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CONFORMALLY RIEMANNIAN STRUCTURES, I;

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Introduction. — We define a conformally Riemannian structure on a differentiable manifold \( M \) of dimension \( n \) to be a differentiable subordinate structure of the tangent bundle to \( M \) whose group \( G \) consists of the non-zero scalar multiples of the orthogonal \( n \times n \) matrices. The method of equivalence of E. Cartan [1], as described by S. Chern [3], associates with a given conformal structure a certain principal fibre bundle on which a set of linear differential forms is defined globally. We obtain such a bundle and set of forms explicitly and show their relation to the normal conformal connection of E. Cartan [2].

The first paragraph contains an exposition of conformal connections in the light of C. Ehresmann's general theory of Cartan connections [4]. In the second paragraph we show how this leads to the normal conformal connection on a manifold admitting a conformally Riemannian structure. The third paragraph summarises the method of Cartan-Chern and we apply this, in the fourth paragraph, to the special case of a conformally Riemannian structure. In the fifth paragraph we show how these ideas are related.

1. Conformal Cartan connections. — We first collect together the information we require on conformal space and on Cartan connections.

Conformal space of dimension \( n \) is defined to be the homogeneous space \( K/K' \), where \( K \) is the linear group on \( n + 2 \) variables \( \{ \xi_0, \xi_1, \ldots, \xi_{n+1} \} \) leaving invariant the quadratic form

\[
\sum_{i=1}^{n} \xi_i^2 + \xi_0 \xi_{n+1}
\]

(1) The word differentiable will always mean differentiable of class \( C^k \).
and $K'$ is the subgroup of $K$ leaving invariant the point $\{1, o, \ldots, o\}$. Explicitly, $K'$ consists of matrices of the form

$$\begin{bmatrix} b & p & c \\ o & A & Aq \\ o & o & a \end{bmatrix}$$

where $A$ is an orthogonal $n \times n$ matrix and the remaining elements satisfy the relations

$$(1.1) \quad ab = 1, \quad ap + \overline{q} = 0, \quad 2ac + \overline{aq} = 0$$

where $\overline{q}$ denotes the transpose of $q$.

The linear group of isotropy $L_n'$ of the conformal space at $\{1, o, \ldots, o\}$ is isomorphic with the group $G$ of non-zero scalar multiples of the orthogonal $n \times n$ matrices. We identify $L_n'$ with $G$ in such a way that the canonical homomorphism $\varphi$ of $K'$ onto $L_n'$ is

$$\begin{bmatrix} b & p & c \\ o & A & Aq \\ o & o & a \end{bmatrix} \mapsto aA.$$

The Lie algebra $\mathfrak{L}K$ is isomorphic with the Lie algebra of the $(n+\alpha) \times (n+\alpha)$ matrices of the type

$$\begin{bmatrix} -\mu & -\overline{q} & 0 \\ \omega & \Omega & \overline{\psi} \\ 0 & -\overline{\psi} & \mu \end{bmatrix}$$

where the $n \times n$ matrix $\Omega$ is skew-symmetric. A representation of the subalgebra $\mathfrak{L}K'$ is obtained by imposing the condition $\omega = 0$. The translation operations on these Lie algebras are obtained by matrix multiplication.

G. EHRESMANN [4] has given necessary and sufficient conditions for the existence of a Cartan connection on $M$ of type $K/K'$, that is, a conformal Cartan connection. These are:

(i) that the tangent bundle of $M$ should admit a subordinate structure with group $L_n'$;

(ii) that there should exist a principal fibre bundle $\mathcal{B} = H(M, K')$ with which the homomorphism $\varphi$ associates the subordinate structure.

Since $K'$ is a subgroup of $K$, $\mathcal{B}$ defines canonically a principal bundle $\mathcal{B} = H(M, K)$. A conformal Cartan connection on $M$ is a connection on $\mathcal{B}$, in the usual sense, such that no horizontal directions on $H$ are tangent to the subspace $H'$.

