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CONFORMALLY RIEMANNIAN STRUCTURES, I ;

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Introduction. — We define a conformally Riemannian structure on a differentiable [1] manifold M of dimension n to be a differentiable subordinate structure of the tangent bundle to M whose group G consists of the non-zero scalar multiples of the orthogonal $n \times n$ matrices. The method of equivalence of E. CARTAN [1], as described by S. CHERN [3], associates with a given conformal structure a certain principal fibre bundle on which a set of linear differential forms is defined globally. We obtain such a bundle and set of forms explicitly and show their relation to the normal conformal connection of E. CARTAN [2].

The first paragraph contains an exposition of conformal connections in the light of C. EHRESMANN's general theory of Cartan connections [4]. In the second paragraph we show how this leads to the normal conformal connection on a manifold admitting a conformally Riemannian structure. The third paragraph summarises the method of Cartan-Chern and we apply this, in the fourth paragraph, to the special case of a conformally Riemannian structure. In the fifth paragraph we show how these ideas are related.

1. Conformal Cartan connections. — We first collect together the information we require on conformal space and on Cartan connections.

Conformal space of dimension n is defined to be the homogeneous space K/K', where K is the linear group on $n+2$ variables $\{\xi_0, \xi_1, \ldots, \xi_{n+1}\}$ leaving invariant the quadratic form

$$\sum_{i=1\ldots n} \xi_i^2 + \xi_0 \xi_{n+1}$$

[1] The word differentiable will always mean differentiable of class C^∞.

and K' is the subgroup of K leaving invariant the point $\{1, 0, \ldots, 0\}$. Explicitly, K' consists of matrices of the form

$$\begin{bmatrix} b & p & c \\ 0 & A & Aq \\ 0 & 0 & a \end{bmatrix}$$

where A is an orthogonal $n \times n$ matrix and the remaining elements satisfy the relations

$$(1.1) \qquad ab = 1, \qquad ap + \tilde{q} = 0, \qquad 2ac + \tilde{q}q = 0$$

$\tilde{q}$ denoting the transpose of q.

The linear group of isotropy L'_n of the conformal space at $\{1, 0, \ldots, 0\}$ is isomorphic with the group G of non-zero scalar multiples of the orthogonal $n \times n$ matrices. We identify L'_n with G in such a way that the canonical homomorphism φ of K' onto L'_n is

$$\begin{bmatrix} b & p & c \\ 0 & A & Aq \\ 0 & 0 & a \end{bmatrix} \xrightarrow{\varphi} aA.$$

The Lie algebra $\mathcal{L}K$ is isomorphic with the Lie algebra of the $(n+2) \times (n+2)$ matrices of the type

$$\begin{bmatrix} -\mu & -\tilde{\psi} & 0 \\ \omega & \Omega & \psi \\ 0 & -\tilde{\omega} & \mu \end{bmatrix}$$

where the $n \times n$ matrix Ω is skew-symmetric. A representation of the subalgebra $\mathcal{L}K'$ is obtained by imposing the condition $\omega = 0$. The translation operations on these Lie algebras are obtained by matrix multiplication.

G. Ehresmann [4] has given necessary and sufficient conditions for the existence of a Cartan connection on M of type K/K', that is, a *conformal Cartan connection*. These are :

(i) that the tangent bundle of M should admit a subordinate structure with group L'_n ;

(ii) that there should exist a principal fibre bundle $\mathcal{H}' = H'(M, K')$ with which the homomorphism φ associates the subordinate structure.

Since K' is a subgroup of K, $\mathcal{H}'$ defines canonically a principal bundle $\mathcal{H} = H(M, K)$. A conformal Cartan connection on M is a connection on $\mathcal{H}$, in the usual sense, such that no horizontal directions on H are tangent to the subspace H'.

