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CONFORMALLY RIEMANNIAN STRUCTURES, I ;

BY

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Introduction. — We define a conformally Riemannian structure on a differentiable ⁽¹⁾ manifold M of dimension n to be a differentiable subordinate structure of the tangent bundle to M whose group G consists of the non-zero scalar multiples of the orthogonal $n \times n$ matrices. The method of equivalence of E. CARTAN [1], as described by S. CHERN [3], associates with a given conformal structure a certain principal fibre bundle on which a set of linear differential forms is defined globally. We obtain such a bundle and set of forms explicitly and show their relation to the normal conformal connection of E. CARTAN [2].

The first paragraph contains an exposition of conformal connections in the light of C. EHRESMANN's general theory of Cartan connections [4]. In the second paragraph we show how this leads to the normal conformal connection on a manifold admitting a conformally Riemannian structure. The third paragraph summarises the method of Cartan-Chern and we apply this, in the fourth paragraph, to the special case of a conformally Riemannian structure. In the fifth paragraph we show how these ideas are related.

1. Conformal Cartan connections. — We first collect together the information we require on conformal space and on Cartan connections.

Conformal space of dimension n is defined to be the homogeneous space K/K' , where K is the linear group on $n+2$ variables $\{\xi_0, \xi_1, \dots, \xi_{n+1}\}$ leaving invariant the quadratic form

$$\sum_{i=1 \dots n} \xi_i^2 + \xi_0 \xi_{n+1}$$

(1) The word differentiable will always mean differentiable of class C^∞ .

and K' is the subgroup of K leaving invariant the point $\{1, 0, \dots, 0\}$. Explicitly, K' consists of matrices of the form

$$\begin{bmatrix} b & p & c \\ 0 & A & Aq \\ 0 & 0 & a \end{bmatrix}$$

where A is an orthogonal $n \times n$ matrix and the remaining elements satisfy the relations

$$(1.1) \quad ab = 1, \quad ap + \tilde{q} = 0, \quad 2ac + \tilde{q}q = 0$$

\tilde{q} denoting the transpose of q .

The linear group of isotropy L'_n of the conformal space at $\{1, 0, \dots, 0\}$ is isomorphic with the group G of non-zero scalar multiples of the orthogonal $n \times n$ matrices. We identify L'_n with G in such a way that the canonical homomorphism φ of K' onto L'_n is

$$\begin{bmatrix} b & p & c \\ 0 & A & Aq \\ 0 & 0 & a \end{bmatrix} \xrightarrow{\varphi} aA.$$

The Lie algebra $\mathcal{L}K$ is isomorphic with the Lie algebra of the $(n+2) \times (n+2)$ matrices of the type

$$\begin{bmatrix} -\mu & -\tilde{\psi} & 0 \\ \omega & \Omega & \psi \\ 0 & -\tilde{\omega} & \mu \end{bmatrix}$$

where the $n \times n$ matrix Ω is skew-symmetric. A representation of the subalgebra $\mathcal{L}K'$ is obtained by imposing the condition $\omega = 0$. The translation operations on these Lie algebras are obtained by matrix multiplication.

G. EHRESMANN [4] has given necessary and sufficient conditions for the existence of a Cartan connection on M of type K/K' , that is, a *conformal Cartan connection*. These are:

- (i) that the tangent bundle of M should admit a subordinate structure with group L'_n ;
- (ii) that there should exist a principal fibre bundle $\mathcal{H}' = H'(M, K')$ with which the homomorphism φ associates the subordinate structure.

Since K' is a subgroup of K , \mathcal{H}' defines canonically a principal bundle $\mathcal{H} = H(M, K)$. A conformal Cartan connection on M is a connection on \mathcal{H} , in the usual sense, such that no horizontal directions on H are tangent to the subspace H' .