We shall construct $\mathcal{B}$ from a cocycle $k_{\alpha \beta}$, whit values in $K'$, defined on an open covering $\{U_\alpha\}$ of $M$. Then $H'$ is the quotient of the sum $\sum_\alpha U_\alpha \times K'$ by the equivalence relation

$$(m_\alpha, k_{\alpha}) \sim (m_\beta, k_{\beta}) \quad \text{if} \quad m_\alpha = m_\beta, \quad k_{\alpha} = (k_{\alpha \beta} m_\alpha) k_{\beta}.$$
A Cartan connection can then be obtained from local 1-forms \( \Gamma \) with values in \( \mathcal{K} \) defined on \( U_\alpha \), provided that on \( U_\alpha \cap U_\beta \) they satisfy the relation.

\[
(1.2) \quad \Gamma_\beta = (k_{2\beta})^{-1} \{ \Gamma_\alpha k_{2\beta} + dk_{2\beta} \}
\]

and possess the further property that \( \Gamma_\alpha \overset{\sim}{=} \in \mathcal{K}' \) if and only if the tangent vector \( \vec{m} \) of \( U_\alpha \) is zero.

Denote by \( h' \) the projection \( H' \to M \) and by \( h^* \) the dual mapping on the differential forms in \( M \). From the local product representation, we have functions \( k_\alpha \) with values in \( \mathcal{K}' \) on \( (h')^{-1} U_\alpha \). The connection form \( \Gamma \) is defined locally in \( H' \) by

\[
(1.3) \quad \Gamma = (k_\alpha)^{-1} \{ (h^* \Gamma_\alpha) k_\alpha + dk_\alpha \}
\]

and this extends uniquely to \( H' \).

2. The normal conformal connection. — Suppose now that a conformally Riemannian structure is given on \( M \), so that the tangent bundle of \( M \) admits a given subordinate structure with group \( G \). We shall construct a particular Cartan connection on \( M \) called the normal conformal connection.

The first condition of Ehresmann is satisfied since the linear isotropic group \( L_n \) is isomorphic to \( G \). We have to construct a bundle \( \mathfrak{w}' = H'(M, K') \) which gives rise to the above subordinate structure, using the homomorphism \( \varphi : K' \to G \).

We are given a covering of \( M \) by open sets, \( U_\alpha \), each admitting a coordinate system \( x_\alpha = \{ x_1^\alpha, \ldots, x_n^\alpha \} \) and a function \( X_\alpha \) with values in the general linear group \( GL(n, \mathbb{R}) \), such that on \( U_\alpha \cap U_\beta \) the function

\[
g_{2\beta} = X_\alpha M_{2\beta} X_\beta^t
\]

where \( M_{2\beta} = [\partial x_2^\beta / \partial x_\beta^\alpha] \), has values in \( G \). If \( dx_2 \) is the natural coframe on \( U_\alpha \), then the coframe

\[
\omega_2 = X_\alpha dx_2
\]

is adapted to the \( G \)-structure, since on \( U_\alpha \cap U_\beta \):

\[
\omega_2 = X_\alpha dx_2 = X_\alpha M_{2\beta} dx_\beta = g_{2\beta}^\alpha \omega_\beta.
\]

From this adapted coframe, we define a local Riemannian metric \( \tilde{g}_2 \) on \( U_\alpha \).

