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TRANSLATION-INVARIANT FUNCTION ALGEBRAS ON ABELIAN GROUPS;

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1. Introduction. Integers, circle, line. — Throughout this paper X will be a locally compact abelian group, with character group Y. For technical convenience we shall always assume both X and Y metrisable, although none of our results, with the single exception of Theorem 4.2, depend essentially on metrisability. The principal object of "abstract" harmonic analysis has always been to describe the translation-invariant vector subspaces of the important Banach spaces on X, especially the subspaces of $L^1(X)$ and $L^2(X)$ and $L^\infty(X)$. We propose to consider here the translation-invariant subalgebras of the most important pointwise Banach algebra on X, namely $C_0(X)$, which consists of all continuous complex-valued functions vanishing at infinity. As usual, the Fourier transform will be an essential tool. We shall study a uniformly closed translation-invariant (henceforth called simply *closed invariant*) subalgebra A of $C_0(X)$ by studying its spectrum $\sigma(A)$, a certain closed subsemigroup of the dual group Y.

Allow us to emphasize the fact that $C_0(X)$ consists of complex functions and that A will not in general be closed under complex conjugation. Indeed the classification of all closed invariant subalgebras of a real $C_0(X)$, or all selfadjoint ones of a complex $C_0(X)$, is quite trivial. Each such subalgebra can be identified with $C_0(X/H)$ for some quotient group X/H.

It will be useful first to look at the classical Fourier series situation, X the reals mod 2π (or the circle) and Y the integers. We can easily prove A invariant not only under translation but also under convolution by, say, all integrable functions. Now let e_n be the exponential defined by $e_n(x) = e^{inx}$ and suppose A contains the function f having $\sum \alpha_n e_n$ as Fourier series. Then A also contains all the exponentials that actually appear in this Fourier series. For $\alpha_n e_n = e_n \star f \in A$; and if $\alpha_n \neq 0$, then $e_n \in A$. Furthermore the series is Fejer summable uniformly to f. Hence the algebra A is completely determined by the exponentials it contains. And since these exponentials are obviously closed under point wise multiplication, they constitute a semigroup that can be identified with a subsemigroup $\sigma(A)$ of the integers. Conversely, it is clear that each subsemigroup S of Y gives rise to a closed invariant subalgebra of $C_0(X)$. (And, in fact, the subsemigroups of the integers can be neatly described. They are either cyclic subgroups or differ from some $\{nm : n > 0\}$ by a finite number of elements.)

The above argument (except for the last parenthesis) applies with mere notational changes to an arbitrary compact abelian X. The classification of closed invariant subalgebras A of $C_0(X)$ reduces to the classification of subsemigroups of the (discrete) dual Y, and as Fourier analysts we can consider the problem solved. (We pass hastily over the quite unsolved algebraic problem of finding all the subsemigroups of an arbitrary abelian group Y. Even for concrete and reasonable groups, like the lattice points in n-space, it isn't easy.)

Returning to Fourier series, suppose we reverse the roles of X and Y. An element f of the invariant algebra A is now a double-ended sequence $\{a_n\}$, with $\lim\limits_{n=\pm\infty} \alpha_n = 0$. Because $\{\alpha_n\}$ is countable, A always contains the complex conjugate sequence $\{\overline{\alpha}_n\}$. (*See* e. g. [9], p. 40). Hence by Weierstrass-Stone, A can be identified with $C_0(Q)$ for some quotient space Q of X. But because A is translation-invariant, and because each $\{\alpha_n\} \in A$ vanishes at infinity, Q must be X itself and A can be no less than all of $C_0(X) =$ all double-ended null sequences. Substantially the same argument applies for any discrete X. Closed subalgebras A are self-adjoint, and hence trivially classifiable. The spectrum of A is always the open subgroup of Y orthogonal to the finite subgroup of X consisting of the periods of the $f \in A$. Using a less obvious method we shall prove in paragraph 5 that closed invariant subalgebras of $C_0(X)$ are self-adjoint for any totally disconnected X. Hence, with respect to the theory of invariant algebras, totally disconnected groups are totally uninteresting.

The real line is not so trivial as the circle and the integers. On the contrary, it displays all the complexities of an arbitrary locally compact abelian group. It is natural to define the spectrum of an $f \in C_0$ as the support of the Fourier transform $\hat{f}$ (in the sense of L. Schwartz's distribution theory) and the spectrum of an invariant subalgebra A as the union of the $\sigma(f)$ for $f \in A$. The set $\sigma(A)$ is (1) perfect, (2) closed under addition, and (3) locally a set of multiplicity (as defined in the uniqueness theory of trigonometric series). Conversely any subset S of the real line having these properties can appear as $\sigma(A)$ for some invariant subalgebra A of C_0.