We shall construct $\mathcal{H}'$ from a cocycle $k'_{\alpha\beta}$, whit values in K', defined on an open covering $\{U_\alpha\}$ of M. Then H' is the quotient of the sum $\sum_\alpha U_\alpha \times K'$ by the equivalence relation

$$(m_\alpha, k'_\alpha) \sim (m_\beta, k'_\beta) \quad \textit{if} \quad m_\alpha = m_\beta, \qquad k'_\alpha = (k'_{\alpha\beta} m_\alpha)\, k'_\beta.$$

A Cartan connection can then be obtained from local 1-forms Γ_α with values in $\mathcal{L}K$ defined on U_α, provided that on $U_\alpha \cap U_\beta$ they satisfy the relation.

$$\Gamma_\beta = (k'_{\alpha\beta})^{-1} \{ \Gamma_\alpha k'_{\alpha\beta} + dk'_{\alpha\beta} \} \tag{1.2}$$

and possess the further property that $\Gamma_\alpha \vec{m} \in \mathcal{L}K'$ if and only if the tangent vector $\vec{m}$ of U_α is zero.

Denote by h' the projection $H' \to M$ and by h'^* the dual mapping on the differential forms in M. From the local product representation, we have functions k'_α with values in K' on $(h')^{-1} U_\alpha$. The connection form Γ is defined locally in H' by

$$\Gamma = (k'_\alpha)^{-1} \{ (h'^* \Gamma_\alpha) k'_\alpha + dk'_\alpha \} \tag{1.3}$$

and this extends uniquely to H.

2. The normal conformal connection. — Suppose now that a conformally Riemannian structure is given on M, so that the tangent bundle of M admits a given subordinate structure with group G. We shall construct a particular Cartan connection on M called the normal conformal connection.

The first condition of Ehresmann is satisfied since the linear isotropic group L'_n is isomorphic to G. We have to construct a bundle $\mathcal{H}' = H'(M, K')$ which gives rise to the above subordinate structure, using the homomorphism $\varphi : K' \to G$.

We are given a covering of M by open sets, U_α, each admitting a coordinate system $x_\alpha = \{ x^1_\alpha, \ldots, x^n_\alpha \}$ and a function X_α with values in the general linear group $GL(n, R)$, such that on $U_\alpha \cap U_\beta$ the function

$$g_{\alpha\beta} = X_\alpha M_{\alpha\beta} X_\beta^{-1}$$

where $M_{\alpha\beta} = [\partial x^i_\alpha / \partial x^j_\beta]$, has values in G. If dx_α is the natural coframe on U_α, then the coframe

$$\omega_\alpha = X_\alpha \, dx_\alpha$$

is adapted to the G-structure, since on $U_\alpha \cap U_\beta$:

$$\omega_\alpha = X_\alpha \, dx_\alpha = X_\alpha M_{\alpha\beta} \, dx_\beta = g_{\alpha\beta} \omega_\beta.$$

From this adapted coframe, we define a *local Riemannian metric* $\tilde{\omega}_\alpha \omega_\alpha$ on U_α.

To construct a cocycle on M which will define a bundle $\mathcal{H}'$, we remark that any matrix of G can be expressed uniquely as aA, where A is an orthogonal $n \times n$ matrix and the real number a is positive. If we split up the functions $g_{\alpha\beta}$ in this way

$$g_{\alpha\beta} = a_{\alpha\beta} A_{\alpha\beta}, \tag{2.1}$$

the following cocycle relations are satisfied

$$a_{\alpha\gamma} = a_{\alpha\beta} a_{\beta\gamma}, \qquad A_{\alpha\gamma} = A_{\alpha\beta} A_{\beta\gamma}.$$

