

# BULLETIN DE LA S. M. F.

F BRICKELL

R. S. CLARK

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*Bulletin de la S. M. F.*, tome 90 (1962), p. 27-38

[http://www.numdam.org/item?id=BSMF\\_1962\\_\\_90\\_\\_27\\_0](http://www.numdam.org/item?id=BSMF_1962__90__27_0)

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## CONFORMALLY RIEMANNIAN STRUCTURES. II ;

BY

F. BRICKELL AND R. S. CLARK

(Southampton).

**Introduction.** — In a previous paper [1], we used the method of É. CARTAN and S. CHERN [2] to obtain a solution of the problem of local equivalence of conformally Riemannian structures and we showed how this solution was associated with the normal conformal connection of É. CARTAN. We now present another solution of the problem and we associate this with the conformal connection of T. Y. THOMAS [3]. We find the relation between the two solutions and this shows how the two connections are related. We use the notation of [1].

**1. A solution of the problem of local equivalence of conformally Riemannian structures.** — Suppose that  $M$  admits a conformally Riemannian structure. Then we are given a covering of  $M$  by coordinate neighbourhoods  $U_\alpha$ , each admitting a function  $X_\alpha$  with values in the linear group  $GL(n, R)$ , such that on  $U_\alpha \cap U_\beta$  the function

$$g_{\alpha\beta} = X_\alpha M_{\alpha\beta} X_\beta^{-1},$$

where  $M_{\alpha\beta} = [\partial x_\alpha^i / \partial x_\beta^j]$ , has values in the group  $G$  of non-zero scalar multiples of the orthogonal  $n \times n$  matrices. We define on  $U_\alpha$  the coframes

$$\omega_\alpha = X_\alpha dx_\alpha, \quad \nu_\alpha = |X_\alpha|^{-1/n} \omega_\alpha,$$

where  $|X_\alpha|$  denotes the absolute value of the determinant of  $X_\alpha$ .

In this paper we shall use the local Riemannian metric  $\tilde{\nu}_\alpha \nu_\alpha$  on  $U_\alpha$ . Let  $\gamma_\alpha$  be the connection form calculated relative to the coframe  $\nu_\alpha$ , so that

$$(1.1) \quad d\nu_\alpha + \gamma_\alpha \wedge \nu_\alpha = 0,$$

and let the curvature form

$$\bar{R}_\alpha = d\gamma_\alpha + \gamma_\alpha \wedge \gamma_\alpha$$

be expressed as

$$\bar{R}_\alpha = \frac{1}{2} \bar{R}^i_{jlk} \nu_\alpha^l \wedge \nu_\alpha^k.$$

Then, with the usual notation, we define the form

$$(1.2) \quad \bar{\Psi}_\alpha = \frac{1}{n-2} \left\{ \frac{\bar{R}}{2(n-1)} \delta_{ih} - \bar{R}_{ih} \right\} \nu_\alpha^i$$

and, finally, we have the conformal curvature form

$$(1.3) \quad \bar{C}_\alpha = \bar{R}_\alpha - \nu_\alpha \wedge \tilde{\Psi}_\alpha - \bar{\Psi}_\alpha \wedge \tilde{\nu}_\alpha.$$

The bundle of frames on  $M$  is defined by the cocycle  $M_{\alpha\beta}$ , whose values lie in  $GL(n, R)$ . The homomorphism  $GL(n, R) \rightarrow R$ , given by

$$\iota \rightarrow \log |\iota|$$

leads to a bundle  $\bar{M}(M, R)$ , with additive group  $R$ , associated with the bundle of frames. It is defined by the cocycle  $\log |M_{\alpha\beta}|$ . Denote by  $\bar{m}$  the projection  $\bar{M} \rightarrow M$  and by  $\bar{m}^*$  the dual mapping on the forms in  $M$ . From the local product structure we have real-valued functions  $f_\alpha$  on  $\bar{U}_\alpha = \bar{m}^{-1} U_\alpha$ , such that on  $\bar{U}_\alpha \cap \bar{U}_\beta$

$$f_\alpha = f_\beta + \bar{m}^*(\log |M_{\alpha\beta}|).$$

Following T. Y. THOMAS [3], we shall work on  $\bar{M}$ , rather than on the original manifold  $M$ .

We first define the local 1-form  $\bar{\omega}_\alpha$  on  $\bar{U}_\alpha$  with values in  $R^n$

$$\bar{\omega}_\alpha = \exp(-f_\alpha/n) (\bar{m}^* \nu_\alpha).$$

A short calculation then shows that on  $\bar{U}_\alpha \cap \bar{U}_\beta$

$$\bar{\omega}_\alpha = \bar{A}_{\alpha\beta} \bar{\omega}_\beta,$$

where  $A_{\alpha\beta} = |g_{\alpha\beta}|^{-1/n} g_{\alpha\beta}$  has values in the group  $Z$  of orthogonal  $n \times n$  matrices and  $\bar{A}_{\alpha\beta} = \bar{m}^* A_{\alpha\beta}$ .