To construct a cocycle on \( M \) which will define a bundle \( \mathfrak{w}' \), we remark that any matrix of \( G \) can be expressed uniquely as \( aA \), where \( A \) is an orthogonal \( n \times n \) matrix and the real number \( a \) is positive. If we split up the functions \( g_{2\beta} \) in this way

\[
g_{2\beta} = a_{2\beta} A_{2\beta},
\]
the following cocycle relations are satisfied
\[ a_{2\gamma} = a_{2\delta} a_{\gamma}, \quad A_{2\gamma} = A_{2\delta} A_{\gamma}. \]

We use these functions to define
\[
k'_{2\beta} = \begin{bmatrix} b_{2\beta} & P_{2\beta} & c_{2\beta} \\ o & A_{2\beta} & A_{2\beta} q_{2\beta} \\ o & o & a_{2\beta} \end{bmatrix},
\]
where \( q_{2\beta} \) is defined by the relation
\[ (2.2) \quad \tilde{q}_{2\beta}^2 = d(\log q_{2\beta}) \]

and the remaining components are determined by the relations (1.1). These new functions \( k'_{2\beta} \) on \( U_2 \cap U_3 \) have values in \( K' \) and it can be shown that they satisfy the cocycle relations, consequently they define a bundle \( \mathcal{C} = H'(M, K') \). Since the cocycle \( g_{2\beta} \) is the image of the cocycle \( k'_{2\beta} \) under the homomorphism \( \varphi \), this bundle \( \mathcal{C} \) satisfies Ehresmann's second condition. In fact, \( k'_{2\beta} \) has values in the subgroup \( K'' \) of \( K' \) defined by \( a > 0 \).

We denote by \( \mathcal{C}' = H''(M, K'') \) the principal bundle with group \( K'' \) defined by the cocycle \( k_{2\beta} \). It is a sub-bundle of \( \mathcal{C} \).

We are now ready to construct on \( U_2 \) the local 1-forms \( \Gamma_2 \) with values in \( \mathcal{C}' K \) which will define the Cartan connection. We shall take this to be
\[
\Gamma_2 = \begin{bmatrix} o & -\tilde{q}_2 \\ \omega_2 & \Omega_2 & \psi_2 \\ o & -\tilde{\omega}_2 & o \end{bmatrix},
\]
where the 1-forms \( \Omega_2 \) and \( \psi_2 \) are still to be determined. We have, of course, to verify that the choices for these remaining components are such that \( \Gamma_2 \) satisfies the relation (1.2); the further condition on \( \Gamma_2 \) is satisfied already since the forms \( \omega_2 \) are linearly independent. Cartan determines \( \Omega_2 \) and \( \psi_2 \) in terms of the local Riemannian metric \( \tilde{\omega}_2 \omega_2 \) on \( U_2 \) by imposing certain conditions on the curvature of the Cartan connection and this will be done by imposing conditions on the local curvature form
\[ d\Gamma_2 + \Gamma_2 \wedge \Gamma_2 \]
consistent with relation (1.2).

This local curvature form has values in \( \mathcal{C} K \) and so it has components
\[
\begin{bmatrix} -B_2 & -\tilde{B}_2 & o \\ T_2 & C_2 & D_2 \\ o & -\tilde{T}_2 & B_2 \end{bmatrix}
\]
where the values of the 2-form \( C_2 \) are skew-symmetric. The first condition \( T_2 = o \) is consistent with (1.2); since
\[ T_2 = d\omega_2 + \Omega_2 \wedge \omega_2, \]
it implies that $\Omega_2$ is the connection form of the local Riemannian metric (calculated relative to the coframe $\omega_2$). It now follows that on $U_2 \cap U_3$:

$$C^g_2 = g^g_{2a} C_a g^a_2.$$ 

Consequently if

$$C_x = \frac{1}{2} C^{i}_{jkh} \omega_x^k \wedge \omega_x^k,$$

the second condition $C_{jkh} = 0$ is consistent with (1.2) and, if $n \geq 3$, it can be shown to determine the form $\phi_2$ uniquely. Thus a Cartan connection has been determined from the conformal structure of $M$; it is the normal conformal connection of E. Cartan.