We use these functions to define

$$k'_{\alpha\beta} = \begin{bmatrix} b_{\alpha\beta} & p_{\alpha\beta} & c_{\alpha\beta} \\ 0 & A_{\alpha\beta} & A_{\alpha\beta} q_{\alpha\beta} \\ 0 & 0 & a_{\alpha\beta} \end{bmatrix},$$

where $q_{\alpha\beta}$ is defined by the relation

$$(2.2) \qquad \tilde{q}_{\alpha\beta}\, \omega_\beta = d(\log a_{\alpha\beta})$$

and the remaining components are determined by the relations (1.1). These new functions $k'_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ have values in K' and it can be shown that they satisfy the cocycle relations, consequently they define a bundle $\mathcal{H}' = H'(M, K')$. Since the cocycle $g_{\alpha\beta}$ is the image of the cocycle $k'_{\alpha\beta}$ under the homomorphism φ, this bundle $\mathcal{H}'$ satisfies Ehresmann's second condition. In fact, $k'_{\alpha\beta}$ has values in the subgroup K'' of K' defined by $a > 0$. We denote by $\mathcal{H}'' = H''(M, K'')$ the principal bundle with group K'' defined by the cocycle $k'_{\alpha\beta}$. It is a sub-bundle of $\mathcal{H}'$.

We are now ready to construct on U_α the local 1-form Γ_α with values in $\mathcal{L}K$ which will define the Cartan connection. We shall take this to be

$$\Gamma_\alpha = \begin{bmatrix} 0 & -\tilde{\psi}_\alpha & 0 \\ \omega_\alpha & \Omega_\alpha & \psi_\alpha \\ 0 & -\tilde{\omega}_\alpha & 0 \end{bmatrix},$$

where the 1-forms Ω_α and ψ_α are still to be determined. We have, of course, to verifiy that the choices for these remaining components are such that Γ_α satisfies the relation (1.2); the further condition on Γ_α is satisfied already since the forms ω^i_α are linearly independent. Cartan determines Ω_α and ψ_α in terms of the local Riemannian metric $\tilde{\omega}_\alpha \omega_\alpha$ on U_α by imposing certain conditions on the curvature of the Cartan connection and this will be done by imposing conditions on the local curvature form

$$d\Gamma_\alpha + \Gamma_\alpha \wedge \Gamma_\alpha$$

consistent with relation (1.2).

This local curvature form has values in $\mathcal{L}K$ and so it has components

$$\begin{bmatrix} -B_\alpha & -\tilde{D}_\alpha & 0 \\ T_\alpha & C_\alpha & D_\alpha \\ 0 & -\tilde{T}_\alpha & B_\alpha \end{bmatrix}$$

where the values of the 2-form C_α are skew-symmetric. The first condition $T_\alpha = 0$ is consistent with (1.2); since

$$T_\alpha = d\omega_\alpha + \Omega_\alpha \wedge \omega_\alpha,$$

it implies that Ω_α is the connection form of the local Riemannian metric (calculated relative to the coframe ω_α). It now follows that on $U_\alpha \cap U_\beta$:

$$C_\beta = g_{\beta\alpha} C_\alpha g_{\alpha\beta}.$$

Consequently if

$$C_\alpha = \frac{1}{2} C^i_{jhk} \omega^h_\alpha \wedge \omega^k_\alpha,$$

the second condition $C^i_{jhi} = 0$ is consistent with (1.2) and, if $n \geqq 3$, it can be shown to determine the form ψ_α uniquely. Thus a Cartan connection has been determined from the conformal structure of M; it is the *normal conformal connection* of E. CARTAN.

We shall need to calculate ψ_α explictly and we suppose that $\psi_\alpha = \psi_{ih}\omega^h_\alpha$. Since

$$C_\alpha = R_\alpha - \omega_\alpha \wedge \tilde{\psi}_\alpha - \psi_\alpha \wedge \tilde{\omega}_\alpha,$$

where $R_\alpha = d\Omega_\alpha + \Omega_\alpha \wedge \Omega_\alpha$ is the curvature form of the local Riemannian metric then, if

$$R_\alpha = \frac{1}{2} R^i_{jhk} \omega^h_\alpha \wedge \omega^k_\alpha,$$

it follows that

$$C^i_{jhk} = R^i_{jhk} + \delta_{ik}\psi_{jh} - \delta_{ih}\psi_{jk} + \delta_{jh}\psi_{ik} - \delta_{jk}\psi_{ih}.$$

The condition $C^i_{jhi} = 0$ then shows that, for $n \geqq 3$,

$$\psi_\alpha = \frac{1}{n-2} \left\{ \frac{R}{2(n-1)} \delta_{ih} - R_{ih} \right\} \omega^h_\alpha$$

where $R_{jh} = R^i_{jhi}$ and $R = R_{jh}\delta^{jh}$. Consequently C_α is the Weyl conformal curvature form for the local Riemannian metric.