But we do not know whether $A_1 \neq A_2$ *implies* $\sigma(A_1) \neq \sigma(A_2)$. This is a spectral synthesis problem, and it seems to be logically distinct from the famous one solved by MALLIAVIN [6]. In effect, MALLIAVIN exhibits two distinct weak* closed invariant subspaces of $L^\infty(X)$ that have the same spectrum. We have not been able to extract from his method two distinct uniformly closed invariant subspaces of $C_0(X)$ that have the same spectrum (although from such a pair of subspaces it would be trivial to construct a corresponding counterexample with subalgebras). It is true that Malliavin's basic function f actually belongs to C_0. But to produce a closed invariant subspace B such that $\sigma(B) = \sigma(f)$ but $f \notin B$, one must be able to assert, for instance, that $\sigma(f)$ supports many "smooth" measures, i. e. measures whose Fourier-Stieltjes transforms belong to C_0. Such an assertion is true for the more ancient L. SCHWARTZ counterexample [10] in vector groups of dimension ≥ 4, and hence we can show (Theorem 4.12) that $\sigma(A)$ does not determine A on these groups. It is natural to conjecture that such subalgebra counterexamples exist for every group X that is neither compact nor totally disconnected, since Malliavin's subspace counterexamples exist for every X that is not compact.

In any event we cannot classify the totality of closed invariant algebras on a general locally compact abelian X by classifying semigroups in the dual group Y. We must instead look for some reasonable subclass of algebras, and a corresponding subclass of semigroups, such that the correspondence is one-one. In n-space, one interesting class of semigroups is the cones, and another is Hille's angular semigroups. These classes are discussed in paragraphs 6 and 8. Specialization to $X =$ the real line yields a simple characterization of the "Phragmen-Lindelöf algebra" : all C_0 functions on the real axis that extend continuously to be analytic and bounded in the open upper half plane.

2. Notations. Definitions. Preliminaries.

X : Locally compact abelian group.
Y : Character group of X.
Both X and Y metrisable.

Additive notation, zero element 0. Same letter y for point of Y and function it induces on X. Similarly with x. Hence $y(x) = x(y)$. (But note $0 \in Y$ induces constant function 1 on X.)

$C_0(X)$: Continuous functions vanishing at infinity.
$C(X)$: Continuous bounded functions.
$L^1(X)$: Integrable functions (for Haar measure).
All functions take complex values.
$\hat{e}$: *Fourier transform* of trigonometric polynomial $e = \alpha_1 y_1 + \ldots + \alpha_n y_n$. Defined as measure on Y concentrated at points $y_1, \ldots, y_n$ with values $\alpha_1, \ldots, \alpha_n$.
$\hat{f}$: *Fourier transform* of $f \in C_0(X)$ or $\in C(X)$. (*See* § 3.)
$\hat{g}$: *Fourier transform* of $g \in L^1(X)$. $\hat{g}(y) = \int \overline{y(x)}\, g(x)\, dx$.
$\sigma(e)$: *Spectrum* of trigonometric polynomial. Finite point set $\{y_1, \ldots, y_n\}$.
$\sigma(f)$: *Spectrum* of $f \in C_0(X)$. (*See* § 3.)
$\sigma(g)$: *Spectrum* of $g \in L^1(X)$. Support of $\hat{g}$. Defined as closure in Y of $\{y : \hat{g}(y) \neq 0\}$.
$\bar{f}$: *Complex conjugate*. $\bar{f}(x) = \overline{f(x)}$.
f' : *Reflection*. $f'(x) = f(-x)$.
$T_x f$: *Translate*. $(T_x f)(x_0) = f(x + x_0)$.
fg : *Pointwise multiplication*. $(fg)(x) = f(x) g(x)$.
$f \star g$: *Convolution*. $(f \star g)(x_0) = \int f(x_0 - x) g(x)\, dx$.

We collect below some formulas for $g \in L^1(X)$.

$$
\begin{aligned}
&\hat{g}(y) = (y \star g)(0). \\
&(g_1 \star g_2)^\wedge = \hat{g}_1 \hat{g}_2. \\
&(T_x g)^\wedge = x\hat{g}. \\
&(yg)^\wedge = T_{-y}\hat{g}. \\
&\sigma(g_1 \star g_2) \subseteq \sigma(g_1) \cap \sigma(g_2) \\
&\sigma(T_x g) = \sigma(g). \\
&\sigma(yg) = y + \sigma(g). \\
&\sigma(g') = \sigma(\bar{g}) = -\sigma(g).
\end{aligned}
$$

In proving a few of the preliminary lemmas in this section and in sections 3 and 4 we shall need to extend the above machinery by substituting measures for $g \in L^1(X)$, or by substituting bounded measurable functions for the $f \in C(X)$. The modifications in definition are obvious, and the formulas remain true. In any event, our excursion into greater generality will be brief : in fact, the major purpose of Lemmas 4.5 and 4.6 will be to show the adequacy of $L^1(X)$ and $C(X)$ for all computations involving invariant subspaces of $C_0(X)$.