We shall construct \mathcal{H}' from a cocycle $k'_{\alpha\beta}$, with values in K' , defined on an open covering $\{U_\alpha\}$ of M . Then H' is the quotient of the sum $\sum_\alpha U_\alpha \times K'$ by the equivalence relation

$$(m_\alpha, k'_\alpha) \sim (m_\beta, k'_\beta) \quad \text{if} \quad m_\alpha = m_\beta, \quad k'_\alpha = (k'_{\alpha\beta} m_\alpha) k'_\beta.$$

A Cartan connection can then be obtained from local 1-forms Γ_α with values in $\mathcal{L}K$ defined on U_α , provided that on $U_\alpha \cap U_\beta$ they satisfy the relation.

$$(1.2) \quad \Gamma_\beta = (k'_{\alpha\beta})^{-1} \{ \Gamma_\alpha k'_{\alpha\beta} + dk'_{\alpha\beta} \}$$

and possess the further property that $\Gamma_\alpha \vec{m} \in \mathcal{L}K'$ if and only if the tangent vector \vec{m} of U_α is zero.

Denote by h' the projection $H' \rightarrow M$ and by h'^* the dual mapping on the differential forms in M . From the local product representation, we have functions k'_α with values in K' on $(h')^{-1}U_\alpha$. The connection form Γ is defined locally in H' by

$$(1.3) \quad \Gamma = (k'_\alpha)^{-1} \{ (h'^* \Gamma_\alpha) k'_\alpha + dk'_\alpha \}$$

and this extends uniquely to H .

2. The normal conformal connection. — Suppose now that a conformally Riemannian structure is given on M , so that the tangent bundle of M admits a given subordinate structure with group G . We shall construct a particular Cartan connection on M called the normal conformal connection.

The first condition of EHRESMANN is satisfied since the linear isotropic group L'_n is isomorphic to G . We have to construct a bundle $\mathcal{X}' = H'(M, K')$ which gives rise to the above subordinate structure, using the homomorphism $\varphi: K' \rightarrow G$.

We are given a covering of M by open sets, U_α , each admitting a coordinate system $x_\alpha = \{x_\alpha^1, \dots, x_\alpha^n\}$ and a function X_α with values in the general linear group $GL(n, R)$, such that on $U_\alpha \cap U_\beta$ the function

$$g_{\alpha\beta} = X_\alpha M_{\alpha\beta} X_\beta^{-1}$$

where $M_{\alpha\beta} = [\partial x_\alpha^i / \partial x_\beta^j]$, has values in G . If dx_α is the natural coframe on U_α , then the coframe

$$\omega_\alpha = X_\alpha dx_\alpha$$

is adapted to the G -structure, since on $U_\alpha \cap U_\beta$:

$$\omega_\alpha = X_\alpha dx_\alpha = X_\alpha M_{\alpha\beta} dx_\beta = g_{\alpha\beta} \omega_\beta.$$

From this adapted coframe, we define a *local Riemannian metric* $\tilde{g}_\alpha \omega_\alpha$ on U_α .

To construct a cocycle on M which will define a bundle \mathcal{X}' , we remark that any matrix of G can be expressed uniquely as aA , where A is an orthogonal $n \times n$ matrix and the real number a is positive. If we split up the functions $g_{\alpha\beta}$ in this way

$$(2.1) \quad g_{\alpha\beta} = a_{\alpha\beta} A_{\alpha\beta},$$

the following cocycle relations are satisfied

$$a_{\alpha\gamma} = a_{\alpha\beta} a_{\beta\gamma}, \quad A_{\alpha\gamma} = A_{\alpha\beta} A_{\beta\gamma}.$$