Finally, we obtain a local formula for the connection form Γ on H''. From the local product structure of H'' we have functions k''_α with values in K'' on $(h'')^{-1} U_\alpha$ and we put

$$k''_\alpha = \begin{bmatrix} b_\alpha & p_\alpha & c_\alpha \\ 0 & A_\alpha & A_\alpha q_\alpha \\ 0 & 0 & a_\alpha \end{bmatrix}, \tag{2.3}$$

where A_α is orthogonal and

$$a_\alpha > 0, \qquad a_\alpha b_\alpha = 1, \qquad a_\alpha p_\alpha + \tilde{q}_\alpha = 0, \qquad 2 a_\alpha c_\alpha + \tilde{q}_\alpha q_\alpha = 0.$$

Since

$$(k''_\alpha)^{-1} = \begin{bmatrix} a_\alpha & \tilde{q}_\alpha \tilde{A}_\alpha & c_\alpha \\ 0 & \tilde{A}_\alpha & \tilde{p}_\alpha \\ 0 & 0 & b_\alpha \end{bmatrix},$$

the formula (1.3) applied to H'' shows that

$$\Gamma = \begin{bmatrix} -\mu & -\tilde{\psi} & 0 \\ \omega & \Omega & \psi \\ 0 & -\tilde{\omega} & \mu \end{bmatrix},$$

where the global forms are defined locally by

$$(2.4)\quad \begin{cases} \omega = \dfrac{1}{a_\alpha} \tilde{A}_\alpha (h''^\star \omega_\alpha), \\ \mu = \dfrac{da_\alpha}{a_\alpha} - \tilde{\omega} q_\alpha, \\ \Omega = \tilde{A}_\alpha \{ (h''^\star \Omega_\alpha) A_\alpha + d\tilde{A}_\alpha \} - \omega \tilde{q}_\alpha + q_\alpha \tilde{\omega}, \\ \psi = dq_\alpha + \Omega q_\alpha - \mu q_\alpha + a_\alpha \tilde{A}_\alpha (h''^\star \psi_\alpha) - (\tilde{q}_\alpha \omega) q_\alpha + \dfrac{1}{2} (\tilde{q}_\alpha q_\alpha) \omega. \end{cases}$$

3. The method of equivalence of E. Cartan and S. Chern. — In this paragraph we shall suppose that G is any closed subgroup of the linear group and that the tangent bundle of a manifold M admits a subordinate structure with group G. In the nomenclature of S. Chern, M admits a G-structure. In [3], Chern gives a procedure for constructing a sequence of fibre bundles and differential forms for a G-structure. We give a short account of his work.

From the definition of a subordinate structure, there exists an open covering of M by coordinate neighbourhoods U_α on which are defined functions X_α, with values in the linear group, such that on $U_\alpha \cap U_\beta$ the functions

$$g_{\alpha\beta} = X_\alpha \mathrm{M}_{\alpha\beta} X_\beta^{-1}$$

have values in G. The coframe

$$\omega_\alpha = X_\alpha \, dx_\alpha$$

on U_α is adapted to the G-structure, since on $U_\alpha \cap U_\beta$,

$$\omega_\alpha = g_{\alpha\beta} \omega_\beta.$$

The first fibre bundle in the sequence is the principal bundle $\mathcal{B} = B(M, G)$ associated with the reduced structure and it is defined by the cocycle $g_{\alpha\beta}$. As usual, we shall denote by b the projection $B \to M$ and by $b^\star$ the dual mapping on the forms in M. Let g_α denote the local functions with values in G on $V_\alpha = b^{-1} U_\alpha$ defined by the local product structure, so that on $V_\alpha \cap V_\beta$,

$$g_\alpha = (b^\star g_{\alpha\beta}) g_\beta.$$

Using the local 1-forms ω_α on U_α, we construct on B a global 1-form θ with values in R^n. It is defined on V_α by

$$\theta = g_\alpha^{-1}(b^\star \omega_\alpha), \tag{3.1}$$

and its exterior derivative is given on V_α by

$$d\theta = g_\alpha^{-1} b^\star(d\omega_\alpha) - g_\alpha^{-1}\, dg_\alpha \wedge \theta.$$