LEMMA 2.1. — *Let B be a uniformly closed linear subspace of $C_0(X)$. Then in order that B be invariant under translation it is necessary and sufficient that B be invariant under convolution by all $g \in L^1(X)$.* (Notice that translation is convolution by point masses.)

PROOF. — Suppose B is translation-invariant, and suppose first that g vanishes outside of some compact set Z. Then for each $f \in B$, convolution by g amounts to integrating the continuous B-valued function $x \to T_x f$ over the compact set Z with respect to the measure $g(-x)\,dx$:

$$f \star g = \int_Z T_x f g(-x)\,dx.$$

By anyone's theory of vector-valued integration this puts $f \star g$ in B. For the most general $g \in L^1(X)$, we can find a monotone sequence of compact Z_n such that $\int_{X-Z_n} |g(x)|\,dx$ approaches zero. Then $\int_{Z_n} T_x f g(-x)\,dx$ converges uniformly to $f \star g$.

In proving the converse we need an *approximate convolution unit.* This is a sequence $g_n \in L^1(X)$ such that

(1) $g_n \geqq 0$;

(2) $\int g_n = 1$;

(3) $g_n \equiv 0$ outside V_n,

where $\{V_n\}$ is a fundamental sequence of neighborhoods of $0 \in X$. (*See* also § 6.)

It is now an easy matter to finish Lemma 2.1 by proving for any $f \in B$ that $(T_x g_n) \star f$ converges uniformly to $T_x f$.

LEMMA 2.2. — *Let $\mathcal{L}$ be the set of Fourier transforms of $g \in L^1(X)$.*

(1) *$\mathcal{L}$ is a uniformly dense subalgebra of $C_0(Y)$.*

(2) *$\mathcal{L}$ is self-adjoint, i. e. closed under complex conjugation.*

(3) *$\mathcal{L}$ is regular in the sense of Silov. That is, for disjoint subsets Z_1 compact and Z_0 closed, there is some $\hat{g} \in \mathcal{L}$ with $\hat{g} \equiv 1$ on Z_1 and $\hat{g} \equiv 0$ on Z_0.*

(4) *For $f \in C_0(X)$, $f \not\equiv 0$, define $N_f = \{g \in L^1(X) : f \star g \equiv 0\}$. Then the set $\mathfrak{N}_f = \{\hat{g} : g \in N_f\}$ is an ideal in $\mathcal{L}$, and there is at least one point $y \in Y$ at which all $\hat{g} \in \mathfrak{N}_f$ vanish.*

(5) *Suppose $\varphi \in L^2(Y)$, φ real, $\varphi \equiv \varphi'$ and $\varphi \equiv 0$ outside some compact K. Then $\varphi \star \varphi \in \mathcal{L}$. In fact, $\varphi \star \varphi = \hat{g}$ for some $g \geqq 0$, and $\sigma(g) \subseteq K + K$.*

Proofs, for instance, in LOOMIS [5]. Statement (4) is Wiener's Tauberian theorem. We shall see in paragraph 4 that for $f \in C_0(X)$, X non-compact, there are infinitely many y at which all $\hat{g} \in \mathfrak{N}_f$ vanish.

We list below three notions of convergence for functions in $C(X)$ or $C_0(X)$.

Define $\|f\|_Z = \sup |f(x)|$, $x \in Z$.

$f_\nu \to f$ *uniformly* means $\|f_\nu - f\|_X \to 0$.

$f_\nu \to f$ *narrowly* means $\|f_\nu\|_X \to \|f\|_X$ and $\|f_\nu - f\|_Z \to 0$ for each compact $Z \subseteq X$.

$f_\nu \to f$ *weak** means $\int_X f'_\nu g \to \int_X f' g$ for each $g \in L^1(X)$.

Clearly uniform convergence implies narrow, and narrow convergence implies weak*. Uniform convergence defines the only complete metrisable topology compatible with multiplication and addition in $C(X)$. Weak* convergence comes from the ordinary weak* topology on $L^\infty(X)$. Narrow convergence also defines a topology, but it is one for which $C(X)$ is not even a topological group under addition (in fact, $f_n \to f$ does not imply $f_n + g \to f + g$). Nonetheless, when Y is considered as a subset of $C(X)$, the narrow topology induces on Y the given topology of Y.

The following lemma shows that we need not always specify the topology on $C(X)$.

