km} < c_2
\]
with \( c_2 \) independent of \( k \). It follows now from the definition of \( \sigma^k(u, \lambda, R) \),
\[
I_1 \leq \left( (km + 1) \ldots ((k + 1) m) \right)^{-1} \left( \frac{cc_2}{k} \right) \sigma^k(Au, \lambda, R).
\]
Similarly
\[
I_2 \leq (cc_2) \sigma^{k+1}(u, \lambda, R).
\]
For \( I_3 \), we have
\[
I_3 \leq \frac{c}{k} \left( \frac{R - r}{R - r - \delta} \right)^{km} \left( \frac{R - r}{\delta} \right)^{m} \frac{(km)!}{((k + 1) m)!} \sigma^k(u, \lambda, R).
\]
Since we have from the definition of \( \delta \),
\[
\left( \frac{R - r}{\delta} \right)^{m} = (k + 1)^{m}
\]
it follows from \( (4.15) \)
\[
I_3 \leq \left( \frac{cc_2}{k} \right) \sigma^k(u, \lambda, R).
\]
Finally we obtain for \( I_4 \),
\[
I_4 \leq \left( \frac{cc_1 c_2}{\lambda} \right) \sum_{p=0}^{k} \left( \frac{c_1}{\lambda} \right)^{k-p} \sigma^p(u, \lambda, R) \quad (\lambda \geq 1).
\]
It follows now for every \( k \geq 0 \),
\[
(1 - \varepsilon c) \sigma^{k+1}(u, \lambda, R) \leq \left( (km + 1) \ldots ((k + 1) m) \right)^{-1} \left( \frac{c_1}{\lambda} \right) \sigma^k(Au, \lambda, R)
\]
\[
+ \left( \frac{c_1}{\lambda} \right) \sum_{p=0}^{k} \left( \frac{c_1}{\lambda} \right)^{k-p} \sigma^p(u, \lambda, R)
\]
for sufficiently large constants $c$, $c_1 > 0$, $c$ being independent of $\varepsilon$, while $c_1$ depends on $\varepsilon$. After we have chosen $\varepsilon = 1/2c$ in (4.18) $c_1$ is a constant dependent only on $A$ and $R$ so that it is possible to find $\lambda$ independent of $k$ such that $\lambda > 2c_1$; thus we obtain the inequality (4.10).

**Lemma 4.4.** — Let $\lambda$ be the same constant as in lemma 4.3; we have then

\[(4.19) \quad \sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \left( \begin{array}{c} k+1 \\ p \end{array} \right) ((mp)!)^{-1} \sigma^p(A^p u, \lambda, R).\]

**Proof.** — The proof is by induction on $k$. For $k = 0$, the Lemma is valid (see Lemma 4.3). Suppose that the lemma is valid up to $k - 1$. Applying the induction hypothesis to the function $A^k u$, we have

\[(4.20) \quad \sigma^k(A u, \lambda, R) \leq \sum_{p=0}^{k} 2^{k-p} \left( \begin{array}{c} k \\ p \end{array} \right) ((pm)!)^{-1} \sigma^p(A^{p+1} u, \lambda, R).\]

Also, we have for $q \leq k$,

\[(4.21) \quad \sigma^q(u, \lambda, R) \leq \sum_{p=0}^{q} 2^{q-p} \left( \begin{array}{c} q \\ p \end{array} \right) ((pm)!)^{-1} \sigma^p(A^{p+1} u, \lambda, R).\]

From Lemma 4.3, we get

\[(4.22) \quad \sigma^{k+1}(u, \lambda, R) \leq [(km + 1) \ldots ((k + 1)m)]^{-1} \times \sum_{p=0}^{k} 2^{k-p} \left( \begin{array}{c} k \\ p \end{array} \right) ((pm)!)^{-1} \sigma^p(A^{p+1} u, \lambda, R) + \sum_{q=0}^{k} \sum_{p=0}^{q} 2^{q-p} \left( \begin{array}{c} q \\ p \end{array} \right) ((pm)!)^{-1} \sigma^p(A^{p+1} u, \lambda, R).\]

Now, let $c_p$ be the coefficient of $\sigma^p(A^p u, \lambda, R)$. Then for $0 \leq p \leq k$

\[
c_p = [(km + 1) \ldots ((k + 1)m)]^{-1} 2^{k-p+1} \left( \begin{array}{c} k \\ p-1 \end{array} \right) ((p-1)m)!^{-1} + \sum_{q=p}^{k} 2^{q-p} \left( \begin{array}{c} q \\ p \end{array} \right) ((mp)!^{-1}.\]