We shall need to calculate $\psi_2$ explicitly and we suppose that $\psi_2 = \psi_{ih} \omega^i_2$. Since

$$R_2 = R - \omega_2 \wedge \psi_2 - \bar{\omega}_2,$$

where $R_2 = d\Omega_2 + \Omega_2 \wedge \Omega_2$ is the curvature form of the local Riemannian metric then, if

$$R_2 = \frac{1}{2} R_{jkh} \omega^k_2 \wedge \omega^k_2,$$

it follows that

$$C^{j}_{ikh} = R^{i}_{jkh} - 3 \partial_{ik} \psi_{jh} - 3 \partial_{ih} \psi_{jk} + 3 \partial_{jh} \psi_{ik} - 3 \partial_{jk} \psi_{ih}.$$ 

The condition $C_{jkh} = 0$ then shows that, for $n \geq 3$,

$$\psi_2 = \frac{1}{n - 2} \left( \frac{R}{n - 2} \right) \partial_{ih} - R_{ih} \right) \omega^k_2,$$

where $R_{jkh} = R_{jhi}$ and $R = R_{jkh} \partial^{jh}$. Consequently $C_2$ is the Weyl conformal curvature form for the local Riemannian metric.

Finally, we obtain a local formula for the connection form $\Gamma$ on $H^r$. From the local product structure of $H^r$ we have functions $l_z$ with values in $K^r$ on $(h^r)^{-1} U_2$, and we put

$$(2.3) \quad l_z^r = \begin{bmatrix} b_z & p_z & c_z \\ 0 & A_z & \tilde{A}_z q_z \\ 0 & 0 & a_z \end{bmatrix},$$

where $A_z$ is orthogonal and

$$a_z > 0, \quad a_z b_z = 1, \quad a_z p_z + \tilde{q}_z = 0, \quad a_z c_z + \tilde{q}_z q_z = 0.$$ 

Since

$$(l_z^r)^{-1} = \begin{bmatrix} a_z & \tilde{q}_z A_z & c_z \\ 0 & A_z & \tilde{A}_z q_z \\ 0 & 0 & b_z \end{bmatrix},$$
the formula (1.3) applied to $H^*$ shows that

$$
\Gamma = \begin{bmatrix}
-\mu & -\tilde{\psi} & 0 \\
\omega & \Omega & \psi \\
0 & -\tilde{\omega} & \mu
\end{bmatrix},
$$

where the global forms are defined locally by

$$
\begin{align*}
\omega &= \frac{1}{\partial_\alpha} \tilde{A}_2 (h^* \alpha_2), \\
\mu &= \frac{d\alpha_2}{\partial_2} - \tilde{\omega} \tilde{q}_2, \\
\Omega &= \tilde{A}_2 \left\{ (h^* \Omega_2) A_2 + d\tilde{A}_2 \right\} - \omega \tilde{q}_2 + q_2 \tilde{q}, \\
\psi &= d\tilde{q}_2 + \Omega q_2 + \mu q_2 + \alpha_2 \tilde{A}_2 (h^* \psi_2) - (\tilde{q}_2 \tilde{q}) q_2 + \frac{1}{3} (\tilde{q}_2 q_2) \omega.
\end{align*}
$$

3. The method of equivalence of E. Cartan and S. Chern. — In this paragraph we shall suppose that $G$ is any closed subgroup of the linear group and that the tangent bundle of a manifold $M$ admits a subordinate structure with group $G$. In the nomenclature of S. Chern, $M$ admits a $G$-structure. In [3], Chern gives a procedure for constructing a sequence of fibre bundles and differential forms for a $G$-structure. We give a short account of his work.

From the definition of a subordinate structure, there exists an open covering of $M$ by coordinate neighbourhoods $U_x$ on which are defined functions $X_x$, with values in the linear group, such that on $U_x \cap U_\beta$ the functions

$$
g_{x \beta} = X_x M_{x \beta} X_\beta^{-1}
$$

have values in $G$. The coframe

$$
\omega_x = X_x dx_x
$$
on $U_x$ is adapted to the $G$-structure, since on $U_x \cap U_\beta$,