We use the cocycle  $\bar{A}_{\alpha\beta}$  on  $\bar{M}$  to define the bundle

$$\bar{B} = \bar{B}(\bar{M}, Z).$$

Denote by  $\bar{b}$  the projection  $\bar{B} \rightarrow \bar{M}$  and let  $\varepsilon_\alpha$  be the local functions, with values in  $Z$ , on  $\bar{V}_\alpha = \bar{b}^{-1} \bar{U}_\alpha$ . Then we have a global 1-form  $\bar{\theta}$  on  $\bar{B}$ , with values in  $R^n$ , defined on  $\bar{V}_\alpha$  by

$$\bar{\theta} = \varepsilon_\alpha^{-1} (\bar{b}^* \bar{\omega}_\alpha).$$

Our next step is to calculate  $d\bar{\theta}$ . The local form  $\bar{m}^*\gamma_\alpha$  on  $\bar{U}_\alpha$  has values in the Lie algebra  $\mathcal{L}Z$ . Consequently it defines on  $\bar{V}_\alpha$  a local connection for  $\bar{\mathcal{B}}$  with form

$$\bar{\Pi}_\alpha = z_\alpha^{-1} dz_\alpha + z_\alpha^{-1} (\bar{b}^* \bar{m}^* \gamma_\alpha) z_\alpha.$$

Using (1.1), it can then be shown that

$$(1.4) \quad d\bar{\theta} = -\bar{\mu}_\alpha \wedge \bar{\theta} - \bar{\Pi}_\alpha \wedge \bar{\theta},$$

where the real-valued 1-form  $\bar{\mu}_\alpha$  on  $\bar{V}_\alpha$  is given by

$$\bar{\mu}_\alpha = \frac{1}{n} \bar{b}^*(df_\alpha).$$

We look for the relation between these forms on overlapping neighbourhoods  $\bar{V}_\alpha \cap \bar{V}_\beta$ . Since  $\bar{\theta}$  is defined globally, it follows that

$$[\bar{\Pi}_\alpha - \bar{\Pi}_\beta + (\bar{\mu}_\alpha - \bar{\mu}_\beta) I] \wedge \bar{\theta} = 0$$

and consequently, since the components  $\bar{\theta}^i$  of  $\bar{\theta}$  are independent, the components of the 1-form

$$(1.5) \quad \bar{\Pi}_\alpha - \bar{\Pi}_\beta + (\bar{\mu}_\alpha - \bar{\mu}_\beta) I$$

must be linear in  $\bar{\theta}^i$ . Further, this is true of  $\bar{\Pi}_\alpha - \bar{\Pi}_\beta$  and  $\bar{\mu}_\alpha - \bar{\mu}_\beta$  separately, since the values of  $\bar{\Pi}_\alpha - \bar{\Pi}_\beta$  are skew-symmetric matrices. We may therefore suppose that

$$\begin{aligned} \bar{\Pi}_\alpha - \bar{\Pi}_\beta &= \bar{\eta}_{jh}^i \bar{\theta}^h, & \text{where } \bar{\eta}_{jh}^i + \bar{\eta}_{ih}^j &= 0, \\ \bar{\mu}_\alpha - \bar{\mu}_\beta &= \bar{\lambda}^h \bar{\theta}^h \end{aligned}$$

and we can then write

$$\bar{\eta}_{jh}^i = \frac{1}{2} (\bar{\eta}_{jh}^i - \bar{\eta}_{ih}^j - \bar{\eta}_{hj}^i - \bar{\eta}_{ij}^h + \bar{\eta}_{hi}^j + \bar{\eta}_{ji}^i).$$

But it follows from (1.5) that

$$\bar{\eta}_{jh}^i - \bar{\eta}_{jh}^i + \bar{\lambda}^h \delta_j^i - \bar{\lambda}^i \delta_h^j = 0$$

and consequently

$$\bar{\eta}_{jh}^i = \delta_h^i \bar{\lambda}^j - \bar{\lambda}^i \delta_j^h.$$

Denoting by  $\bar{\lambda}_{\alpha\beta}$  the column vector  $\bar{\lambda}^i$ , we have the required relations on  $\bar{V}_\alpha \cap \bar{V}_\beta$

$$(1.6) \quad \begin{cases} \bar{\Pi}_\alpha - \bar{\Pi}_\beta = \bar{\theta} \bar{\lambda}_{\alpha\beta} - \bar{\lambda}_{\alpha\beta} \bar{\theta}, \\ \bar{\mu}_\alpha - \bar{\mu}_\beta = \bar{\theta} \bar{\lambda}_{\alpha\beta}. \end{cases}$$

A set of linearly independent components of  $\bar{\theta}$ ,  $\bar{\Pi}_\alpha$  and  $\bar{\mu}_\alpha$  forms a moving coframe on  $\bar{V}_\alpha$ . This covering of  $\bar{B}$  by moving coframes determines a structure on  $\bar{B}$  in the sense of S. CHERN [2]. From (1.6), its group  $D$  is isomorphic with the additive group  $R^n$ .