Since

\[
\sum_{q=p}^{k} 2^{q-p} \left( \begin{array}{c} q \\ p \end{array} \right) \leq 2^{k-p+1} \left( \begin{array}{c} k \\ p \end{array} \right)
\]

we get

\[
c_p \leq 2^{k-p+1} \left( \begin{array}{c} k+1 \\ p \end{array} \right) ((pm)!)^{-1}.\]
On the other hand, for \( p = k + i \), we have evidently,
\[
\sigma_{k+i} = \left[ (k + i) m \right]^{-1}.
\]
Hence, it follows
\[
\sigma^{k+i}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} \left[ ((pm)!)^{-1} \sigma^p(A^p u, \lambda, R) \right];
\]
this is the inequality which we wanted to prove; thus the induction is completed.

**Proof of Theorem 1.** — Let \( u \in C^\infty(\Omega) \) such that
\[
\| A^k u \|_{L^2(\Omega')} \leq (km)! c^{k+1}
\]
for \( \Omega' \) an open set of \( \Omega \) and for all \( k \geq 0 \) with a constant \( c \) independent of \( k \).

Since the analyticity is a local property, we may suppose that the origin of \( \mathbb{R}^n \) belongs to \( \Omega' \) and it is sufficient to prove analyticity at the origin. Take \( R < 1 \) with \( \Omega_R \subset \Omega' \), then
\[
\sigma^p(A^k u, \lambda, R) = \| A^k u \|_{L^2(\Omega_R)} \leq km! c^{k+1}.
\]
Now from Lemma 4.3, we have
\[
\sigma^{k+1}(u, \lambda, R) \leq \sum_{p=0}^{k+1} 2^{k-p+1} \binom{k+1}{p} \left[ ((pm)!)^{-1} \sigma^p(A^p u, \lambda, R) \right]
\]
\[
\leq \sum_{p=0}^{k+1} 2^{k-p+1} c^{p+1} \binom{k+1}{p} = c (c + 2)^{k+1}.
\]
From the definition of \( \sigma^{k+1}(u, \lambda, R) \) we obtain
\[
\| u \|_{k+1, m, \Omega_R} \leq ((k + 1) m)! c^{k+1}
\]
with a certain constant \( c \) independent of \( k \).

Then, Lemma 3.3 permits us to estimate \( \| u \|_p \) for \( p = 0, 1, \ldots \) by \( \| u \|_{k+1, m} \) for \( k = 0, 1, \ldots \) and we have
\[
\| u \|_p, \Omega_R \leq p! c^{p+1}
\]
for all \( p (= 0, 1, \ldots) \), where \( c \) is a constant depending only on \( A \) and \( \Omega_R \).

Now, by Sobolev's lemma [13], we see that \( u \) is analytic at the origin. Hence, the proof of Theorem 1 is completed.
5. Regularity of the kernel of $\tilde{L}^s$. — We denote by $D(\tilde{L}^s)$ the domain of $\tilde{L}^s$, that is the set of elements $f \in L^s(\Omega)$ such that $\int |\lambda^s|^2 d \| E_\lambda f \| < \infty$.

Then, under our hypothesis on $\tilde{L}$, it is easy to see that

\begin{equation}
D(\tilde{L}^s) \subseteq D(\tilde{L}^{s'}) \quad \text{if} \quad Rl s \geq Rl s',
\end{equation}

(5.1) for every complex number $s$,

\begin{equation}
\tilde{L}^s f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k) \quad \text{if} \quad f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k).
\end{equation}

(5.2)

Let $f \in \bigcap_{k=0}^{\infty} D(\tilde{L}^k)$. It follows from (5.1), (5.2) that $f \in D(\tilde{L}^{s+k})$

and $\tilde{L}^s f \in D(\tilde{L}^k)$ for every complex number $s$ and integer $k \geq 0$. We have then

\begin{equation}
\tilde{L}^s \tilde{L}^s f = \tilde{L}^s \tilde{L}^k f = \tilde{L}^{s+k} f
\end{equation}

(5.3)

(for these properties, see [16], § 228; [18], p. 222).

Proposition 5.1. — For any complex number $s$, $\tilde{L}^s$ defines a kernel $L^s(x, y)$, that is, a distribution in the product space $\Omega \times \Omega$.

Proof. — We first consider the case $Rl s < 0$. In this case, $\tilde{L}^s$ is a continuous map of $L^2(\Omega)$ into itself. For, by hypothesis on $\tilde{L} = \int \lambda dE_\lambda$, we have a positive constant $c_0$ such that $\lambda > c_0$ on the spectrum, hence $\lambda^{Rl s} \leq c_0^{Rl s}$ for $\lambda \leq 1$ and $\lambda^{Rl s} \leq 1$ for $\lambda > 1$, since $Rl s < 0$.