LEMMA 2.3. — *Let B be an invariant linear subspace of $C_0(X)$. Let $\overline{B}$ be the uniform closure, $\overline{\overline{B}}$ the narrow closure, and $\overline{\overline{\overline{B}}}$ the weak* closure, all taken within $C_0(X)$. Then $\overline{B}$ equals $\overline{\overline{B}}$ equals $\overline{\overline{\overline{B}}}$.*

PROOF. — Clearly it is sufficient to assume B already uniformly closed and to prove B equals $\overline{\overline{\overline{B}}}$. Suppose B is a proper subspace of $\overline{\overline{\overline{B}}}$. Since both subspaces are uniformly closed, then by Hahn-Banach there exists some integrable Radon measure μ on X that is orthogonal to all of B but not orthogonal to all of $\overline{\overline{\overline{B}}}$. Suppose $\int f(x)\, d\mu(-x) \neq 0$, for some $f \in \overline{\overline{\overline{B}}}$. Then the bounded (continuous) function $f \star \mu$ is not identically zero. Hence by duality there is some $g \in L^1(X)$ such that

$$\int (f \star \mu)(x) g(-x)\, dx \neq 0.$$

This integral also can be written

$$\int f(x) (\mu \star g)(-x)\, dx \neq 0.$$

Suppose now $f_\alpha \to f$ weak*, $f_\alpha \in B$. Since $\mu \star g \in L^1(X)$, then

$$\int f_\alpha (\mu \star g)' \to \int f (\mu \star g)' \neq 0.$$

But

$$\int f_\alpha(\mu \star g)' = \int (f_\alpha \star g)\, d\mu' = 0 \qquad \text{because} \quad f_\alpha \star g \in B.$$

Contradiction.

LEMMA 2.4. — *Let $f_n \rightarrow f_1$ narrowly in $C(X)$. Then for each $f \in C(X)$, $f_n f \rightarrow f_1 f$ narrowly. And for each $f \in C_0(X)$, $f_n f \rightarrow f_1 f$ uniformly.*

Proof straightforward.

COROLLARY 2.5. — *Let the points $y_n \in Y$ converge to the zero point of Y. Then for each $f \in C(X)$, $y_n f \rightarrow f$ narrowly. And for each $f \in C_0(X)$, $y_n f \rightarrow f$ uniformly.*

3. **Spectrum of a function.** — This section summarizes certain well-known facts about spectral synthesis and spectral analysis.

If we want to define the spectrum $\sigma(f)$ for an $f \in C_0(X)$ as the support of $\hat{f}$, first we must define the Fourier transform $\hat{f}$. In this paper we shall use the transpose of the already defined Fourier transform on $L^1(X)$. Specifically, $\hat{f}$ is defined as an element of the dual space of the algebra $\mathcal{L}$ of ordinary Fourier transforms by the rule

$$(\hat{f}, \hat{g}) = \int f(-x)\, g(x)\, dx. \tag{3.1}$$

Now we say that $\hat{f}$ " vanishes identically " on an open subset V of Y if $\langle \hat{f}, \hat{g} \rangle = 0$ whenever $\sigma(g) \subseteq V$, and we define the *spectrum* $\sigma(f)$ for $f \in C_0(X)$ as the complement of the largest open subset of Y on which $\hat{f}$ vanishes identically. [Later, in order to make use of spectral analysis and synthesis, we shall adopt the same definitions of $\hat{f}$ and $\sigma(f)$ for bounded continuous f.] Notice that the spectrum of f as used in this paper has little to do with the spectrum in the sense of Banach algebras. There is an ancient historical connection between the two uses, but this connection is irrelevant for us here.

We list below some easily verified formulas for the spectrum of $f \in C_0(X)$, or $f \in C(X)$, like the formulas listed in paragraph 2 for $g \in L^1(X)$.

$\sigma(f_1 f_2) \subseteq$ closure of $[\sigma(f_1) + \sigma(f_2)]$.
$\sigma(fg) \subseteq$ closure of $[\sigma(f) + \sigma(g)]$.
$\sigma(f \star g) \subseteq \sigma(f) \cap \sigma(g)$.
$\sigma(T_x f) = \sigma(f)$.
$\sigma(yf) = y + \sigma(f)$.
$\sigma(f') = \sigma(\hat{f}) = -\sigma(f)$.

Lemmas 3.2, 3.3, 3.4 below state that our definition of the spectrum for a bounded continuous function is not in conflict with previously defined spectra for other kinds of functions. The proofs are straightforward.

LEMMA 3.2. — *If* $f \in C_0(X)$, $g \in L^1(X)$ *and* $f \equiv g$, *then* $\sigma(f) = \sigma(g)$.

LEMMA 3.3. — *If* $f \in C(X)$, *e is a trigonometric polynomial, and* $f \equiv e$, *then* $\sigma(f) = \sigma(e)$.

LEMMA 3.4. — *If* $\psi \in L^1(Y)$, *if* $\psi \equiv 0$ *outside a closed subset K of Y, and if* $\hat{\psi}$ *is the "inverse Fourier transform"*

$$\hat{\psi}(x) = \int x(y)\,\psi(y)\,dy,$$

then $\hat{\psi} \in C_0(X)$ *and* $\sigma(\hat{\psi}) \subseteq K$.