$$
\omega_x = g_{x \beta} \omega_\beta.
$$

The first fibre bundle in the sequence is the principal bundle $\pi = B(M, G)$ associated with the reduced structure and it is defined by the cocycle $g_{x \beta}$. As usual, we shall denote by $\pi$ the projection $B \to M$ and by $\pi^*$ the dual mapping on the forms in $M$. Let $g_x$ denote the local functions with values in $G$ on $V_x = \pi^{-1} U_x$ defined by the local product structure, so that on $V_x \cap V_\beta$,

$$
g_x = (\pi^* g_{x \beta}) g_\beta.
$$
Using the local 1-forms $\omega_x$ on $U_x$, we construct on $B$ a global 1-form $\theta$ with values in $R^n$. It is defined on $V_x$ by

\begin{equation}
\theta = g_x^{-1}(b^*\omega_x),
\end{equation}

and its exterior derivative is given on $V_x$ by

\begin{equation}
d\theta = g_x^{-1} b^*(d\omega_x) - g_x^{-1} dg_x \wedge \theta.
\end{equation}

We can express $g_x^{-1} b^*(d\omega_x)$ as \( \frac{1}{2} C^i_{jk} \theta^i \wedge \theta^k \) and so, if we put

$$\Pi_x = g_x^{-1} dg_x + \varepsilon_x,$$

where $\varepsilon_x = \varepsilon^i_{jk} \theta^i$ is a 1-form on $V_x$ with values in the Lie algebra $\mathcal{L}G$ whose coefficients are to be determined, the above formula for $d\theta$ becomes

\begin{equation}
(3.2) \quad d\theta + \Pi_x \wedge \theta = \frac{1}{2} (\varepsilon^i_{jk} - \varepsilon^i_{kh} + C^i_{kh}) \theta^i \wedge \theta^k.
\end{equation}

We impose as many linear relations with constant coefficients between the quantities \( \frac{1}{2} (\varepsilon^i_{jk} - \varepsilon^i_{kh} + C^i_{kh}) \) as possible. These quantities are then determined uniquely. This implies that if the coefficients of the form $\eta \wedge \theta$ satisfy the same linear relations, where $\eta$ is any 1-form $\eta^i \theta^i$ with values in $\mathcal{L}G$, then $\eta \wedge \theta = 0$. The relations may, or may not, determine the coefficients $\varepsilon^i_{jk}$. If they do and if the coefficients of $\eta \wedge \theta$ satisfy the same relations, then $\eta = o$.

Thus on $V_x$ we have the formula

\begin{equation}
(3.2) \quad d\theta + \Pi_x \wedge \theta = \tau_x
\end{equation}

and on $V_\beta$

\begin{equation}
(3.2) \quad d\theta + \Pi_\beta \wedge \theta = \tau_\beta,
\end{equation}

where the coefficients of $\tau_x$ and $\tau_\beta$ are determined by the imposed linear relations. Since on $V_x \cap V_\beta$, $\tau_x - \tau_\beta = (\Pi_x - \Pi_\beta) \wedge \theta$, the coefficients of the form $(\Pi_x - \Pi_\beta) \wedge \theta$ also satisfy these linear relations. But the form $\Pi_x - \Pi_\beta$ has values in $\mathcal{L}G$ and, since

\begin{equation}
g_{x\beta}^{-1} dg_x - g_{\beta\gamma}^{-1} dg_\gamma = g_{x\beta}^{-1} b^*(g_{x\beta}^{-1} dg_\gamma) g_\gamma,
\end{equation}

it is linear in $\theta^i$. Consequently

$$(\Pi_x - \Pi_\beta) \wedge \theta = 0$$
and we have a global 2-form $\tau$ on $B$ defined on $V_2$ by $\tau = \tau_2$. If the imposed relations determine the coefficients $\varepsilon_{jkr}$, then $\Pi_2 = \Pi_3$ and we have a global 1-form $\Pi$ on $B$ defined on $V_2$ by $\Pi = \Pi_2$.