Thus, $\lambda^s$ is bounded on the spectrum of $\tilde{L}$. Hence $\tilde{L}^s$ is a continuous linear map of $L^s(\Omega)$ into itself. A fortiori, $\tilde{L}^s$ is a continuous linear map of $\mathcal{D}(\Omega)$ into $\mathcal{D}'(\Omega)$. By the kernel theorem of L. Schwartz [19], $\tilde{L}^s$ defines a kernel.

For general $s$, we take a positive integer $m$ such that $Rl (s - m) < 0$. Then, as seen above, $\tilde{L}^{s-m}$ is a continuous map of $\mathcal{D}(\Omega)$ into $\mathcal{D}'(\Omega)$ while $\tilde{L}^m$, $m^{th}$ iterate of $L$ with $C^\infty$ coefficients, is evidently a continuous map of $\mathcal{D}(\Omega)$ into itself.

Now, the proposition follows from (5.3), by remarking that

\begin{equation}
\tilde{L}^s \varphi = \tilde{L}^{s-m} \tilde{L}^m \varphi \quad \text{for} \quad \varphi \in \mathcal{D}(\Omega) \quad \text{since} \quad \mathcal{D}(\Omega) \subseteq \bigcap_{k=0}^{\infty} D(\tilde{L}^k).
\end{equation}

From now on, we denote by $L^s(x, y)$ the kernel of $\tilde{L}^s$. 
Proposition 5.2. — For every complex number \( s \), the kernel \( L^s(x, y) \) is regular.

Proof. — We have to prove that \( L^s \) maps continuously \( \mathcal{D}(\Omega) \) into \( \mathcal{E}(\Omega) \) and can be extended to a continuous linear map of \( \mathcal{E}'(\Omega) \) into \( \mathcal{D}'(\Omega) \).

Suppose that for every \( s \), \( L^s \) maps continuously \( \mathcal{D}(\Omega) \) into \( \mathcal{E}(\Omega) \). Let \( \varphi, \psi \) be in \( \mathcal{D}(\Omega) \). We have then \( (L^s \varphi, \psi) = (\varphi, L^\bar{s} \psi) \), \( \bar{s} \) denoting the conjugate complex of \( s \), this implies that \( L^s \) can be identified on the dense subspace \( \mathcal{D}(\Omega) \) of \( \mathcal{E}'(\Omega) \) with the transpose of \( L^\bar{s} \), while the transpose of \( L^\bar{s} \) is a continuous map of \( \mathcal{E}'(\Omega) \) into \( \mathcal{D}'(\Omega) \) when \( L^\bar{s} \) is a continuous map of \( \mathcal{D}(\Omega) \) into \( \mathcal{E}(\Omega) \). Hence, \( L^s \) can be extended to a continuous map of \( \mathcal{E}'(\Omega) \) into \( \mathcal{D}'(\Omega) \).

It remains now to prove that \( L^s \) maps continuously \( \mathcal{D}(\Omega) \) into \( \mathcal{E}(\Omega) \).

Remark first that the image of \( \mathcal{D}(\Omega) \) by \( L^s \) is contained in \( \mathcal{E}(\Omega) \). For, if \( \varphi \in \mathcal{D}(\Omega) \), then \( \varphi \in \bigcap_{k=0}^{\infty} D(L^k) \), so that by (5.2) we have \( L^s \varphi \in \bigcap_{k=0}^{\infty} D(L^k) \).

From the regularity theorem for a linear elliptic operator with \( C^\infty \) coefficients ([7], [15]), it follows that \( L^s \varphi \) is of class \( C^\infty \).

As for the continuity of the mapping \( L^s \), it is sufficient [17] to verify that the image of every bounded set in \( \mathcal{D}(\Omega) \) by \( L^s \) is also a bounded set in \( \mathcal{E}(\Omega) \).