We can express $g_\alpha^{-1} b^\star(d\omega_\alpha)$ as $\frac{1}{2} C^i{}_{hk}\theta^h \wedge \theta^k$ and so, if we put

$$\Pi_\alpha = g_\alpha^{-1}\, dg_\alpha + \varepsilon_\alpha,$$

where $\varepsilon_\alpha = \varepsilon^i_{jh}\theta^h$ is a 1-form on V_α with values in the Lie algebra $\mathfrak{L}G$ whose coefficients are to be determined, the above formula for $d\theta$ becomes

$$d\theta + \Pi_\alpha \wedge \theta = \frac{1}{2}(\varepsilon^i_{kh} - \varepsilon^i_{hk} + C^i{}_{hk})\theta^h \wedge \theta^k. \tag{3.2}$$

We impose as many linear relations with constant coefficients between the quantities $\frac{1}{2}(\varepsilon^i_{kh} - \varepsilon^i_{hk} + C^i{}_{hk})$ as possible. These quantities are then determined uniquely. This implies that if the coefficients of the form $\eta \wedge \theta$ satisfy the same linear relations, where η is any 1-form $\eta^i_{jh}\theta^h$ with values in $\mathfrak{L}G$, then $\eta \wedge \theta = 0$. The relations may, or may not, determine the coefficients ε^i_{jh}. If they do and if the coefficients of $\eta \wedge \theta$ satisfy the same relations, then $\eta = 0$.

Thus on V_α we have the formula

$$d\theta + \Pi_\alpha \wedge \theta = \tau_\alpha$$

and on V_β

$$d\theta + \Pi_\beta \wedge \theta = \tau_\beta,$$

where the coefficients of τ_α and τ_β are determined by the imposed linear relations. Since on $V_\alpha \cap V_\beta$,

$$\tau_\alpha - \tau_\beta = (\Pi_\alpha - \Pi_\beta) \wedge \theta,$$

the coefficients of the form $(\Pi_\alpha - \Pi_\beta) \wedge \theta$ also satisfy these linear relations. But the form $\Pi_\alpha - \Pi_\beta$ has values in $\mathfrak{L}G$ and, since

$$g_\alpha^{-1}\, dg_\alpha - g_\beta^{-1}\, dg_\beta = g_\beta^{-1} b^\star(g_{\alpha\beta}^{-1}\, dg_{\alpha\beta})g_\beta, \tag{3.3}$$

it is linear in θ^i. Consequently

$$(\Pi_\alpha - \Pi_\beta) \wedge \theta = 0$$

and we have a global 2-form τ on B defined on V_α by $\tau = \tau_\alpha$. If the imposed relations determine the coefficients ε^i_{jh}, then $\Pi_\alpha = \Pi_\beta$ and we have a global 1-form Π on B defined on V_α by $\Pi = \Pi_\alpha$.

But in the general case,

$$\Pi_\alpha - \Pi_\beta = \lambda^\nu_{\alpha\beta} \Lambda_\nu \qquad (\nu = 1, \ldots, d_1)$$

where Λ_ν are a basis for the d_1-dimensional vector space of 1-forms on B, with values in $\mathcal{L} G$, which satisfy the equation $\eta \wedge \theta = 0$ and whose components are linear in θ^i with constant coefficients. The functions $\lambda_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ form a cocycle on B with values in the additive group R^{d_1} and so they define a principal bundle