Suppose that  $M'$  is another manifold carrying a conformally Riemannian structure. It also will give rise to a manifold  $\bar{B}'$  carrying a  $D$ -structure. In the next paragraph we shall prove the

**THEOREM.** — *The conformally Riemannian structures on  $M, M'$  are locally equivalent at points  $m, m'$  if, and only if, the  $D$ -structures on  $\bar{B}, \bar{B}'$  are locally equivalent at two points which project to  $m, m'$ .*

In [1], we obtained a solution of the local equivalence problem for conformally Riemannian structures by applying the Chern process to these structures. It follows from the above theorem that another solution can be obtained by applying the Chern process to the  $D$ -structures on  $\bar{B}, \bar{B}'$  and this we now proceed to do.

The group  $D$  is isomorphic with the additive group  $R^n$  and it is convenient to work with the isomorphic bundle

$$\bar{B}^1 = \bar{B}^1(\bar{B}, R^n)$$

defined by the cocycle  $\bar{\lambda}_{\alpha\beta}$ . Denote by  $\bar{b}^1$  the projection  $\bar{B}^1 \rightarrow \bar{B}$  and by  $\bar{\lambda}_\alpha$  the local function, with values in  $R^n$ , on  $\bar{V}_\alpha^1 = (\bar{b}^1)^{-1} \bar{V}_\alpha$ . Then we have global 1-forms on  $\bar{B}^1$  defined on  $\bar{V}_\alpha^1$  by

$$\begin{aligned} \bar{\theta}^1 &= \bar{b}^{1*} \theta, \\ \bar{\Pi}^1 &= \bar{b}^{1*} \bar{\Pi}_\alpha - \bar{\theta}^1 \tilde{\lambda}_\alpha + \bar{\lambda}_\alpha \tilde{\theta}^1, \\ \bar{\mu}^1 &= \bar{b}^{1*} \bar{\mu}_\alpha - \tilde{\theta}^1 \bar{\lambda}_\alpha \end{aligned}$$

and we obtain a decomposition for their exterior derivatives. It can be shown that

$$d\bar{\theta}^1 = -\bar{\Pi}^1 \wedge \bar{\theta}^1 - \bar{\mu}^1 \wedge \bar{\theta}^1.$$

Following the general method, we calculate  $d\bar{\Pi}^1$ ,  $d\bar{\mu}^1$  and then put

$$\bar{\chi}_\alpha = d\bar{\lambda}_\alpha + \bar{\chi}'_\alpha + \bar{\chi}''_\alpha + \bar{\chi}'''_\alpha,$$

where  $\bar{\chi}'_\alpha, \bar{\chi}''_\alpha, \bar{\chi}'''_\alpha$  are 1-forms on  $\bar{V}_\alpha^1$ , with values in  $R^n$ , which are linear in the components of  $\bar{\theta}^1, \bar{\Pi}^1, \bar{\mu}^1$  respectively. We can then show that these forms are determined uniquely by requiring that

$$\begin{aligned} d\bar{\Pi}^1 &= \bar{\chi}_\alpha \wedge \tilde{\theta}^1 + \bar{\theta}^1 \wedge \tilde{\chi}_\alpha - \bar{\Pi}^1 \wedge \bar{\Pi}^1 + \bar{\Phi}, \\ d\bar{\mu}^1 &= -\tilde{\chi}_\alpha \wedge \bar{\theta}^1, \end{aligned}$$

where the form  $\bar{\Phi} = \frac{1}{2} \bar{\Phi}^i_{jlk} \bar{\theta}^{1k} \wedge \bar{\theta}^{1l}$  satisfies the relations

$$\bar{\Phi}^i_{jli} = 0.$$

Explicitly, we find that

$$\begin{aligned} \bar{\chi}_x^1 &= \bar{b}^{1*} (\bar{z}_x \bar{b}^* (\exp(f_x/n) (\bar{m}^* \bar{\psi}_x))) - (\bar{\lambda}_x \bar{\theta}^1) \bar{\lambda}_x + \frac{1}{2} (\bar{\lambda}_x \bar{\lambda}_x) \bar{\theta}^1, \\ \bar{\chi}_x^2 &= -\bar{\Pi}^1 \bar{\lambda}_x, \\ \bar{\chi}_x^3 &= -\bar{\mu}^1 \bar{\lambda}_x, \end{aligned}$$

and that  $\bar{\Phi} = \bar{b}^{1*} (\bar{z}_x^{-1} (\bar{b}^* \bar{m}^* \bar{C}_x) \bar{z}_x)$ . The local forms  $\bar{\psi}_x$  and  $\bar{C}_x$  arise from the Riemannian metric  $\bar{v}_x \nu_x$  on  $U_x$  and were defined in (1.2), (1.3).