But in the general case,

$$\Pi_2 = \Pi_3 = \lambda_{2j} \Lambda_j \quad (\nu = 1, \ldots, d_1)$$

where $\Lambda_j$ are a basis for the $d_1$-dimensional vector space of 1-forms on $B$, with values in $\mathcal{C}^G$, which satisfy the equation $\gamma_i \wedge \theta = 0$ and whose components are linear in $\theta^i$ with constant coefficients. The functions $\lambda_{2j}$ on $V_2 \cap V_3$ form a cocycle on $B$ with values in the additive group $\mathbb{R}^l$ and so they define a principal bundle

$$\mathcal{B}^1 = B^1(B, \mathbb{R}^l).$$

Denote by $b^i$ the projection $B^1 \to B$ and by $\lambda_2$ the local functions with values in $\mathbb{R}^l$ on $V_2 = (b^i)^{-1} V_2$. Since on $V_2 \cap V_3$,

$$\lambda_2 - \lambda_3 = b^i \lambda_{2j}^i,$$

we have global 1-forms $\gamma_1$, $\Pi$ on $B$ defined by

$$\gamma_1 = b^i \gamma_1^i,$$

$$\Pi = b^i \Pi_2 - \lambda_{2j} (b^i \Lambda_j).$$

We now use the same procedure to construct a decomposition for $d\gamma_1$ and $d\Pi$ and thus obtain further local forms $\gamma_2$ on $V_2$. Defining a third bundle

$$\mathcal{B}^2 = B^2(B, \mathbb{R}^l),$$

we then construct global forms $\gamma_2$, $\Pi_2$, $\gamma_2^* = \varphi \gamma_2$ on $B^2$. And so on. If the new forms are defined globally at any stage, the process terminates. The final bundle space $B^r$ then carries a structure whose group is the identity. This solves the problem of local equivalence in the sense now to be explained.

Suppose that $M'$ is a second manifold carrying a $G$-structure and denote quantities arising from $M'$, corresponding to those already defined for $M$, by an accent. The two $G$-structures on $M$ and $M'$ are locally equivalent at points $m$ and $m'$ if there exists a local diffeomorphism of some neighbourhood $U_2$ of $m$ onto a neighbourhood $U_2'$ of $m'$ such that

$$(\gamma_2')^* = \varphi \gamma_2$$

where $^*$ denotes the dual mapping defined by the diffeomorphism and $\varphi$ is some differentiable function on $U_2$ with values in $G$. Two such diffeomorphisms are said to give the same local equivalence of the structures at $m$, $m'$ if they coincide in some neighbourhood of $m$. It follows from the work of E. Cartan [1] that the local equivalences for the $G$-structures on $M, M'$ can be obtained from the local equivalences for the identity-structures on $B^r, B^r$. CARTAN gives a finite algorithm for finding the latter.
b. Application of the method of Cartan-Chern to conformal structure.

We now return to our original notation and suppose that $G$ is the group of non-zero scalar multiples of the orthogonal $n \times n$ matrices. Its Lie algebra $\mathfrak{g}G$ is isomorphic with the algebra of $n \times n$ matrices $A$ such that

$$A + \tilde{A} = \rho I,$$

where $\rho$ is any scalar.

We first construct the bundle $\mathcal{B} = B(M, G)$ and the form $\theta$ on $B$ as in the preceding paragraph. We can then find local forms $\Pi_x$ on $V_x$ in many ways so that the equation (3.2) becomes

$$d\theta + \Pi_x \wedge \theta = 0.$$

In order to make a definite choice, we put

$$\Pi_x = g_x^{-1} dg_x + g_x^{-1} (b^* \Omega_x) g_x$$

where, as in paragraph 2, $\Omega_x$ is the connection form of the local Riemannian metric $\mathcal{O}_x \omega_x$ on $U_x$. $\Pi_x$ is then the corresponding local connection form on $V_x$.