$$\mathcal{B}^1 = B^1(B, R^{d_1}).$$

Denote by b^1 the projection $B^1 \to B$ and by λ_α the local functions with values in R^{d_1} on $V^1_\alpha = (b^1)^{-1} V_\alpha$. Since on $V^1_\alpha \cap V^1_\beta$,

$$\lambda_\alpha - \lambda_\beta = b^{1\star} \lambda_{\alpha\beta},$$

we have global 1-forms θ^1, Π^1 on B^1 defined by

$$\theta^1 = b^{1\star}\theta,$$
$$\Pi^1 = b^{1\star}\Pi_\alpha - \lambda^\nu_\alpha (b^{1\star}\Lambda_\nu).$$

We now use the same procedure to construct a decomposition for $d\theta^1$ and $d\Pi^1$ and thus obtain further local forms χ_α on V^1_α. Defining a third bundle

$$\mathcal{B}^2 = B^2(B^1, R^{d_2}),$$

we then construct global forms θ^2, Π^2, χ^2 on B^2. And so on. If the new forms are defined globally at any stage, the process terminates. The final bundle space B^r then carries a structure whose group is the identity. This solves the problem of local equivalence in the sense now to be explained.

Suppose that M' is a second manifold carrying a G-structure and denote quantities arising from M', corresponding to those already defined for M, by an accent. The two G-structures on M and M' are locally equivalent at points m and m' if there exists a local diffeomorphism of some neighbourhood U_α of m onto a neighbourhood $U'_{\alpha'}$ of m' such that

$$(\omega'_{\alpha'})^\star = g\,\omega_\alpha$$

where * denotes the dual mapping defined by the diffeomorphism and g is some differentiable function on U_α with values in G. Two such diffeomorphisms are said to give the same local equivalence of the structures at m, m' if they coïncide in some neighbourhood of m. It follows from the work of E. Cartan [1] that the local equivalences for the G-structures on M, M' can be obtained from the local equivalences for the identity-structures on B^r, B'^r. Cartan gives a finite algorithm for finding the latter.

4. Application of the method of Cartan-Chern to conformal structure. — We now return to our original notation and suppose that G is the group of non-zero scalar multiples of the orthogonal $n\times n$ matrices. Its Lie algebra $\mathcal{L}G$ is isomorphic with the algebra of $n\times n$ matrices A such that

$$A+\tilde{A}=\rho I,$$

where ρ is any scalar.

We first construct the bundle $\mathcal{B}=B(M, G)$ and the form θ on B as in the preceeding paragraph. We can then find local forms Π_α on V_α in many ways so that the equation (3.2) becomes

$$d\theta+\Pi_\alpha\wedge\theta=0.$$

In order to make a definite choice, we put

$$(4.1)\qquad \Pi_\alpha=g_\alpha^{-1}\,dg_\alpha+g_\alpha^{-1}\,(b^*\Omega_\alpha)g_\alpha$$

where, as in paragraph 2, Ω_α is the connection form of the local Riemannian metric $\tilde{\omega}_\alpha\omega_\alpha$ on U_α. Π_α is then the corresponding local connection form on V_α.

Suppose that $\eta=\eta^i_{jh}\theta^h$ is any local 1-form with values in $\mathcal{L}G$ and such that $\eta\wedge\theta=0$. Then

$$\eta^i_{jh}+\eta^j_{ih}=2\lambda^h\delta_{ij},\qquad \eta^i_{jh}-\eta^i_{hj}=0.$$

These equations show that

$$\begin{aligned}\eta^i_{jh}&=\frac{1}{2}(\eta^h_{ij}+\eta^i_{hj}-\eta^h_{ji}-\eta^j_{hi}+\eta^i_{jh}+\eta^j_{ih})\\&=\lambda^j\,\delta_{ih}-\lambda^i\delta_{jh}+\lambda^h\,\delta_{ij}\end{aligned}$$

and so it follows that

$$\eta=\theta\tilde{\lambda}-\lambda\tilde{\theta}+(\tilde{\lambda}\theta)I.$$

Thus any such form is determined by a function λ with values in R^n. In particular, $\Pi_\alpha-\Pi_\beta$ will be determined by functions $\lambda_{\alpha\beta}$ on $V_\alpha\cap V_\beta$,

$$(4.2)\qquad \Pi_\alpha-\Pi_\beta=\theta\tilde{\lambda}_{\alpha\beta}-\lambda_{\alpha\beta}\tilde{\theta}+(\tilde{\lambda}_{\alpha\beta}\theta)\,I.$$