From the general theory of [1], §3, the local forms  $\bar{\chi}_x$  define a global form  $\bar{\chi}^1$  on  $\bar{B}^1$ . The manifold  $\bar{B}^1$  and the forms  $\bar{\theta}^1, \bar{\Pi}^1, \bar{\mu}^1, \bar{\chi}^1$  solve the problem of local equivalence for  $D$ -structures in the sense previously explained in [1], §3. Hence they solve the problem of local equivalence for conformally Riemannian structures.

**2. The proof of the theorem of paragraph 1.** — Suppose that  $M$  and  $M'$  are manifolds with a conformally Riemannian structure. Suppose that we are given a diffeomorphism  $\varphi$  of a coordinate neighbourhood  $U_x$  of  $M$  onto a coordinate neighbourhood  $U'_x$  of  $M'$  which provides a local equivalence for these structures at  $m, m'$ . Then

$$(2.1) \quad \varphi^*(\omega'_x) = g \omega_x,$$

where  $g$  is some function on  $U_x$  with values in  $G$ . Suppose that, in terms of local coordinates,  $\varphi$  is given by

$$x'^i = x'^i(x^1, \dots, x^n).$$

We avoid indices by writing this as  $x' = x'(x)$  and we shall denote the matrix  $[dx'^i/dx^j]$  by  $\partial x'/\partial x$ . The equation (2.1) shows that on  $U_x$

$$(2.2) \quad X'_x(x'(x)) \frac{\partial x'}{\partial x} = g(x) X_x(x).$$

Using the local product structure of  $\bar{B}$  and  $\bar{B}'$ , we define a local diffeomorphism  $\bar{\varphi}$  of  $\bar{V}_x$  onto  $\bar{V}'_x$  by the mapping

$$U_x \times R \times Z \rightarrow U'_x \times R \times Z$$

given by

$$(m_x, f_x, z_x) \rightarrow \left( \varphi m_x, f_x + \log \left| \frac{\partial x'}{\partial x} \right|, \frac{g}{|g|^{1/n}} z_x \right),$$

where the functions are to be evaluated at  $m_x$ . In terms of the local product structure,

$$\bar{\theta} = \bar{s}_x^{-1} \exp(-f_x/n) |X_x|^{-1/n} Y_x dx.$$

We have a similar expression for  $\bar{\theta}'$  and consequently

$$\bar{\varphi}^* \bar{\theta}' = (\bar{s}_x^{-1} g^{-1} |g|^{1/n}) \left( \exp(-f_x/n) \left| \frac{\partial x'}{\partial x} \right|^{-1/n} \right) \frac{X'_{x'}(x'(x))}{|X'_{x'}(x'(x))|^{1/n}} \frac{\partial x'}{\partial x} dx.$$

Using (2.2) we find that on  $\bar{V}_x$

$$(2.3) \quad \bar{\varphi}^* \bar{\theta}' = \bar{\theta}.$$

From (1.4) it now follows that

$$[(\bar{\Pi}_x - \bar{\varphi}^* \bar{\Pi}'_x) + (\bar{\mu}_x - \bar{\varphi}^* \bar{\mu}'_x) I] \wedge \bar{\theta} = 0$$

and so the argument leading to (1.6) shows that

$$(2.4) \quad \begin{cases} \bar{\Pi}_x - \bar{\varphi}^* \bar{\Pi}'_x = \bar{\theta} \tilde{\zeta} - \zeta \tilde{\theta}, \\ \bar{\mu}_x - \bar{\varphi}^* \bar{\mu}'_x = \tilde{\theta} \zeta, \end{cases}$$

where  $\zeta$  is some function on  $\bar{V}_x$  with values in  $R^n$ . From (2.3), (2.4) it follows that the diffeomorphism  $\bar{\varphi}$  provides a local equivalence for the  $D$ -structures on  $\bar{B}$ ,  $\bar{B}'$  at a pair of points which project to  $m$ ,  $m'$ .