By using the right kind of "approximate unit" (*see* § 6) one can also define $\sigma(f)$ for an arbitrary $f \in C_0(X)$ as $\lim \sigma(f_n)$ for well-behaved f_n converging to f, for instance $f_n \in C_0(X) \cap L^1(X)$. The limit is taken in the natural uniform topology on the space formed from the closed subsets Z of Y. (Z_1 and Z_2 are near to each other if $Z_1 \subseteq Z_2 + V$ and $Z_2 \subseteq Z_1 + V$, for V a neighborhood of 0 in Y.) The sequence f_n that converges to f must be specially chosen, however. It is not necessarily true that $\sigma(f_n)$ approaches $\sigma(f_0)$ when f_n approaches f_0, no matter what degree of smoothness and smallness we require. Nonetheless, we can always assert that $\lim\inf \sigma(f_n) = \sigma(f_0)$.

LEMMA 3.5. — *Let* $f_0 \in C(X)$ *have compact spectrum* $S_0 = \sigma(f_0)$. *Then the mapping* $f \to \sigma(f)$ *is lower-semicontinuous at* f_0 *for the weak* topology on* $C(X)$ *and the natural topology and partial order on closed subsets of Y. Specifically, for each neighborhood W of* 0 *in Y, there exist*

$$g_1, \ldots, g_N \in L^1(X)$$

such that if $\left| \int f' g_n - \int f'_0 g_n \right| < 1$ *for each* g_n, *and if* $S = \sigma(f)$, *then* $S_0 \subseteq S + W$.

PROOF. — Given a neighborhood V of 0 in Y, there are $g_1, \ldots, g_N \in L^1(X)$ such that if $\left| \int (f - f_0) g_n \right| < 1$ then $\sigma(f)$ intersects $s_j + V$. Suppose we have already chosen $s_1 + V, \ldots, s_n + V$ to cover $S_0 = \sigma(f_0)$. Let

$$t_j \in \sigma(f) \cap (s_j + V), \text{ that is } \quad t_j = s_j + v_j.$$

Then $s_j \in S + V$. Hence

$$s_j + V \subseteq S + V + V.$$

Hence

$$S_0 \subseteq U(s_j + V) \subseteq S + V + V.$$

We isolate below the only part of 3.5 that we shall use. (It is not strictly speaking a corollary, but rather a step in the proof of 3.5.)

COROLLARY 3.6. — *Let $f_n \to f \in C(X)$ in the weak* sense. Suppose $\sigma(f_0)$ intersects the open set V. Then eventually all $\sigma(f_n)$ intersect V.*

Other equivalent definitions of the spectrum can be constructed from the two lemmas below. For a complete discussion of these matters, *see* HERZ [3].

LEMMA 3.7. (Spectral analysis). — *The $y \in \sigma(f)$ are exactly the characters that can be approximated by finite linear combinations of translates of f. This statement is true for any $f \in C(X)$ with weak* approximation, or for any $f \in C_0(X)$ with narrow approximation.*

LEMMA 3.8. (" Loose " spectral synthesis.) — *Given $f \in C(X)$ let W be a neighborhood of o in Y. Then f is the narrow limit of trigonometric polynomials e with $\sigma(e) \subseteq \sigma(f) + W$. Conversely, suppose K is a closed subset of Y such that, for any neighborhood W of o, f is the weak* limit of trigonometric polynomials e with $\sigma(e) \subseteq K + W$. Then $\sigma(f) \subseteq K$.*

4. Spectrum of an invariant algebra. — The spectrum of a closed invariant linear subspace B of $C_0(X)$ is defined to be the union of the $\sigma(f)$ for $f \in B$. We shall show in this section that if A is a closed invariant subalgebra of $C_0(X)$, $\sigma(A)$ is a closed subsemigroup of Y, and locally a set of multiplicity. Any such subsemigroup of Y appears as the $\sigma(A)$ of at least one closed invariant A, but perhaps more than one. The only case in which we have been able to show uniqueness is the case where ordinary spectral synthesis holds for $\sigma(A)$.

LEMMA 4.1. — *Let B be a closed invariant linear subspace of $C_0(X)$. Then for any point $y \in \sigma(B)$, and for any neighborhood V of y, there exists some non-zero $f \in B$ with $\sigma(f) \subseteq V$.*

PROOF. — By definition, $y \in \sigma(f_0)$ for some $f_0 \in B$. Choose $g \in L^1(X)$ with $\sigma(g) \subseteq V$ and $\hat{g}(y) \neq$ o. Then $g \star f_0 = f \in B$, and $f \not\equiv$ o, and

$$\sigma(f) \subseteq \sigma(g) \subseteq V.$$

THEOREM 4.2. — *The spectrum $\sigma(B)$ for a closed invariant linear subpace B of $C_0(X)$, defined as the union of the $\sigma(f)$ for $f \in B$, is a closed subset of Y.*

PROOF. — Let y be a point in the closure of $\sigma(A)$ and choose y_n in $\sigma(A)$ with $y_n \to y$. Choose neighborhoods V_n of the y_n which are such that their closures are pairwise disjoint and any neighborhood of y eventually contains all the closures $\overline{V}_n$. By Lemma 4.1 there are functions $f_n \not\equiv 0$ in A with

$$\sigma(f_n) \subseteq V_n \qquad \text{and} \qquad \sum_n \| f_n \|_X < \infty.$$

The function $f = \sum f_n$ is in A and we shall show that $y \in \sigma(f)$. Let W be any neighborhood of y, and choose $\overline{V}_n \subseteq W$. Since $\sigma(f_n) \subseteq V_n$, we can find $g \in L^1(X)$ with

$$\sigma(g) \subseteq V_n \qquad \text{and} \qquad \int f_n g' \neq 0.$$

And since $\int f g' = \int f_n g'$, the theorem is proved.