Suppose that $\eta = \eta_{jh}^l \theta^h$ is any local 1-form with values in $\mathfrak{g} G$ and such that $\eta \wedge \theta = 0$. Then

$$\eta_{jh}^l + \eta_{ih}^l = \eta^j h \delta_{ij}, \quad \eta_{jh}^l - \eta_{hj}^l = 0.$$

These equations show that

$$\eta_{jh}^l = \frac{1}{2} (\eta_{ij}^l + \eta_{hj}^l - \eta_{ij}^h - \eta_{hj}^l + \eta_{ij}^l + \eta_{ih}^j)$$

$$= \lambda^j \delta_{ij} - \lambda^h \delta_{ij} + \lambda^h \delta_{ij}$$

and so it follows that

$$\eta = 0 \lambda - \lambda \bar{\theta} + (\bar{\lambda} \theta) I.$$

Thus any such form is determined by a function $\lambda$ with values in $R^n$. In particular, $\Pi_x - \Pi_\beta$ will be determined by functions $\lambda_{x^3}$ on $V_x \cap V_\beta$,

$$(4.2) \quad \Pi_x - \Pi_\beta = 0 \lambda_{x^3} - \lambda_{x^3} \bar{\theta} + (\bar{\lambda}_{x^3} \theta) I.$$
A calculation of their exterior derivatives gives
\[
\begin{align*}
\d d\Pi &= -\Pi^1 \wedge \theta^1, \\
\d\Pi &= d\Pi_x \wedge \tilde{\theta}^1 + \theta^1 \wedge d\tilde{\Pi}_x - (d\Pi_x \wedge \tilde{\theta}^1) - \Pi^1 \wedge \theta^1 + \phi
\end{align*}
\]
where \(\phi\) involves mixed products of components from \(\Pi^1\) and \(\theta^1\) and \(\phi\) involves products of components of \(\theta^1\). Following the general method, we put
\[
\chi_x = d\Pi_x + \tilde{\theta}_x + \gamma^x_x,
\]
where \(\chi_x\) and \(\gamma^x_x\) are 1-forms on \(V_x\) with values in \(R^a\) which are linear in the components of \(\theta^1\) and \(\Pi^1\) respectively. We can show that \(\chi_x\) and \(\gamma^x_x\) are uniquely determined by requiring that
\[
d\Pi = \chi_x \wedge \tilde{\theta}^1 + \theta^1 \wedge \gamma_x - (d\chi_x \wedge \theta^1) - \Pi^1 \wedge \theta^1 + \phi,
\]
where the form \(\Phi = \frac{1}{3} \Phi_{ijk} \theta^i \wedge \theta^j \wedge \theta^k\) satisfies the relations
\[
\Phi_{ijk} = 0.
\]
Explicitly, we find that
\[
\begin{align*}
\chi_x &= b_{i}^j (\tilde{x}_x (\tilde{b} \phi_x)) = (\tilde{x}_x (\theta^j)) \tilde{\theta}_x + \frac{1}{3} (\tilde{\theta}_x \theta^j) \theta^1, \\
\gamma^x_x &= -\tilde{\Pi}_x \theta^1
\end{align*}
\]
and that \(\Phi = b_{i}^j (\tilde{x}_x (\tilde{b} \phi_x))\). The local forms \(\psi_x\) and \(C_x\), which arise from the Riemannian metric on \(U_x\), have been defined in paragraph 2.

From the general theory of paragraph 3, the local forms \(\chi_x\) define a global form \(\chi^1\) on \(B^1\) and hence \(\Phi\) is also defined globally. The forms \(\theta^1, \Pi^1, \chi^1\) contain \(n + \frac{1}{2} n (n - 1) + 1 + n\) linearly independent components and so they define an identity-structure on \(B^1\). This structure, as explained in paragraph 3, solves the problem of local equivalence.