Let \( s \) be such that \( Rl's < 0 \). Let \( B \) be a bounded set in \( \mathcal{D}(\Omega) \). Then, by definition [17], the image \( \tilde{L}^s(B) \) of \( B \) by \( \tilde{L}^s \) is bounded in \( \mathcal{D}(\Omega) \), a fortiori, bounded in \( L^2(\Omega) \). Now \( \tilde{L}^s \) is a continuous map of \( L^2(\Omega) \) into itself, so that \( \tilde{L}^s \tilde{L}^s(B) \) is bounded in \( L^2(\Omega) \). On the other hand, \( \tilde{L}^s(B) \) is a family of \( C^\infty \) functions belonging to the domain of \( \tilde{L}^k \); hence it follows from (5.3) that \( \tilde{L}^s \tilde{L}^s(B) \) is bounded in \( L^2(\Omega) \), from this, we see, according to Lemma 3.2 and Sobolev's lemma [13], that \( \tilde{L}^s(B) \) is a family of \( C^\infty \) functions whose derivatives of orders \( mk - \left[ \frac{n}{2} \right] - 1 \) are uniformly bounded on every compact of \( \Omega \). Since \( k \) is arbitrary, this proves that \( \tilde{L}^s(B) \) is bounded in \( \mathcal{E}(\Omega) \).

For general \( s \), as in the proof of Proposition 3.1, choose \( m \) so large that \( Rl(s - m) < 0 \) and remark that \( \tilde{L}^s \varphi = \tilde{L}^{s-m} \tilde{L}^m \varphi \) for \( \varphi \in \mathcal{D}(\Omega) \), then \( \tilde{L}^m \) and \( \tilde{L}^{s-m} \) map respectively \( \mathcal{D}(\Omega) \) into \( \mathcal{D}(\Omega) \) and \( \mathcal{E}(\Omega) \) continuously. This completes the proof.

6. Estimates for the Green's function of the associated parabolic operator. — Consider the family of operators \( G_t = \int e^{-lt} dE \), for \( t > 0 \).
$G_t$ is a bounded and Hermitian operator in $L^2(\Omega)$. Associated with these operators we have a $C^\omega$ function in $\mathbb{R} \times \Omega \times \Omega$,

$$G(t, x, y) = \int e^{-\lambda t} d\nu(\lambda, x, y),$$

where $d\nu(\lambda, x, y)$ denotes the spectral function of $\tilde{L}[8]$.

We have then

$$\left( \frac{\partial}{\partial t} + L_x \right) G(t, x, y) = 0 \quad \text{and} \quad \left( \frac{\partial}{\partial t} + L_y \right) \bar{G}(t, x, y) = 0$$

for $t > 0$.

The next lemma shows that the function $G(t, x, y)$ and its derivatives fall off exponentially as $t \to \infty$.

**Lemma 6.1.** — Let $H$ be a compact in $\Omega \times \Omega$. Under our assumption that $\tilde{L}$ is strictly positive operator ($\lambda > c_0 > 0$ on the spectrum), we have

$$\left| \left( \frac{\partial}{\partial t} \right)^\alpha D_x^\beta D_y^\gamma G(t, x, y) \right| \leq ce^{-c_0 t/2}$$

for $t > 1$ and uniformly for $(x, y) \in H$, where $c$ depends on $p, \alpha, \beta$ and $H$.

**Proof.** — Denote by $\tilde{L}$ the elliptic operator with conjugate complex coefficients of $L$.

Consider the operator:

$$L_x + \tilde{L}_y = L \left( x, \frac{\partial}{\partial x} \right) + \tilde{L} \left( y, \frac{\partial}{\partial y} \right)$$

which is evidently elliptic with $C^\omega$ coefficients in the product space $\Omega \times \Omega$.

Now, by Lemma 3.2 and Sobolev's lemma [13] applied to $(L_x + \tilde{L}_y)$, it is easy to see that the desired estimate is a simple consequence of the following: let $U$ be a relatively compact open subset in $\Omega$ such that $H \subset U \times U$. Then for every positive integers $k^1, k^2$, we have

$$\left| \left( L_x + \tilde{L}_y \right)^k \left( \frac{\partial}{\partial t} \right)^k G(t, x, y) \right| \leq ce^{-c_0 t/2}$$

for $t > 1$ and for $(x, y) \in U \times U$. Since

$$L_x G(t, x, y) = \tilde{L}_y G(t, x, y) = -\frac{\partial}{\partial t} G(t, x, y) \quad \text{for} \quad t > 0,$$

it is sufficient to estimate $\left( \frac{\partial}{\partial t} \right)^k G(t, x, y)$ for every positive integer $k$. 


Let \( m \) be a sufficiently large positive integer such that \( \tilde{L}^{-m} \) has a kernel \( K(x, y) \) of the Carleman type ([4], [5], [8]). For \( x \in \Omega \), let \( K_x \in L^2 \) denote the function \( K(x, \cdot) \).