LEMMA 4.3. — *If B is the closed invariant subspace of $C_0(X)$ generated by f_0, then $\sigma(f_0) = \sigma(B)$.*

PROOF. — Clearly $\sigma(f_0) \subseteq \sigma(B)$. For the converse, choose any f in B. Then there is a sequence f_n of linear combinations of the $T_x f_0$ converging uniformly to f. Also $\sigma(f_n) \subseteq \sigma(f_0)$ since $\sigma(T_x f) \subseteq \sigma(f_0)$. Thus $\sigma(g) \subseteq \sigma(f)$ since $f_n \to g$ uniformly and $\sigma(f)$ is closed.

In Lemma 2.2 we defined N_f to be the set of all $g \in L^1(X)$ such that $g \star f \equiv 0$. Clearly this is equivalent to demanding that g be orthogonal to the closed linear subspace B of $C_0(X)$ generated by f. Furthermore, N_f is a convolution ideal in $L^1(X)$, and its Fourier transform $\mathfrak{N}_f$ is a pointwise ideal in $\mathcal{L}$. Wiener's Tauberian theorem, Lemma 2.2 (4), asserts that the *zeros* of $\mathfrak{N}_f$ [i. e. the set of y such that $\hat{g}(y) = 0$ for all $\hat{g} \in \mathfrak{N}_f$] is non-empty. For an arbitrary closed invariant subspace B of $C_0(X)$, whether or not it is generated by one function, let us define N_B to be

$$\left\{ g \in L^1(X) : \quad \int f' g = 0 \quad \text{for all } f \in B \right\}$$

and $\mathfrak{N}_B$ to be the pointwise ideal $\mathcal{L}$ consisting of Fourier transforms of the $g \in N_B$.

LEMMA 4.4. — *For any closed invariant linear subspace B of $C_0(X)$, the spectrum $\sigma(B)$ coincides with the zeros of $\mathfrak{N}_B$.*

PROOF. — We shall show first that any point not a common zero of the functions in $\mathfrak{N}_B$ cannot be in $\sigma(B)$. Let y be such a point and g_0 a function in N_B with $\hat{g}_0(y) \neq 0$. Choose g_1 in $L^1(X)$ with $\hat{g}_0 \hat{g}_1 = 1$ on a neighborhood U of y. Let g be any function in $L^1(X)$ with $\sigma(g) \subseteq U$. Then

$$(g_0 \star g_1 \star g)\hat{} = \hat{g}_0 \hat{g}_1 \hat{g} = \hat{g}$$

so $g_0 \star g_1 \star g = g$, and since N_B is a convolution ideal, g is in N_B. Thus each f in B is orthogonal to each g in $L^1(X)$ with $\sigma(g) \subseteq U$ and so y is not in $\sigma(B)$.

We shall show next that any point not in $\sigma(B)$ is not a common zero of the functions in $\mathfrak{N}_B$. Let y be a point not in $\sigma(B)$. Because of Lemma 4.2 there is a neighborhood V of y disjoint from $\sigma(B)$. Let g be a function in $L^1(X)$ with $\sigma(g) \subseteq V$ and $\hat{g}(y) \neq 0$. Then for each f in B, $\sigma(f) \cap \sigma(g)$ is empty so g is orthogonal to all of B and is thus in N_B. Since $\hat{g}(y) \neq 0$, y is not a common zero of the functions in $\mathfrak{N}_B$. This completes the proof of Lemma 4.4.

Since the full Banach dual of $C_0(X)$ is the space $M(X)$ of integrable Radon measures it is natural to ask what would happen if $M(X)$ were substituted for $L^1(X)$ in developing the notion of the spectrum.

LEMMA 4.5. — *Let B be a closed invariant subspace of $C_0(X)$ and let $B^\perp$ be the subspace of $M(X)$ orthogonal to B. Then $B^\perp$ is determined by the absolutely continuous measures it contains, i. e. by N_B. And $\sigma(B)$ is exactly the common zeros of the Fourier transforms of the measures in $B^\perp$.*

PROOF. — Suppose B_1 and B_2 are closed invariant subspaces with $B_1 \underset{\neq}{\subset} B_2$. Then by Lemma 2.3 the weak* closures in $L^\infty(X)$ are similarly related, $\overline{B_1} \underset{\neq}{\subset} \overline{B_2}$. But then the orthogonals in $L^1(X)$ have the reverse relation $N(B_1) \underset{\neq}{\supset} N(B_2)$. Hence the correspondance $B \to N(B)$ is a lattice anti-isomorphism and the first assertion of the lemma is proved.