We use these functions to define

$$k'_{\alpha\beta} = \begin{bmatrix} b_{\alpha\beta} & p_{\alpha\beta} & c_{\alpha\beta} \\ 0 & A_{\alpha\beta} & A_{\alpha\beta} q_{\alpha\beta} \\ 0 & 0 & a_{\alpha\beta} \end{bmatrix},$$

where $q_{\alpha\beta}$ is defined by the relation

$$(2.2) \quad \tilde{q}_{\alpha\beta} \omega_\beta = d(\log a_{\alpha\beta})$$

and the remaining components are determined by the relations (1.1). These new functions $k'_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ have values in K' and it can be shown that they satisfy the cocycle relations, consequently they define a bundle $\mathcal{H}' = H'(M, K')$. Since the cocycle $g_{\alpha\beta}$ is the image of the cocycle $k'_{\alpha\beta}$ under the homomorphism φ , this bundle \mathcal{H}' satisfies EHRESMANN'S second condition. In fact, $k'_{\alpha\beta}$ has values in the subgroup K'' of K' defined by $a > 0$. We denote by $\mathcal{H}'' = H''(M, K'')$ the principal bundle with group K'' defined by the cocycle $k'_{\alpha\beta}$. It is a sub-bundle of \mathcal{H}' .

We are now ready to construct on U_α the local 1-form Γ_α with values in $\mathcal{L}K$ which will define the Cartan connection. We shall take this to be

$$\Gamma_\alpha = \begin{bmatrix} 0 & -\tilde{\psi}_\alpha & 0 \\ \omega_\alpha & \Omega_\alpha & \psi_\alpha \\ 0 & -\tilde{\omega}_\alpha & 0 \end{bmatrix},$$

where the 1-forms Ω_α and ψ_α are still to be determined. We have, of course, to verify that the choices for these remaining components are such that Γ_α satisfies the relation (1.2); the further condition on Γ_α is satisfied already since the forms ω_α^i are linearly independent. CARTAN determines Ω_α and ψ_α in terms of the local Riemannian metric $\tilde{\omega}_\alpha \omega_\alpha$ on U_α by imposing certain conditions on the curvature of the Cartan connection and this will be done by imposing conditions on the local curvature form

$$d\Gamma_\alpha + \Gamma_\alpha \wedge \Gamma_\alpha$$

consistent with relation (1.2).

This local curvature form has values in $\mathcal{L}K$ and so it has components

$$\begin{bmatrix} -B_\alpha & -\tilde{D}_\alpha & 0 \\ T_\alpha & C_\alpha & D_\alpha \\ 0 & -\tilde{T}_\alpha & B_\alpha \end{bmatrix}.$$

where the values of the 2-form C_α are skew-symmetric. The first condition $T_\alpha = 0$ is consistent with (1.2); since

$$T_\alpha = d\omega_\alpha + \Omega_\alpha \wedge \omega_\alpha,$$

it implies that Ω_x is the connection form of the local Riemannian metric (calculated relative to the coframe ω_x). It now follows that on $U_\alpha \cap U_\beta$:

$$C_\beta = g_{\beta\alpha} C_\alpha g_{\alpha\beta}.$$

Consequently if

$$C_\alpha = \frac{1}{2} C^i_{jhk} \omega_x^h \wedge \omega_x^k,$$

the second condition $C^i_{jhi} = 0$ is consistent with (1.2) and, if $n \geq 3$, it can be shown to determine the form ψ_x uniquely. Thus a Cartan connection has been determined from the conformal structure of M ; it is the *normal conformal connection* of E. CARTAN.

We shall need to calculate ψ_x explicitly and we suppose that $\psi_x = \psi_{ih} \omega_x^h$. Since

$$C_\alpha = R_\alpha - \omega_x \wedge \tilde{\Psi}_\alpha - \psi_x \wedge \tilde{\omega}_\alpha,$$

where $R_\alpha = d\Omega_x + \Omega_x \wedge \Omega_x$ is the curvature form of the local Riemannian metric then, if

$$R_\alpha = \frac{1}{2} R^i_{jhk} \omega_x^h \wedge \omega_x^k,$$

it follows that

$$C^i_{jhk} = R^i_{jhk} + \delta_{ik} \psi_{jh} - \delta_{ih} \psi_{jk} + \delta_{jh} \psi_{ik} - \delta_{jk} \psi_{ih}.$$

The condition $C^i_{jhi} = 0$ then shows that, for $n \geq 3$,

$$\psi_x = \frac{1}{n-2} \left\{ \frac{R}{2(n-1)} \delta_{ih} - R_{ih} \right\} \omega_x^h$$

where $R_{jh} = R^i_{jhi}$ and $R = R_{jh} \delta^{jh}$. Consequently C_α is the Weyl conformal curvature form for the local Riemannian metric.