Now
\[
\left| \left( \frac{\partial}{\partial t} \right)^k G(t, x, y) \right| = \left| \int e^{-\lambda t} \, \mathcal{M} \, (\lambda, x, y) \right| \\
= \left| \int e^{-\lambda t} (-\lambda)^k \, \mathcal{M} \, (\lambda, x, y) \right| \\
= \left| \int e^{-\lambda t} (-\lambda)^k \mathcal{M} \, \mathcal{N} \, (E_t, K_x, K_y) \right| \\
\leq e^{-c_0 t/2} \int e^{-\gamma^2 t/2} \, \mathcal{M} \, \mathcal{N} \, (d(E_t K_x, K_y)) 
\]
since \( \lambda > c_0 \) and \( t \geq 1 \). Now the variation of \( (E_t K_x, K_y) \) in \( \mathcal{R} \) is majorised by \( \| K_x \|_{1L^2} \| K_y \|_{1L^2} ([16], \S 126) \) and \( \| K_x \|_{1L^2} \| K_y \|_{1L^2} \leq c(U) \) for \( (x, y) \in U \times U \), where \( c(U) \) is a constant depending only on \( U \) and \( \tilde{L} \).

It follows that
\[
\left| \left( \frac{\partial}{\partial t} \right)^k G(t, x, y) \right| \leq c \, e^{-c_0 t/2} 
\]
for \( t > 1 \) and \( (x, y) \in H \) with a constant \( c \) depending on \( k, H \) and \( \tilde{L} \). Thus Lemma 5.1 is proved.

We next consider the behaviour of \( G(t, x, y) \) and its derivatives as \( t \to 0 \). The required information is given by the results of G. Bergendal [1] and S. D. Eidelman [6].

Let \( K \) be a relatively compact open subset of \( \Omega \). Consider now the parabolic operator \( \left( \frac{\partial}{\partial t} + L \right) \) on \( \mathcal{R} \times K \) associated with \( L \). According to S. D. Eidelman, we have a fundamental solution \( \mathcal{E}(t, x, y) \) of \( \left( \frac{\partial}{\partial t} + L_x \right) \).

It is of class \( C^\infty \) in \( (t, x, y) \) when \( t > 0 \) and satisfies near \( t = 0 \) the following estimate.

**Lemma 6.2** (S. D. Eidelman). — For \( 0 < t < 1 \) and \( (x, y) \in K \times K \), we have
\[
\left| \left( \frac{\partial}{\partial t} \right)^p \mathcal{D}^2 \mathcal{D}^3 \mathcal{E}(t, x, y) \right| \leq c t^{-p(m+|x|+|y|)(|t|+|x-y|)^{\frac{1}{2}}+\beta},
\]
where \( \mu = 1/(m-1) \) and \( c \) depends only on \( L, K \), while \( c \) depends also on \( p, \alpha, \beta \).

As for the behaviour of \( G(t, x, y) \) we have

**Lemma 6.3** (G. Bergendal). — Let \( H \) be a compact subset of \( \Omega \times \Omega \) such that \( H \subset K \times K \). Let \( E(t, x, y) \) be the same as in lemma 6.2. Then there exist positive constants \( c, c_1 \) such that
\[
\left| \left( \frac{\partial}{\partial t} \right)^p \mathcal{D}^2 \mathcal{D}^3 \left[ G(t, x, y) - E(t, x, y) \right] \right| \leq c \, e^{-c_1 t^2}
\]
for \( o < t < 1 \) and for \((x, y) \in H\), where \( c_i \) depends only on \( L \) and \( H \), while \( c \) depends also on \( p, a, \beta \).

For \( p + |x| + |\beta| = 0 \), this is proved in [1]. The general case can be proved in a similar fashion (see [2], § 2.3).

7. A representation for the kernel \( L^s(x, y) \) in terms of the Green's function \( G(t, x, y) \).

**Proposition 7.1.** — Let \( s \) be a complex number such that \( Re s < -n/m \). Then we have

\[
L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} G(t, x, y) \, dt.
\]

The integral on the right converges uniformly on every compact subset of \( \Omega \times \Omega \) and represents a continuous function of \((x, y) \in \Omega \times \Omega\), where we denote by \( \Gamma(-s) \) the Gamma function.

**Proof.** — From Lemma 6.1 we have for \( t \geq 1 \) and for \((x, y) \in H\),

\[
|G(t, x, y)| \leq c e^{-ct^s}
\]

while for \( o < t < 1 \) and for \((x, y) \in H\), it follows from Lemma 6.2 and Lemma 6.3,

\[
|G(t, x, y)| \leq |E(t, x, y)| + |(E - G)(t, x, y)| \leq c t^{-n/m} + c e^{-ct^{-s}}
\]

with positive constants \( c, c_i \) depending on \( H \).