Conversely, suppose that we are given a local equivalence for the  $D$ -structures on  $\bar{B}$ ,  $\bar{B}'$  at a pair of points which project to  $m$ ,  $m'$ . We then have a local diffeomorphism  $\bar{\varphi}$  of a neighbourhood  $W$  of  $\bar{B}$  onto a neighbourhood  $W'$  of  $\bar{B}'$  which satisfies the equations (2.3), (2.4) on  $W$ . We may suppose that  $\bar{m} \bar{b} W$  is a neighbourhood  $U_x$  of  $m$  which admits coordinates  $x$  and that  $W$  admits coordinates  $(x, w)$ . Similar remarks apply to  $W'$  and so  $\bar{\varphi}$  is represented by equations

$$x' = x'(x, w), \quad w' = w'(x, w).$$

But since the components  $\bar{\theta}'_h dx^h$  of  $\bar{\theta}'$  are independent, it follows from (2.3) that  $\partial x' / \partial w = 0$  and so

$$x' = x'(x).$$

This equation defines a diffeomorphism  $\varphi$  of  $U_x$  onto  $U'_x$  which takes  $m$  to  $m'$ . The equation

$$X'_{x'}(x'(x)) \frac{\partial x'}{\partial x} = a(x) X_x(x)$$

defines a non-singular matrix  $a(x)$ . By expressing (2.3) in terms of local coordinates, we can show that the function  $a$  on  $U_x$  has values in  $G$ .

Consequently the diffeomorphism  $\varphi$  provides a local equivalence for the conformally Riemannian structures at points  $m, m'$ .

**3. The relation between the two solutions.**— In paragraph 1 we found that the manifold  $\bar{B}^1$  and the forms  $\bar{\theta}^1, \bar{\Pi}^1, \bar{\mu}^1, \bar{\chi}^1$  solved the problem of local equivalence for conformally Riemannian structures. Another solution was found in [1], § 4, to be given by a manifold  $B^1$  and forms  $\theta^1, \Pi^1, \chi^1$ . We now obtain a relationship between these solutions by setting up a diffeomorphism mapping  $\bar{B}^1$  onto  $B^1$  and then finding the relation between the two sets of forms.

We require some preliminary results and we first calculate the functions  $\bar{\lambda}_{\alpha\beta}$  explicitly. Defining the functions  $\sigma_{\alpha\beta}, \rho_{\alpha\beta}$ , with values in  $R^n$ , on  $U_\alpha \cap U_\beta$  by the equations

$$(3.1) \quad d \log |M_{\alpha\beta}| = \bar{\sigma}_{\alpha\beta} dx_\beta = \bar{\rho}_{\alpha\beta} y_\beta,$$

it follows that

$$(3.2) \quad \bar{\rho}_{\alpha\beta} = \bar{\sigma}_{\alpha\beta} |X_\beta|^{1/n} X_\beta^1.$$

Then, using the definitions of paragraph 1, we find that

$$\begin{aligned} n \bar{\theta}^1 \bar{\lambda}_{\alpha\beta} &= n (\bar{\mu}_\alpha - \bar{\mu}_\beta) = \bar{b}^* (df_\alpha - df_\beta) \\ &= \bar{b}^* \bar{m}^* (d \log |M_{\alpha\beta}|) \\ &= \bar{b}^* \bar{m}^* (\bar{\nu}_\beta \rho_{\alpha\beta}). \end{aligned}$$

Consequently, if we substitute

$$\bar{\theta}^1 = (\bar{b}^* \bar{m}^* \bar{\nu}_\beta) (\bar{b}^* \exp(-f_\beta/n)) \bar{\nu}_\beta,$$

it follows that

$$(3.3) \quad n \bar{\lambda}_{\alpha\beta} = \bar{\nu}_\beta \bar{b}^* (\exp(f_\beta/n)) \bar{m}^* \rho_{\alpha\beta}.$$

We next define functions  $\rho_\alpha$ , with values in  $R^n$ , on  $U_\alpha$  by the equations

$$(3.4) \quad d \log |X_\alpha|^{1/n} = \bar{\rho}_\alpha y_\alpha.$$

Then, from the equations (2.2) of [1] for  $q_{\alpha\beta}$ ,

$$\begin{aligned} \bar{q}_{\alpha\beta} |X_\beta|^{1/n} y_\beta &= \bar{q}_{\alpha\beta} \omega_\beta = d \log |g_{\alpha\beta}|^{1/n} \\ &= \frac{1}{n} \{ d \log |M_{\alpha\beta}| + d \log |X_\alpha| - d \log |X_\beta| \} \\ &= \frac{1}{n} \bar{\rho}_{\alpha\beta} y_\beta + \bar{\rho}_\alpha y_\alpha - \bar{\rho}_\beta y_\beta \\ &= \left\{ \frac{1}{n} \bar{\rho}_{\alpha\beta} + \bar{\rho}_\alpha A_{\alpha\beta} |M_{\alpha\beta}| - \bar{\rho}_\beta \right\} y_\beta \end{aligned}$$



and consequently on  $U_\alpha \cap U_\beta$

$$(3.5) \quad |X_\beta|^{1/n} q_{\alpha\beta} = \frac{1}{n} \rho_{\alpha\beta} + \tilde{A}_{\alpha\beta} \rho_\alpha |M_{\alpha\beta}| - \rho_\beta.$$