Finally, we obtain a local formula for the connection form Γ on H^n . From the local product structure of H^n we have functions k''_α with values in K^n on $(k''_\alpha)^{-1} U_\alpha$ and we put

$$(2.3) \quad k''_\alpha = \begin{bmatrix} b_\alpha & p_\alpha & c_\alpha \\ 0 & A_\alpha & A_\alpha q_\alpha \\ 0 & 0 & a_\alpha \end{bmatrix},$$

where A_α is orthogonal and

$$a_\alpha > 0, \quad a_\alpha b_\alpha = 1, \quad a_\alpha p_\alpha + \tilde{q}_\alpha = 0, \quad 2a_\alpha c_\alpha + \tilde{q}_\alpha q_\alpha = 0.$$

Since

$$(k''_\alpha)^{-1} = \begin{bmatrix} a_\alpha & \tilde{q}_\alpha \tilde{A}_\alpha & c_\alpha \\ 0 & \tilde{A}_\alpha & \tilde{p}_\alpha \\ 0 & 0 & b_\alpha \end{bmatrix},$$

the formula (1.3) applied to H'' shows that

$$\Gamma = \begin{bmatrix} -\mu & -\tilde{\psi} & 0 \\ \omega & \Omega & \psi \\ 0 & -\tilde{\omega} & \mu \end{bmatrix},$$

where the global forms are defined locally by

$$(2.4) \quad \left\{ \begin{array}{l} \omega = \frac{1}{a_x} \tilde{A}_x (h^{**} \omega_x), \\ \mu = \frac{da_x}{a_x} - \tilde{\omega} q_x, \\ \Omega = \tilde{A}_x \{ (h^{**} \Omega_x) A_x + d\tilde{A}_x \} - \omega \tilde{q}_x + q_x \tilde{\omega}, \\ \psi = dq_x + \Omega q_x - \mu q_x + a_x \tilde{A}_x (h^{**} \psi_x) - (\tilde{q}_x \omega) q_x + \frac{1}{2} (\tilde{q}_x q_x) \omega. \end{array} \right.$$

3. The method of equivalence of E. Cartan and S. Chern. — In this paragraph we shall suppose that G is any closed subgroup of the linear group and that the tangent bundle of a manifold M admits a subordinate structure with group G . In the nomenclature of S. CHERN, M admits a G -structure. In [3], CHERN gives a procedure for constructing a sequence of fibre bundles and differential forms for a G -structure. We give a short account of his work.

From the definition of a subordinate structure, there exists an open covering of M by coordinate neighbourhoods U_x on which are defined functions X_x , with values in the linear group, such that on $U_x \cap U_\beta$ the functions

$$g_{x\beta} = X_x M_{x\beta} X_\beta^{-1}$$

have values in G . The coframe

$$\omega_x = X_x dx_x$$

on U_x is adapted to the G -structure, since on $U_x \cap U_\beta$,

$$\omega_x = g_{x\beta} \omega_\beta.$$

The first fibre bundle in the sequence is the principal bundle $\mathcal{B} = B(M, G)$ associated with the reduced structure and it is defined by the cocycle $g_{x\beta}$. As usual, we shall denote by b the projection $B \rightarrow M$ and by b^* the dual mapping on the forms in M . Let g_x denote the local functions with values in G on $V_x = b^{-1} U_x$ defined by the local product structure, so that on $V_x \cap V_\beta$,

$$g_x = (b^* g_{x\beta}) g_\beta.$$

Using the local 1-forms ω_α on V_α , we construct on B a global 1-form θ with values in R^n . It is defined on V_α by

$$(3.1) \quad \theta = g_\alpha^{-1}(b^*\omega_\alpha),$$

and its exterior derivative is given on V_α by

$$d\theta = g_\alpha^{-1}b^*(d\omega_\alpha) - g_\alpha^{-1}dg_\alpha \wedge \theta.$$