From these estimates, it is easy to see that the integral converges uniformly for \((x, y) \in H\) when \( Re s < -n/m \) and represents a continuous function of \((x, y)\) since \( G(t, x, y) \) is of class \( C^\infty \) for \( t > 0 \).

We shall prove now the equality stated in proposition 7.1. For \( \varphi, \psi \in \mathcal{O}(\Omega) \), consider

\[
P = \frac{1}{\Gamma(-s)} \left( \int_0^\infty t^{-s-1} G(t, x, y) \, dt, \varphi(x) \overline{\psi(y)} \right),
\]

where \( \langle \quad, \quad \rangle \) denote the scalar product between \( \mathcal{O}'(\Omega \times \Omega) \) and \( \mathcal{O}(\Omega \times \Omega) \).

By what has been seen,

\[
P = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} \, dt \int_{\Omega \times \Omega} G(t, x, y) \varphi(x) \overline{\psi(y)} \, dx \, dy
\]

\[
= \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} \, dt \int_{\mathbb{R}^n} e^{-ct} \, d(E_2 \varphi, \psi),
\]
where the integration $\int e^{-\lambda t} d(E_\lambda \varphi, \psi)$ is taken in the sense of the Radon-Stieltjes integral with respect to the complex-valued function of bounded variation $(E_\lambda \varphi, \psi)$ in $-\infty < \lambda < \infty$.

Let

$$(E_\lambda \varphi, \psi) = [p_1(\lambda) - p_2(\lambda)] + i[p_3(\lambda) - p_4(\lambda)]$$

be the canonical resolution of $(E_\lambda \varphi, \psi)$ with the real valued monotone increasing functions of bounded variation $p_k(\lambda), k = 1, 2, 3, 4$ ([20], p. 203).

Then we have

$$\int_{c_0}^{\infty} e^{-\lambda t} d(E_\lambda \varphi, \psi) = \sum_{k=1}^{4} \int_{c_0}^{\infty} e^{-\lambda t} d\varphi_k(\lambda)$$

where $\varepsilon_1 = - \varepsilon_2 = - i \varepsilon_3 = i \varepsilon_4 = 1$.

Consider now

$$\int_{c}^{\infty} t^{-s-1} dt \int_{c_0}^{\infty} e^{-\lambda t} d\varphi_k(\lambda).$$

Since $t^{-s-1} e^{-\lambda t}$ is a continuous function of $(t, \lambda)$ in the integration domain: $0 < t < \infty, c_0 < \lambda < \infty$ and the obvious estimate $|t^{-s-1} e^{-\lambda t}| \leq t^{-Rl_s-1} e^{-ct}$ implies that it is integrable there with respect to the product measure $dt d\varphi_k(\lambda)$ when $Rl_s < 0$.

By *Fubini's theorem*, we have,

$$\int_{c}^{\infty} t^{-s-1} dt \int_{c_0}^{\infty} e^{-\lambda t} d\varphi_k(\lambda) = \int_{c_0}^{\infty} d\varphi_k \int_{c}^{\infty} t^{-s-1} e^{-\lambda t} dt.$$

Noting that $\int_{0}^{\infty} t^{-s-1} e^{-\lambda t} dt = \Gamma(-s)\lambda^s$ and summing up the above integral with respect to $k$, we have

$$P = \sum_{k=1}^{4} \int \lambda^s d\varphi_k(\lambda)$$

which is equal to

$$\int \lambda^s d(E_\lambda \varphi, \psi) = (L^s \varphi, \psi).$$

This completes the proof.

**8. Proof of theorem 2.** — As in paragraph 5, we see that it is sufficient to prove Theorem 2 for $Rl_s < - \frac{n}{m}$. Since we have already proved that $L^s(x, y)$ is regular, it is sufficient to prove that $L^s(x, y)$ is of class $C^\infty$ outside the diagonal [17].
For $\Re s < -\frac{n}{m}$, we have by Proposition 7.1,

\[(8.1) \quad L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{s-1} G(t, x, y) \, dt.\]

If $(x, y)$ belongs to a compact set $H$ in the complement of the diagonal we see from Lemmas 6.1, 6.2 and 6.3 that

\[(8.2) \quad \left| \left( \frac{\partial}{\partial t} \right)^k D_x^k D_y^l G(t, x, y) \right| \leq c e^{-c_1(t+|t^s|)} \quad (0 < t < \infty)
\]

with positive constants $c, c_1$, where $c_1$ is independent of $k, x, \beta$.