We do not calculate these functions explicitly at present. They form a cocycle on B and this defines a principal bundle $\mathcal{B}^1=B^1(B, R^n)$.

On B^1 we define global forms θ^1, Π^1 where

$$\begin{cases}\theta^1=b^{1*}\theta\\ \Pi^1=b^{1*}\Pi_\alpha-\theta^1\tilde{\lambda}_\alpha+\lambda_\alpha\tilde{\theta}^1-(\tilde{\lambda}_\alpha\theta^1)I.\end{cases}$$

A calculation of their exterior derivatives gives

$$\begin{cases} d\theta^1 = -\Pi^1 \wedge \theta^1, \\ d\Pi^1 = d\lambda_\alpha \wedge \tilde{\theta}^1 + \theta^1 \wedge d\tilde{\lambda}_\alpha - (d\lambda_\alpha \wedge \tilde{\theta}^1) I - \Pi^1 \wedge \Pi^1 + (a) + (b), \end{cases}$$

where (a) involves mixed products of components from Π^1 and θ^1 and (b) involves products of components of θ^1. Following the general method, we put

$$\chi_\alpha = d\lambda_\alpha + \chi'_\alpha + \chi''_\alpha,$$

where χ'_α and χ''_α are 1-forms on V^1_α with values in R^n which are linear in the components of θ^1 and Π^1 respectively. We can show that χ'_α and χ''_α are *uniquely* determined by requiring that

$$d\Pi^1 = \chi_\alpha \wedge \tilde{\theta}^1 + \theta^1 \wedge \tilde{\chi}_\alpha - (\tilde{\chi}_\alpha \wedge \theta^1) I - \Pi^1 \wedge \Pi^1 + \Phi,$$

where the form $\Phi = \frac{1}{2} \Phi^i_{jhk} \theta^{1h} \wedge \theta^{1k}$ satisfies the relations

$$\Phi^i_{jhi} = 0.$$

Explicitly, we find that

$$\begin{cases} \chi'_\alpha = b^{1\star}(\tilde{g}_\alpha(b^\star \psi_\alpha)) = (\tilde{\lambda}_\alpha \theta^1)\lambda_\alpha + \frac{1}{2}(\tilde{\lambda}_\alpha \lambda_\alpha)\theta^1, \\ \chi''_\alpha = -\tilde{\Pi}^1 \lambda_\alpha \end{cases}$$

and that $\Phi = b^{1\star}(g_\alpha^{-1}(b^\star C_\alpha) g_\alpha)$. The local forms ψ_α and C_α, which arise from the Riemannian metric on U_α, have been defined in paragraph 2.

From the general theory of paragraph 3, the local forms χ_α define a global form χ^1 on B^1 and hence Φ is also defined globally. The forms θ^1, Π^1, χ^1 contain $n + \frac{1}{2} n(n-1) + 1 + n$ linearly independent components and so they define an identity-structure on B^1. This structure, as explained in paragraph 3, solves the problem of local equivalence.

5. **The relation between the two theories.** — Starting from a given conformally Riemannian structure on M, we constructed, in paragraph 2, global forms ω, μ, Ω and ψ on H'' which defined a normal conformal connection. In paragraph 4, we carried out the Chern process for the conformal structure and obtained global forms θ^1, Π^1 and χ^1 on B^1. We shall set up a diffeomorphism mapping H'' onto B^1 and then find the relation between these two sets of forms.