We can express $g_\alpha^{-1}b^*(d\omega_\alpha)$ as $\frac{1}{2}C^i_{hk}\theta^h \wedge \theta^k$ and so, if we put

$$\Pi_\alpha = g_\alpha^{-1}dg_\alpha + \varepsilon_\alpha,$$

where $\varepsilon_\alpha = \varepsilon^i_{jh}\theta^h$ is a 1-form on V_α with values in the Lie algebra $\mathcal{L}G$ whose coefficients are to be determined, the above formula for $d\theta$ becomes

$$(3.2) \quad d\theta + \Pi_\alpha \wedge \theta = \frac{1}{2}(\varepsilon^i_{kh} - \varepsilon^i_{hk} + C^i_{hk})\theta^h \wedge \theta^k.$$

We impose as many linear relations with constant coefficients between the quantities $\frac{1}{2}(\varepsilon^i_{kh} - \varepsilon^i_{hk} + C^i_{hk})$ as possible. These quantities are then determined uniquely. This implies that if the coefficients of the form $\eta \wedge \theta$ satisfy the same linear relations, where η is any 1-form $\eta^i_{jh}\theta^h$ with values in $\mathcal{L}G$, then $\eta \wedge \theta = 0$. The relations may, or may not, determine the coefficients ε^i_{jh} . If they do and if the coefficients of $\eta \wedge \theta$ satisfy the same relations, then $\eta = 0$.

Thus on V_α we have the formula

$$d\theta + \Pi_\alpha \wedge \theta = \tau_\alpha$$

and on V_β

$$d\theta + \Pi_\beta \wedge \theta = \tau_\beta,$$

where the coefficients of τ_α and τ_β are determined by the imposed linear relations. Since on $V_\alpha \cap V_\beta$,

$$\tau_\alpha - \tau_\beta = (\Pi_\alpha - \Pi_\beta) \wedge \theta,$$

the coefficients of the form $(\Pi_\alpha - \Pi_\beta) \wedge \theta$ also satisfy these linear relations. But the form $\Pi_\alpha - \Pi_\beta$ has values in $\mathcal{L}G$ and, since

$$(3.3) \quad g_\alpha^{-1}dg_\alpha - g_\beta^{-1}dg_\beta = g_\beta^{-1}b^*(g_\alpha^{-1}dg_\alpha)g_\beta,$$

it is linear in θ^i . Consequently

$$(\Pi_\alpha - \Pi_\beta) \wedge \theta = 0$$

and we have a global 2-form τ on B defined on V_α by $\tau = \tau_\alpha$. If the imposed relations determine the coefficients ε^i_{jh} , then $\Pi_\alpha = \Pi_\beta$ and we have a global 1-form Π on B defined on V_α by $\Pi = \Pi_\alpha$.

But in the general case,

$$\Pi_\alpha - \Pi_\beta = \lambda_{\alpha\beta}^\nu \Lambda_\nu \quad (\nu = 1, \dots, d_1)$$

where Λ_ν are a basis for the d_1 -dimensional vector space of 1-forms on B , with values in ${}^x G$, which satisfy the equation $\tau_i \wedge \theta^i = 0$ and whose components are linear in θ^i with constant coefficients. The functions $\lambda_{\alpha\beta}$ on $V_\alpha \cap V_\beta$ form a cocycle on B with values in the additive group R^{d_1} and so they define a principal bundle

$$\mathcal{B}^1 = B^1(B, R^{d_1}).$$

Denote by b^1 the projection $B^1 \rightarrow B$ and by λ_α the local functions with values in R^{d_1} on $V_\alpha^1 = (b^1)^{-1} V_\alpha$. Since on $V_\alpha^1 \cap V_\beta^1$,

$$\lambda_\alpha - \lambda_\beta = b^{1*} \lambda_{\alpha\beta},$$

we have global 1-forms θ^1, Π^1 on B^1 defined by

$$\begin{aligned} \theta^1 &= b^{1*} \theta, \\ \Pi^1 &= b^{1*} \Pi_\alpha - \lambda_{\alpha}^\nu (b^{1*} \Lambda_\nu). \end{aligned}$$