It now follows from (8.1) and (8.2) that $L^s(x, y)$ is of class $C^\infty$ outside the diagonal, since we may differentiate under the integral sign any number of times.

9. **Proof of theorem 3.** — In this section $c, c_i (i = 1, 2, \ldots)$ will denote positive constants independent of $k$. We suppose that $L$ has analytic coefficients.

To prove Theorem 3, it is sufficient to prove the following two statements:

(i) $L^s(x, y)$ is an analytic function in the complement of the diagonal in $\Omega \times \Omega$.

(ii) For each $\varphi \in D(\Omega)$, $\check{L}^s \varphi$ is an analytic function in every open set where $\varphi$ is analytic.

**Proof of (i).** — $(L_x + L_y)^k$ is a linear elliptic operator of order $m$ with analytic coefficients in $\Omega \times \Omega$. Applying Theorem 1, we see that to prove (i) it is sufficient to prove the following: for each compact set $H$ in the complement of the diagonal, there exists a constant $c$ independent of $k$ such that

\[(9.1) \quad \sup_{(x, y) \in H} |(L_x + L_y)^k L^s(x, y)| \leq (mk)! c^{k+1}.
\]

It is sufficient to consider the case $\Re s < -\frac{n}{m}$.

As in paragraph 8, we start from the integral representation of $L^s(x, y)$:

\[(9.2) \quad L^s(x, y) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{s-1} G(t, x, y) \, dt.
\]

If $(x, y) \in H$, we have the estimate (8.2) which permits us to differentiate under the integral sign, so that we have

\[(9.3) \quad (L_x + L_y)^k L^s(x, y) = \frac{(-1)^k \alpha^k}{\Gamma(-s)} \int_0^\infty t^{s-1} \left( \frac{\partial}{\partial t} \right)^k G(t, x, y) \, dt.
\]
For we have
\[ \left( \frac{\partial}{\partial t} + L_x \right) G(t, x, y) = \left( \frac{\partial}{\partial t} + \bar{L}_y \right) G(t, x, y) = 0 \]
for \( t > 0 \). Let us first suppose that \( s \) is not a negative integer. By integration by parts in (9.3) [which is permitted by (8.2)] we obtain
\[ (9.4) \quad \left( L_x + \bar{L}_y \right)^k L^s(x, y) = \frac{2^k}{\Gamma(-s)} (-s-1)(-s-2)\ldots(-s-k) \int_0^\infty t^{-s-k-1} G(t, x, y) dt. \]
Now as a special case of (8.2) we have
\[ |G(t, x, y)| \leq c e^{-c_1(t+|x-y|)} \]
uniformly for \((x, y) \in H\) with positive constants \( c, c_1 \) depending on \( H \).

Remembering that \( \mu = (m-1)^{-1} \), it follows from a simple calculation that
\[ (9.5) \quad \sup_{(x,y) \in H} \left| \int_0^\infty t^{-s-k-1} G(t, x, y) dt \right| \leq ((m-1)k)! c^{k+1}, \]
c being independent of \( k \), which gives evidently, from (9.4),
\[ \sup_{(x,y) \in H} \left| \left( L_x + \bar{L}_y \right)^k L^s(x, y) \right| \leq \frac{2^k}{|\Gamma(-s)|} |(-s-1)(-s-2)\ldots(-s-k)| ((m-1)k)! c^{k+1} \leq (mk)! c^k. \]
If \( s \) is a negative integer, we see that the integral
\[ \int_0^\infty t^{-s-1} \left( \frac{\partial}{\partial t} \right)^k G(t, x, y) dt \quad (x, y) \in H \]
vanishes for all large \( k \) and (9.1) is trivially valid. So (i) is proved.

Proof of (ii). — Let \( \varphi \in \mathcal{O}(\Omega) \). We suppose \( \varphi \) is analytic in an open subset \( \Omega_0 \) of \( \Omega. \) We shall show that \( \bar{L}^s \varphi \) is analytic in \( \Omega_0. \)

Let \( \Omega_1, \Omega_2 \) be any relatively compact open subsets of \( \Omega_0 \) such that
\[ \Omega_1 \subset \Omega_2 \subset \Omega_0. \]
Let \( \alpha \in \mathcal{O}(\Omega_0) \) and \( \alpha \equiv 1 \) on \( \Omega_2. \) One has then
\[ \bar{L}^s(\varphi) = \bar{L}^s(\alpha \varphi) + \bar{L}^s((1-\alpha) \varphi). \]