We must first calculate the functions $\lambda_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ explicitly. From (4.2), we have

$$(5.1) \qquad \operatorname{trace}(\Pi_\alpha - \Pi_\beta) = n\tilde{\lambda}_{\alpha\beta}\theta.$$

Since the values of Ω_α are skew-symmetric matrices, it follows from (4.1), (3.3) and (2.1) that

$$\begin{aligned}\text{trace}(\Pi_\alpha - \Pi_\beta) &= \text{trace}(g_\alpha^{-1}\,dg_\alpha - g_\beta^{-1}\,dg_\beta)\\ &= \text{trace}\, b^\star(g_{\alpha\beta}^{-1}\,dg_{\alpha\beta})\\ &= \text{trace}\, b^\star(d(\log a_{\alpha\beta})I + \tilde{A}_{\alpha\beta}\,dA_{\alpha\beta})\\ &= nb^\star(d(\log a_{\alpha\beta})).\end{aligned}$$

Then using (2.2) and (3.1), we find that

$$\text{trace}(\Pi_\alpha - \Pi_\beta) = nb^\star(\tilde{q}_{\alpha\beta}\,\omega_\beta) = n(b^\star\tilde{q}_{\alpha\beta})g_\beta\,\theta.$$

Comparing this result with (5.1), it follows that

$$\lambda_{\alpha\beta} = \tilde{g}_\beta(b^\star q_{\alpha\beta}), \tag{5.2}$$

We recall from paragraph 2 that the bundle $\mathcal{H}''$ is defined by means of the cocycle $k'_{\alpha\beta}$ on M. Consequently H'' is the quotient of the sum $\sum U_\alpha \times K''$ by the equivalence relation

$$(m_\alpha, k''_\alpha) \sim (m_\beta, k''_\beta) \qquad \text{if} \quad m_\alpha = m_\beta, \qquad k''_\alpha = k'_{\alpha\beta}k''_\beta.$$

In paragraph 4 we defined $\mathcal{B}$ by means of the cocycle $g_{\alpha\beta}$ on M and $\mathcal{B}^1$ by means of the cocycle $\lambda_{\alpha\beta}$ on B. Combining these definitions and using (5.2), it follows that B^1 is the quotient of the sum $\sum U_\alpha \times G \times R^n$ by the equivalence relation

$$(m_\alpha, g_\alpha, \lambda_\alpha) \sim (m_\beta, g_\beta, \lambda_\beta)$$

if $m_\alpha = m_\beta$, $g_\alpha = g_{\alpha\beta}g_\beta$, $\lambda_\alpha = \lambda_\beta + \tilde{g}_\beta q_{\alpha\beta}$. The functions $k'_{\alpha\beta}$, $g_{\alpha\beta}$ and $q_{\alpha\beta}$ are all to be evaluated at $m_\alpha = m_\beta$.

We now set up a local diffeomorphism of $U_\alpha \times K''$ onto $U_\alpha \times G \times R^n$.

$$(m_\alpha, k''_a) \to (m_\alpha, a_\alpha A_\alpha, q_\alpha)$$

where a_α, A_α and q_α are obtained from the decomposition (2.3) for any element k''_α of K''. It can be shown that these local diffeomorphisms commute with the above equivalence relations and so they define a global diffeomorphism of H'' onto B^1. Denoting the dual mapping on the forms in B^1 by $\bigstar$, it follows that

$$\begin{cases} \bigstar\,\lambda_\alpha = q_\alpha, \\ \bigstar\,(b^{1\star}g_\alpha) = a_\alpha A_\alpha, \\ \bigstar\,(b^{1\star}b^\star\zeta_\alpha) = h''^\star\zeta_\alpha \qquad \text{for any form } \zeta_\alpha \text{ on } U_\alpha. \end{cases}$$

Using the definitions of the forms θ^1, Π^1, χ^1 on B^1 from paragraph 4 and the definitions of the forms ω, μ, Ω, ψ on H'' from paragraph 2, it is then easy to see that

$$\star\,\theta^1 = \omega, \qquad \star\,\Pi^1 = \Omega + \mu I, \qquad \star\,\chi^1 = \psi.$$

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