We now use the same procedure to construct a decomposition for $d\theta^1$ and $d\Pi^1$ and thus obtain further local forms γ_α on V_α^1 . Defining a third bundle

$$\mathcal{B}^2 = B^2(B^1, R^{d_2}),$$

we then construct global forms $\theta^2, \Pi^2, \gamma^2$ on B^2 . And so on. If the new forms are defined globally at any stage, the process terminates. The final bundle space B^r then carries a structure whose group is the identity. This solves the problem of local equivalence in the sense now to be explained.

Suppose that M' is a second manifold carrying a G -structure and denote quantities arising from M' , corresponding to those already defined for M , by an accent. The two G -structures on M and M' are locally equivalent at points m and m' if there exists a local diffeomorphism of some neighbourhood U_x of m onto a neighbourhood $U'_{x'}$ of m' such that

$$(\omega'_{x'})^* = g^* \omega_x$$

where $*$ denotes the dual mapping defined by the diffeomorphism and g^* is some differentiable function on U_x with values in G . Two such diffeomorphisms are said to give the same local equivalence of the structures at m, m' if they coincide in some neighbourhood of m . It follows from the work of E. CARTAN [1] that the local equivalences for the G -structures on M, M' can be obtained from the local equivalences for the identity-structures on B^r, B'^r . CARTAN gives a finite algorithm for finding the latter.

4. Application of the method of Cartan-Chern to conformal structure.

— We now return to our original notation and suppose that G is the group of non-zero scalar multiples of the orthogonal $n \times n$ matrices. Its Lie algebra $\mathcal{L}G$ is isomorphic with the algebra of $n \times n$ matrices A such that

$$A + \tilde{A} = \rho I,$$

where ρ is any scalar.

We first construct the bundle $\mathcal{B} = B(M, G)$ and the form θ on B as in the preceding paragraph. We can then find local forms Π_α on V_α in many ways so that the equation (3.2) becomes

$$d\theta + \Pi_\alpha \wedge \theta = 0.$$

In order to make a definite choice, we put

$$(4.1) \quad \Pi_\alpha = g_\alpha^{-1} dg_\alpha + g_\alpha^{-1} (b^* \Omega_\alpha) g_\alpha$$

where, as in paragraph 2, Ω_α is the connection form of the local Riemannian metric $\tilde{\omega}_\alpha \omega_\alpha$ on U_α . Π_α is then the corresponding local connection form on V_α .

Suppose that $\eta = \eta_{jh}^i \theta^h$ is any local 1-form with values in $\mathcal{L}G$ and such that $\eta \wedge \theta = 0$. Then

$$\eta_{jh}^i + \eta_{ih}^j = 2\lambda^h \delta_{ij}, \quad \eta_{jh}^i - \eta_{hj}^i = 0.$$

These equations show that

$$\begin{aligned} \eta_{jh}^i &= \frac{1}{2} (\eta_{ij}^h + \eta_{hj}^i - \eta_{ji}^h - \eta_{hi}^j + \eta_{jh}^i + \eta_{ih}^j) \\ &= \lambda^i \delta_{ih} - \lambda^i \delta_{jh} + \lambda^h \delta_{ij} \end{aligned}$$

and so it follows that

$$\eta = \theta \tilde{\lambda} - \lambda \tilde{\theta} + (\tilde{\lambda} \theta) I.$$

Thus any such form is determined by a function λ with values in R^n . In particular, $\Pi_\alpha - \Pi_\beta$ will be determined by functions $\lambda_{\alpha\beta}$ on $V_\alpha \cap V_\beta$,

$$(4.2) \quad \Pi_\alpha - \Pi_\beta = \theta \tilde{\lambda}_{\alpha\beta} - \lambda_{\alpha\beta} \tilde{\theta} + (\tilde{\lambda}_{\alpha\beta} \theta) I.$$

We do not calculate these functions explicitly at present. They form a cocycle on B and this defines a principal bundle $\mathcal{B}^1 = B^1(B, R^n)$.