Suppose $\hat{\mu}(y) \neq 0$ for some $\mu \in B^\perp$. Then convolution by any $g \in L^1(X)$ for which $\hat{g}(y) \neq 0$ yields $g_1 = g \star \mu \in L^1(X)$ with $\hat{g}_1(y) \neq 0$. Hence the lemma is completely proved.

Having justified the use of $L^1(X)$ rather than $M(X)$, we should perhaps justify our use of $C(X)$ rather than $L^\infty(X)$, e. g. in Lemmas 3.7 and 3.8.

LEMMA 4.6. — *Let B be a weak* closed invariant subspace of $L^\infty(X)$. Then B is determined by the continuous functions it contains.*

PROOF. — Convolution by an integrable approximate unit, as in the proof of Lemma 2.1.

Note that B is not always determined by the C_0 functions it contains. Consider, for instance, on a non-compact X the one-dimensional weak* closed subspace consisting of constant functions.

THEOREM 4.7. — *Let A be a closed invariant subalgebra of $C_0(X)$. Then $\sigma(A)$ is a subsemigroup of Y.*

PROOF. — Given $y_1 \in \sigma(A)$ and $y_2 \in \sigma(A)$ we want $y_1 + y_2 \in \sigma(A)$. We shall exhibit $f \in A$ with $\sigma(f)$ arbitrarily close to $y_1 + y_2$. Let U be any neighborhood of 0 in Y. By Lemma 2.1 there exist $f_k \in A$ ($k = 1, 2$) with $f_k \not\equiv 0$ and $\sigma(f) \subseteq y_k + U$. Choose $x \in X$ to make $f = f_1(\mathrm{T}_x f_2) \not\equiv 0$. Then $\sigma(f) \subseteq$ closure of $(\sigma(f_1) + \sigma(\mathrm{T}_x f_2)) =$ closure of

$$(\sigma(f_1) + \sigma(f_2)) \subseteq y_1 + y_2 + U + U.$$

And $y_1 + y_2 + U + U$ can be an arbitrarily small neighborhood of $y_1 + y_2$.

But not all closed subsemigroups S of Y can appear as $\sigma(A)$. For instance, the subgroup consisting of 0 alone can never appear unless X is compact. Let us introduce the following definitions. A subset Z of Y is called a *set of multiplicity* if it contains the spectrum of some $f \not\equiv 0$ in $C_0(X)$. And Z is said to be *locally a set of multiplicity* if every open set that intersects Z intersects it in a set of multiplicity. Because of Lemma 4.1, the spectrum of a closed translation invariant subspace of $C_0(X)$ is locally a set of multiplicity.

The following lemma states that on the circle our definition coincides with the one used in the uniqueness theory of trigonometric series. (*See*, for instance, ZYGMUND [13], p. 291, where a set of multiplicity is also called an *M*-set.)

LEMMA 4.8. — *Let Y be the reals mod 2π and let Z be a closed subset of Y. Then Y is a set of multiplicity if and only if there exists a non-zero trigonometric series converging to zero at all points of the complement of Z in $[0, 2\pi]$.*

PROOF. — BARI [2], p. 22.

LEMMA 4.9. — *A subset Z of Y that is locally of multiplicity cannot contain an isolated point unless Y is discrete.*

PROOF. — We can assume without loss that the isolated point is 0. But then the only f with $\sigma(f) = \{0\}$ is $f \equiv 1$. Hence X must be compact, and Y discrete.

LEMMA 4.10. — *Let S be a closed subset of Y that is locally a set of multiplicity. Let $\alpha(S)$ consist of all $f \in C_0(X)$ that have $\sigma(f) \subseteq S$. Then $\alpha(S)$ is a closed invariant subspace of $C_0(X)$ and $\sigma(\alpha(S)) = S$. Furthermore if S is a subsemigroup of Y, then $\alpha(S)$ is a subalgebra of $C_0(X)$.*

PROOF. — It is clear that $\alpha(S)$ is a linear subspace. $\alpha(S)$ is translation invariant since $\sigma(T_x f) = \sigma(f)$. If S is a subsemigroup of Y, $\alpha(S)$ is a subalgebra since

$$\sigma(f_1 f_2) \subseteq \text{closure } (\sigma(f_1) + \sigma(f_2)).$$

Let f be in the closure of $\alpha(S)$. Then there is a sequence f_n in $\alpha(S)$ converging uniformly to f. Since for each n, $\sigma(S_n) \subseteq S$, and S is closed, $\sigma(f) \subseteq S$. Thus f is in $\alpha(S)$ and $\alpha(S)$ must be closed.