We are now ready to construct a diffeomorphism mapping  $\bar{B}^1$  onto  $B^1$ . Using (3.3), it follows that  $\bar{B}^1$  is the quotient of the sum  $\Sigma U_\alpha \times R \times Z \times R^n$  by the equivalence relation

$$(m_\alpha, f_\alpha, z_\alpha, \bar{\lambda}_\alpha) \sim (m_\beta, f_\beta, z_\beta, \bar{\lambda}_\beta)$$

if

$$\begin{aligned} m_\alpha &= m_\beta, & f_\alpha &= f_\beta + \log |M_{\alpha\beta}|, & z_\alpha &= A_{\alpha\beta} z_\beta, \\ \bar{\lambda}_\alpha &= \bar{\lambda}_\beta + \frac{1}{n} \tilde{\xi}_\beta \exp(f_\beta/n) \rho_{\alpha\beta}. \end{aligned}$$

We saw in paragraph 5 of [1] that  $B^1$  is the quotient of the sum  $\Sigma U_\alpha \times G \times R^n$  by the equivalence relation

$$(m_\alpha, g_\alpha, \lambda_\alpha) \sim (m_\beta, g_\beta, \lambda_\beta)$$

if

$$m_\alpha = m_\beta, \quad g_\alpha = g_\beta g_\beta, \quad \lambda_\alpha = \lambda_\beta + \tilde{g}_\beta q_{\alpha\beta}.$$

All the functions on  $U_\alpha \cap U_\beta$  involved above are to be evaluated at  $m_\alpha$ .

We set up a local diffeomorphism of  $U_\alpha \times R \times Z \times R^n$  onto  $U_\alpha \times G \times R^n$

$$(m_\alpha, f_\alpha, z_\alpha, \bar{\lambda}_\alpha) \rightarrow (m_\alpha, \exp(f_\alpha/n) |X_\alpha|^{1/n} z_\alpha, \bar{\lambda}_\alpha + \tilde{\xi}_\alpha \exp(f_\alpha/n) \rho_\alpha).$$

Using (3.5), it can be shown that these local diffeomorphisms commute with the above equivalence relations and so they define a global diffeomorphism of  $\bar{B}^1$  onto  $B^1$ . Denoting the dual mapping on the forms in  $B^1$  by  $\#$ , it follows that

$$\begin{aligned} \# \lambda_\alpha &= \bar{\lambda}_\alpha + \bar{b}^{1*} (\tilde{\xi}_\alpha \bar{b}^* (\exp(f_\alpha/n) \bar{m}^* \rho_\alpha)), \\ \# b^{1*} g_\alpha &= \bar{b}^{1*} (\tilde{z}_\alpha \bar{b}^* (\exp(f_\alpha/n) \bar{m}^* |X_\alpha|^{1/n})), \\ \# (b^{1*} b^* \zeta_\alpha) &= \bar{b}^{1*} \bar{b}^* \bar{m}^* \zeta_\alpha \quad \text{for any form } \zeta_\alpha \text{ on } U_\alpha. \end{aligned}$$

It can be shown that the connection forms  $\Omega_\alpha, \gamma_\alpha$  for the local Riemannian metrics  $\tilde{\omega}_\alpha \omega_\alpha, \tilde{\nu}_\alpha \nu_\alpha$  satisfy the equation

$$\Omega_\alpha - \gamma_\alpha = \nu_\alpha \tilde{\rho}_\alpha - \rho_\alpha \tilde{\nu}_\alpha.$$

Using the definitions of the forms  $\theta^1, \Pi^1, \chi^1$  on  $B^1$  from [1], § 4, and the definitions of the forms  $\bar{\theta}^1, \bar{\Pi}^1, \bar{\mu}^1, \bar{\chi}^1$ , on  $\bar{B}^1$  from paragraph 1 of this paper, it follows that the required relation is

$$\# \theta^1 = \bar{\theta}^1, \quad \# \Pi^1 = \bar{\Pi}^1 + \bar{\mu}^1 I, \quad \# \chi^1 = \bar{\chi}^1.$$

4. **The conformal connection of T. Y. THOMAS.** — We found that the solution of the local equivalence problem for conformally Riemannian structures given in [1] was related to the normal conformal connection of É. Cartan. We now show that the solution given in this paper is related to the conformal connection of T. Y. THOMAS [3].