On B^1 we define global forms θ^1, Π^1 where

$$\begin{cases} \theta^1 = b^{1*} \theta \\ \Pi^1 = b^{1*} \Pi_\alpha - \theta^1 \tilde{\lambda}_\alpha + \lambda_\alpha \tilde{\theta}^1 - (\tilde{\lambda}_\alpha \theta^1) I. \end{cases}$$

A calculation of their exterior derivatives gives

$$\begin{cases} d\theta^1 = -\mathbf{\Pi}^1 \wedge \theta^1, \\ d\mathbf{\Pi}^1 = d\tilde{\lambda}_x \wedge \tilde{\theta}^1 + \theta^1 \wedge d\tilde{\lambda}_x - (d\tilde{\lambda}_x \wedge \tilde{\theta}^1)I - \mathbf{\Pi}^1 \wedge \mathbf{\Pi}^1 + \textcircled{a} + \textcircled{b}, \end{cases}$$

where \textcircled{a} involves mixed products of components from $\mathbf{\Pi}^1$ and θ^1 and \textcircled{b} involves products of components of θ^1 . Following the general method, we put

$$\gamma_x = d\lambda_x + \gamma'_x + \gamma''_x,$$

where γ'_x and γ''_x are 1-forms on V_x^1 with values in R^n which are linear in the components of θ^1 and $\mathbf{\Pi}^1$ respectively. We can show that γ'_x and γ''_x are *uniquely* determined by requiring that

$$d\mathbf{\Pi}^1 = \gamma_x \wedge \tilde{\theta}^1 + \theta^1 \wedge \tilde{\gamma}_x - (\tilde{\gamma}_x \wedge \theta^1)I - \mathbf{\Pi}^1 \wedge \mathbf{\Pi}^1 + \Phi,$$

where the form $\Phi = \frac{1}{2} \Phi^i_{jlk} \theta^{1j} \wedge \theta^{1k}$ satisfies the relations

$$\Phi^i_{jhi} = 0.$$

Explicitly, we find that

$$\begin{cases} \gamma'_x = b^{1*}(\tilde{g}_x(b^* \psi_x)) = (\tilde{\lambda}_x \theta^1) \lambda_x + \frac{1}{2}(\tilde{\lambda}_x \lambda_x) \theta^1, \\ \gamma''_x = -\tilde{\mathbf{\Pi}}^1 \lambda_x \end{cases}$$

and that $\Phi = b^{1*}(g_x^{-1}(b^* C_x)g_x)$. The local forms ψ_x and C_x , which arise from the Riemannian metric on U_x , have been defined in paragraph 2.

From the general theory of paragraph 3, the local forms γ_x define a global form γ^1 on B^1 and hence Φ is also defined globally. The forms θ^1 , $\mathbf{\Pi}^1$, γ^1 contain $n + \frac{1}{2}n(n-1) + 1 + n$ linearly independent components and so they define an identity-structure on B^1 . This structure, as explained in paragraph 3, solves the problem of local equivalence.

5. The relation between the two theories. — Starting from a given conformally Riemannian structure on M , we constructed, in paragraph 2, global forms ω , μ , Ω and ψ on H^n which defined a normal conformal connection. In paragraph 4, we carried out the Chern process for the conformal structure and obtained global forms θ^1 , $\mathbf{\Pi}^1$ and γ^1 on B^1 . We shall set up a diffeomorphism mapping H^n onto B^1 and then find the relation between these two sets of forms.