Now, \((1-\alpha) \varphi \in \mathcal{O}(\Omega) \) and its support does not intersect \( \Omega_1; \) by what has been seen in (i), \( \bar{L}^s(x, y) \) is an analytic function of \((x, y)\) outside the diagonal in \( \Omega \times \Omega, \) so that it follows immediately from the integral representation of \( L^s(x, y) \) that \( \bar{L}^s((1-\alpha) \varphi) \) is analytic in \( \Omega_1. \)
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It remains to show that $L^s(\alpha \varphi)$ is analytic in $\Omega_1$. It is sufficient to consider the case $\Re s \leq -\frac{n}{m}$. Then we have for each integer $k \geq 0$

\begin{equation}
L^k L^s(\alpha \varphi) (x) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} dt \int_\Omega G(t, x, y) (\alpha L^k \varphi)_y dy \\
+ \frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} L_p^x \int_0^\infty t^{-s-1} dt \\
\times \int_\Omega G(t, x, y) ([L, \alpha] L^{k-p-1} \varphi)_y dy,
\end{equation}

where $[L, \alpha]$ is the commutator of $L$ and $\alpha$.

Consider the second term in the above expression, which we write as

\begin{equation}
\frac{1}{\Gamma(-s)} \sum_{p=0}^{k-1} F_p (x),
\end{equation}

where

\begin{equation}
F_p (x) = L_p^x \int_0^\infty t^{-s-1} dt \int_\Omega G(t, x, y) ([L, \alpha] L^{k-p-1} \varphi)_y dy.
\end{equation}

Now $[L, \alpha]$ is a differential operator of order $(m - 1)$ whose coefficients have their supports in $(\Omega - \Omega_2)$, so that if we consider $x$ in $\Omega_1$ we may perform the differentiation $L_p^x$ under the integral sign as in paragraph 8 and we obtain,

\begin{equation}
F_p (x) = (-s-1)(-s-2)\ldots(-s-p) \int_0^\infty t^{-s-p-1} dt \\
\times \int_\Omega G(t, x, y) ([L, \alpha] L^{k-p-1} \varphi)_y dy, \quad \text{for } s \text{ non-integral}
\end{equation}

$= 0$ for all large $p$ if $s$ is a negative integer.

Since the coefficients of $[L, \alpha]$ have their supports in $(\Omega_0 - \Omega_2)$ and $\varphi$ is analytic in $\Omega_2$ by hypothesis, we have

\begin{equation}
\sup_{x \in \Omega_0} |[L, \alpha] L^{k-p-1} \varphi| \leq (k - p)! \cdot c^{k-p+1}
\end{equation}

with $c$ independent of $k$ and $p$. Further we have (see § 8)

\begin{equation}
\sup_{[x, y] \in \Omega \times (\Omega_0 - \Omega_2)} |G(t, x, y)| \leq c e^{-c_1(t + t^{-p})},
\end{equation}

we obtain from (9.8), (9.9) and (9.10)

\begin{equation}
\sup_{x \in \Omega} |F_p (x)| \leq (pm)! (k - p)! \cdot c^{k+1}
\end{equation}

with a constant $c$ independent of $k, p$. 
Consequently, we have

\[(9.11) \sup_{x \in \Omega_0} \left| \frac{1}{\Gamma(-s)} \sum_{\rho=0}^{k-1} F_{\rho}(x) \right| \leq (km)! \cdot c^{k+1}. \]

On the other hand, since \( \varphi \) is analytic in \( \Omega_0 \), we have

\[
\sup_{x \in \Omega_1} |xL^k \varphi| \leq (km)! \cdot c^{k+1}
\]

and from the results of paragraph 6 [see (7.2), (7.3)], we have

\[
\sup_{(x,y) \in \Omega_1 \times \Omega_0} \left| G(t, x, y) \right| \leq c t^{-n/m} e^{-c t}
\]

so that it follows for \( Rl/s < -n/m \),

\[(9.12) \sup_{x \in \Omega_1} \left| \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1} dt \int_{\Omega_0} G(t, x, y) (xL^k \varphi) \, dy \right| \leq (km)! \cdot c^{k+1}. \]

From (9.6), (9.11) and (9.12) we obtain finally

\[
\sup_{x \in \Omega_1} \left| L^k \tilde{L}^s (x\varphi) \right| \leq (km)! \cdot c^{k+1},
\]

with \( c \) independent of \( k \); now from Theorem 1 we see that \( \tilde{L}^s (x\varphi) \) is analytic in \( \Omega_1 \), which was an arbitrary open subset of \( \Omega_0 \). This proves (ii) and the proof of Theorem 3 is thus completed.

**BIBLIOGRAPHY.**


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