Suppose that  $M$  admits a conformally Riemannian structure. In terms of the natural coframe  $dx_\alpha$ , the local Riemannian metric  $\tilde{\nu}_\alpha \nu_\alpha$  on  $U_\alpha$  can be expressed as

$$d\tilde{x}_\alpha G_\alpha dx_\alpha,$$

where  $G_\alpha = |X_\alpha|^{-2/n} \tilde{X}_\alpha X_\alpha$ . Relative to the coframe  $dx_\alpha$ , the connection form  $K_\alpha$  is obtained from the Christoffel symbols calculated from  $G_\alpha = [G_{ij}]$ . We find that

$$(4.1) \quad K_\alpha = \Gamma_\alpha^{-1} \{ \gamma_\alpha X_\alpha + dX_\alpha \} - \tilde{\nu}_\alpha \nu_\alpha I.$$

If the curvature form is

$$\frac{1}{2} F^i_{jlk} dx^j \wedge dx^k$$

and if, with the usual notation, we define the 1-form with values in  $R^n$

$$Q_\alpha = \frac{1}{n-2} \left\{ \frac{F}{2(n-1)} G_{ih} - F_{ih} \right\} dx^h_\alpha,$$

then it can be shown that

$$(4.2) \quad Q_\alpha = |X_\alpha|^{-1/n} \tilde{X}_\alpha \bar{\psi}_\alpha.$$

Consider the functions on  $U_\alpha \cap U_\beta$ , with values in  $GL(n+2, R)$ , given by

$$\begin{bmatrix} 1 & \tilde{\sigma}_{\alpha\beta} & \frac{1}{2} \tilde{\sigma}_{\alpha\beta} G_\beta^{-1} \sigma_{\alpha\beta} \\ 0 & M_{\alpha\beta} & M_{\alpha\beta} G_\beta^{-1} \sigma_{\alpha\beta} \\ 0 & 0 & |M_{\alpha\beta}|^{2/n} \end{bmatrix}.$$

Their images  $\bar{M}_{\alpha\beta}$  under  $\bar{m}^*$  are functions on  $\bar{U}_\alpha \cap \bar{U}_\beta$ . These form a cocycle on  $\bar{M}$  and define a principal bundle

$$\mathcal{P} = P(\bar{M}, GL(n+2, R)).$$

The connection introduced by THOMAS is a connection in this bundle and is defined, relative to the above cocycle, by local forms  $\Lambda_\alpha$  on  $\bar{U}_\alpha$  given by

$$\Lambda_\alpha = \begin{bmatrix} -\frac{1}{n} df_\alpha & n\bar{m}^* \tilde{Q}_\alpha & 0 \\ -\frac{1}{n} \bar{m}^*(dx_\alpha) & \bar{m}^* K_\alpha - \frac{1}{n} (df_\alpha) I & n\bar{m}^*(G_\alpha^{-1} Q_\alpha) \\ 0 & -\frac{1}{n} \bar{m}^*(d\tilde{x}_\alpha G_\alpha) & -\frac{1}{n} df_\alpha \end{bmatrix}.$$

The bundle  $\mathcal{E}$  has a subordinate structure whose group  $K$  consists of matrices

$$(4.3) \quad \begin{bmatrix} 1 & \tilde{q} & \frac{1}{2} \tilde{q} \bar{q} \\ 0 & y & y \tilde{q} \\ 0 & 0 & 1 \end{bmatrix},$$

where  $y$  is an orthogonal matrix. A reduction of the cocycle is effected by the local functions  $\bar{X}_\alpha$  on  $\bar{U}_\alpha$ , with values in  $GL(n+2, R)$ , defined by

$$\bar{X}_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp(-f_\alpha/n) \bar{m}^* (|\bar{X}_\alpha|^{-1/n} \bar{X}_\alpha) & 0 \\ 0 & 0 & \exp(-2f_\alpha/n) \end{bmatrix},$$

since it can be shown that on  $\bar{U}_\alpha \cap \bar{U}_\beta$

$$\bar{k}_{\alpha\beta} = \bar{X}_\alpha \bar{M}_{\alpha\beta} \bar{X}_\beta^{-1} = \begin{bmatrix} 1 & \tilde{q}_{\alpha\beta} & \frac{1}{2} \tilde{q}_{\alpha\beta} \bar{q}_{\alpha\beta} \\ 0 & \bar{A}_{\alpha\beta} & \bar{A}_{\alpha\beta} \bar{q}_{\alpha\beta} \\ 0 & 0 & 1 \end{bmatrix},$$

where, from (3.2),  $\bar{q}_{\alpha\beta} = \exp(f_\beta/n) \bar{m}^* \rho_{\alpha\beta}$ . Let  $\bar{\mathcal{E}} = \bar{H}(\bar{M}, \bar{K})$  be the principal bundle associated with this subordinate structure and defined by the cocycle  $\bar{k}_{\alpha\beta}$  on  $\bar{M}$ . Denote by  $\bar{h}$  the projection  $\bar{H} \rightarrow \bar{M}$  and let the local functions  $\bar{k}_\alpha$  on  $\bar{h}^{-1}\bar{U}_\alpha$  be

$$\bar{k}_\alpha = \begin{bmatrix} 1 & \tilde{q}_\alpha & \frac{1}{2} \tilde{q}_\alpha \bar{q}_\alpha \\ 0 & y_\alpha & y_\alpha \tilde{q}_\alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