5. The relation between the two theories. — Starting from a given conformally Riemannian structure on \(M\), we constructed, in paragraph 2, global forms \(\omega, \mu, \Omega\) and \(\psi\) on \(H^\prime\) which defined a normal conformal connection. In paragraph 4, we carried out the Chern process for the conformal structure and obtained global forms \(\theta^1, \Pi^1\) and \(\chi^1\) on \(B^1\). We shall set up a diffeomorphism mapping \(H^\prime\) onto \(B^1\) and then find the relation between these two sets of forms.

We must first calculate the functions \(\gamma_{x3}\) on \(V_x \cap V_2^3\) explicitly. From (4.2), we have
\[
\text{trace } (\Pi_x - \Pi_2) = n \gamma_{x3} \theta^1.
\]
Since the values of $\Omega_x$ are skew-symmetric matrices, it follows from (4.1), (3.3) and (2.1) that

$$\text{trace } (\Pi_x - \Pi_\beta) = \text{trace } (g_{x}^{-1} dg_x - g_\beta^{-1} dg_\beta) = \text{trace } b^* (g_{x}^{-1} dg_x) = \text{trace } b^* (d (\log a_{x}) I + \tilde{A}_x dA_x) = nb^* (d (\log a_{x})).$$

Then using (2.3) and (3.1), we find that

$$\text{trace } (\Pi_x - \Pi_\beta) = nb^* (\tilde{q}_{x} \omega_\beta) = n(b^* \tilde{q}_{x} \omega_\beta).$$

Comparing this result with (5.1), it follows that

$$(\lambda_{x} - \lambda_\beta) = \tilde{g}_{x} (b^* q_{x}).$$

We recall from paragraph 2 that the bundle $\mathfrak{g}^\sigma$ is defined by means of the cocycle $k_{x} \beta$ on $M$. Consequently $H^\sigma$ is the quotient of the sum $\sum U_x \times K^\sigma$ by the equivalence relation

$$(m_x, k_x) \sim (m_\beta, k_\beta) \quad \text{if } m_x = m_\beta, \quad k_x = k_\beta.$$ 

In paragraph 4 we defined $\mathfrak{g}$ by means of the cocycle $g_{x} \beta$ on $M$ and $\mathfrak{g}^1$ by means of the cocycle $\lambda_{x} \beta$ on $B$. Combining these definitions and using (5.2), it follows that $B^1$ is the quotient of the sum $\sum U_x \times G \times R^n$ by the equivalence relation

$$(m_x, g_x, \lambda_x) \sim (m_\beta, g_\beta, \lambda_\beta)$$

if $m_x = m_\beta, \ g_x = g_\beta \tilde{g}_\beta, \ \lambda_x = \lambda_\beta + \tilde{g}_\beta q_{x}$. The functions $k_{x} \beta, \ g_{x} \beta$ and $q_{x}$ are all to be evaluated at $m_x = m_\beta$.

We now set up a local diffeomorphism of $U_x \times K^\sigma$ onto $U_x \times G \times R^n$.

$$(m_x, k_x) \rightarrow (m_x, a_x A_x, q_x)$$

where $a_x, A_x$ and $q_x$ are obtained from the decomposition (2.3) for any element $k_x^\sigma$ of $K^\sigma$. It can be shown that these local diffeomorphisms commute with the above equivalence relations and so they define a global diffeomorphism of $H^\sigma$ onto $B^1$. Denoting the dual mapping on the forms in $B^1$ by $\star$, it follows that

$$\star \lambda_x = q_x,$$

$$\star (b^* g_x) = a_x A_x,$$

$$\star (b^* b^* \xi) = b^* \xi$$

for any form $\xi$ on $U_x$. 

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Using the definitions of the forms $\theta^1$, $\Pi^1$, $\chi^1$ on $B^i$ from paragraph 4 and the definitions of the forms $\omega$, $\mu$, $\Omega$, $\psi$ on $H^2$ from paragraph 2, it is then easy to see that

$$\star \theta^1 = \omega, \quad \star \Pi^1 = \Omega + \mu I, \quad \star \chi^1 = \psi.$$

REFERENCES.


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