We must first calculate the functions $\lambda_x \lambda_\beta$ on $V_x \cap V_\beta$ explicitly. From (4.2), we have

$$(5.1) \quad \text{trace}(\mathbf{\Pi}_x - \mathbf{\Pi}_\beta) = n \tilde{\lambda}_x \lambda_\beta \theta.$$

Since the values of Ω_α are skew-symmetric matrices, it follows from (4.1), (3.3) and (2.1) that

$$\begin{aligned} \text{trace } (\Pi_\alpha - \Pi_\beta) &= \text{trace } (g_\alpha^{-1} dg_\alpha - g_\beta^{-1} dg_\beta) \\ &= \text{trace } b^*(g_{\alpha\beta}^{-1} dg_{\alpha\beta}) \\ &= \text{trace } b^*(d(\log a_{\alpha\beta})I + \tilde{A}_{\alpha\beta} dA_{\alpha\beta}) \\ &= nb^*(d(\log a_{\alpha\beta})). \end{aligned}$$

Then using (2.2) and (3.1), we find that

$$\text{trace } (\Pi_\alpha - \Pi_\beta) = nb^*(\tilde{q}_{\alpha\beta}\omega_\beta) = n(b^*\tilde{q}_{\alpha\beta})g_\beta^0.$$

Comparing this result with (3.1), it follows that

$$(3.2) \quad \lambda_{\alpha\beta} = \tilde{g}_\beta(b^*q_{\alpha\beta}),$$

We recall from paragraph 2 that the bundle \mathcal{K}'' is defined by means of the cocycle $k''_{\alpha\beta}$ on M . Consequently H'' is the quotient of the sum $\sum U_\alpha \times K''$ by the equivalence relation

$$(m_\alpha, k''_\alpha) \sim (m_\beta, k''_\beta) \quad \text{if } m_\alpha = m_\beta, \quad k''_\alpha = k''_{\alpha\beta}k''_\beta.$$

In paragraph 4 we defined \mathcal{B} by means of the cocycle $g_{\alpha\beta}$ on M and \mathcal{B}^1 by means of the cocycle $\lambda_{\alpha\beta}$ on B . Combining these definitions and using (3.2), it follows that B^1 is the quotient of the sum $\sum U_\alpha \times G \times R^n$ by the equivalence relation

$$(m_\alpha, g_\alpha, \lambda_\alpha) \sim (m_\beta, g_\beta, \lambda_\beta)$$

if $m_\alpha = m_\beta$, $g_\alpha = g_{\alpha\beta}g_\beta$, $\lambda_\alpha = \lambda_\beta + \tilde{g}_\beta q_{\alpha\beta}$. The functions $k''_{\alpha\beta}$, $g_{\alpha\beta}$ and $q_{\alpha\beta}$ are all to be evaluated at $m_\alpha = m_\beta$.

We now set up a local diffeomorphism of $U_\alpha \times K''$ onto $U_\alpha \times G \times R^n$.

$$(m_\alpha, k''_\alpha) \rightarrow (m_\alpha, a_\alpha A_\alpha, q_\alpha)$$

where a_α , A_α and q_α are obtained from the decomposition (2.3) for any element k''_α of K'' . It can be shown that these local diffeomorphisms commute with the above equivalence relations and so they define a global diffeomorphism of H'' onto B^1 . Denoting the dual mapping on the forms in B^1 by \star , it follows that

$$\left\{ \begin{array}{l} \star \lambda_\alpha = q_\alpha, \\ \star (b^{1*}g_\alpha) = a_\alpha A_\alpha, \\ \star (b^{1*}b^*\zeta_\alpha) = h''^*\zeta_\alpha \end{array} \right. \quad \text{for any form } \zeta_\alpha \text{ on } U_\alpha.$$

Using the definitions of the forms θ^1 , Π^1 , χ^1 on B^1 from paragraph 4 and the definitions of the forms ω , μ , Ω , ψ on H^n from paragraph 2, it is then easy to see that

$$\star \theta^1 = \omega, \quad \star \Pi^1 = \Omega + \mu I, \quad \star \chi^1 = \psi.$$

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