Our next aim is to prove that  $\bar{H}$  and  $\bar{B}^1$  are diffeomorphic.  $\bar{H}$  is the quotient of the sum  $\Sigma \bar{U}_\alpha \times \bar{K}$  by the equivalence relation

$$(\bar{m}_\alpha, \bar{k}_\alpha) \sim (\bar{m}_\beta, \bar{k}_\beta)$$

if  $\bar{m}_\alpha = \bar{m}_\beta$  and  $\bar{k}_\alpha = \bar{k}_{\alpha\beta} \bar{k}_\beta$ . From (3.3) it follows that

$$n \bar{\lambda}_{\alpha\beta} = \tilde{z}_\beta \bar{b}^* \bar{q}_{\alpha\beta}.$$

Consequently  $\bar{B}^1$  is the quotient of the sum  $\Sigma \bar{U}_\alpha \times Z \times R^n$  by the equivalence relation

$$(\bar{m}_\alpha, z_\alpha, \bar{\lambda}_\alpha) \sim (\bar{m}_\beta, z_\beta, \bar{\lambda}_\beta)$$

if

$$\bar{m}_\alpha = \bar{m}'_\alpha, \quad z_\alpha = A_{\alpha\beta} z_\beta, \quad \bar{l}_\alpha = \bar{l}'_\alpha + \frac{1}{n} \tilde{z}'_\beta \bar{q}_{\alpha\beta}.$$

All the functions on  $\bar{U}_\alpha \cap \bar{U}_\beta$  involved above are to be evaluated at  $\bar{m}_\alpha$ .

We set up a local diffeomorphism of  $\bar{U}_\alpha \times \bar{K}$  onto  $\bar{U}_\alpha \times Z \times R^n$

$$(\bar{m}_\alpha, \bar{k}_\alpha) \rightarrow \left( \bar{m}_\alpha, \gamma_\alpha, \frac{1}{n} \bar{q}_\alpha \right),$$

where  $\gamma_\alpha$  and  $\bar{q}_\alpha$  are obtained from  $\bar{k}_\alpha$  using the decomposition (4.3). It can be shown that these diffeomorphisms commute with the above equivalence relations and so they define a diffeomorphism of  $\bar{H}$  onto  $\bar{B}^1$ . Denote by  $\star$  the dual mapping on the forms in  $\bar{B}^1$ . Our solution of the local equivalence problem for conformally Riemannian manifolds led to global forms  $\bar{\theta}^1, \bar{\Pi}^1, \bar{\mu}^1, \bar{\chi}^1$  on  $\bar{B}^1$ . These give rise to global forms on  $\bar{H}$ .

Consider now the connection on  $\mathcal{X}$  given by THOMAS. It can be defined relative to the cocycle  $\bar{k}_{\alpha\beta}$  by forms  $\bar{\Lambda}_\alpha$  on  $\bar{U}_\alpha$  given by

$$\bar{\Lambda}_\alpha = \bar{X}_\alpha \{ \Lambda_\alpha \bar{X}_\alpha^{-1} + d\bar{X}_\alpha^{-1} \}.$$

Using (4.1), (4.2), it can be shown that

$$\bar{\Lambda}_\alpha = \begin{bmatrix} -\frac{1}{n} df_\alpha & n \exp(f_\alpha/n) \bar{m}'^* \bar{\psi}_\alpha & 0 \\ -\frac{1}{n} \bar{\theta}_\alpha & \bar{m}'^* \gamma_\alpha & n \exp(f_\alpha/n) \bar{m}'^* \bar{\psi}_\alpha \\ 0 & -\frac{1}{n} \tilde{\theta}_\alpha & -\frac{1}{n} df_\alpha \end{bmatrix}.$$

The global connection form  $\Lambda$  on  $P$  is defined on the subspace  $\bar{H}$ . On  $\bar{h}^{-1}\bar{U}_\alpha$  it is given by

$$\Lambda = \bar{k}_\alpha^{-1} \{ (\bar{h}'^* \bar{\Lambda}_\alpha) \bar{k}_\alpha + d\bar{k}_\alpha \}$$

and a straight-forward calculation then shows that on  $\bar{H}$

$$\Lambda = \star \begin{bmatrix} -\bar{\mu}^1 & n \tilde{\chi}^1 & 0 \\ -\frac{1}{n} \bar{\theta}^1 & \bar{\Pi}^1 & n \bar{\chi}^1 \\ 0 & -\frac{1}{n} \tilde{\theta}^1 & \bar{\mu}^1 \end{bmatrix}.$$

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(Manuscrit reçu le 11 juillet 1961.)

F. BRICKELL and R. S. CLARK,  
Department of Mathematics,  
University of Southampton,  
Southampton (Grande-Bretagne).