Now only $\sigma(\alpha(S)) \supseteq S$ needs proving. Let $y \in S$ and consider any neighborhood V of y. Since S is locally a set of multiplicity there is some $f \in C_0(X)$, $\neq 0$, with $\sigma(f) \subseteq V \cap S$. Hence y belongs to the closure of the (closed) set $\sigma(\alpha(S))$ and the lemma is proved.

Let us now assemble some of the above lemmas in a theorem.

THEOREM 4.11. — *If A is a closed invariant subalgebra of $C_0(X)$, then $\sigma(A)$ is a closed subsemigroup of Y and locally a set of multiplicity. Conversely, each closed subsemigroup of Y that is locally a set of multiplicity is the $\sigma(A)$ of at least one such A.*

The above theorem does not assert that the closed subsemigroups of Y that are locally sets of multiplicity are in one-one correspondence with the closed invariant subalgebras of $C_0(X)$. For although $\sigma(\alpha(S)) = S$, it can happen that $\alpha(\sigma(A)) \supsetneq A$.

THEOREM 4.12. — *Let X be a 4-dimensional vector group. Then there exist closed invariant subalgebras $A_1 \neq A_2$ of $C_0(X)$ such that $\sigma(A_1) = \sigma(A_2)$.*

PROOF. — Y is also a 4-dimensional vector group. Let $S = S_1 \cup S_2$, where S_1 is the sphere $\{y : (y_1 - 4)^2 + y_2^2 + y_3^2 + y_4^2 = 1\}$ and S_2 is the half-space $\{y : y_1 \geq 6\}$. Clearly S is a semigroup, for $S_1 + S_1 \subseteq S_2$, $S_2 + S_2 \subseteq S_2$ and $S_1 + S_2 \subseteq S_2$. Let A be the closed subalgebra of $C_0(X)$ generated by $\alpha(S_2)$ and by all $f \in C_0(X)$ that are Fourier-Stieltjes transforms of measures carried by S_1. The uniform mass distribution is one such measure, its Fourier transform behaving like $|x|^{-\frac{1}{2}}$ at infinity. [The set of $f \in C_0(X)$ that are Fourier-Stieltjes transforms of measures carried by S_1 can also be described as the weak* closure in $C_0(X)$ of the trigonometric polynomials that have sepctra in S_1.] On the other hand, Laurent SCHWARTZ [10] has exhibited an $f_0 \in C_0(X)$ with $\sigma(f_0) \subseteq S_1$ but with f_0 not the limit, even in the weak* topology relative to $L^1(X)$, of Fourier transforms of measures carried by S_1. Hence $f_0 \notin A$, although $f_0 \in \alpha(\sigma(A))$. This completes the proof.

Spectral synthesis is said to hold for a closed subset Z of Y if every $f \in C(X)$ with $\sigma(f) \subseteq Z$ is a weak* limit of trigonometric polynomials having spectrum in Z. Equivalently, if the unique weak* closed invariant subspace B of $C(X)$ having $Z = \bigcup_{f \in B} \sigma(f)$ is

$$B = \{f : f \in C(X),\ \sigma(f) \subseteq S\}.$$

Still another equivalent characterization is that every $g_0 \in L^1(X)$ with $\hat{g}_0 \equiv 0$ on Z is the limit, in the L^1 norm, of $g \in L^1(X)$ whose Fourier transforms vanish identically on neighborhoods of Z. For a detailed discussion of spectral synthesis, *see* HERZ [3]. The Schwartz counterexample [10] actually showed the failure of spectral synthesis for any n-sphere in R^{n+1}, $n \geqq 2$. And recently MALLIAVIN [6] has proved that every non-discrete locally compact abelian group contains a compact subset for which spectral synthesis is impossible.

Lemma 3.8 states, of course, that every f with $\sigma(f) \subseteq Z$ is spectrally synthesizable by means of trigonometric polynomials with spectra in an arbitrarily small neighborhood of Z. But may not be synthesizable from within Z itself.

THEOREM 4.13. — *Let S be a closed subset of Y that is locally a set of multiplicity and for wich spectral synthesis holds. Then $\alpha(S)$ is the unique closed invariant subspace of $C_0(X)$ having S as spectrum.*

PROOF. — Let B be a closed invariant subspace of $C_0(X)$ with $\sigma(B) = S$. Let $\overline{B}$ be the weak* closure of B in $C(X)$. If $f \in \overline{B}$, $\sigma(f) \subseteq S$ so since $\sigma(B) = S$, $\bigcup_{f \in \overline{B}} \sigma(f) = S$. Thus since spectral synthesis holds for S,

$$\overline{B} = \{f : f \in C(X), \sigma(f) \subseteq S\} \qquad \text{and thus} \qquad \overline{B} \cap C_0(X) = \alpha(S).$$

But by Lemma 2.3, $\overline{B} \cap C_0(X) = B$, which finishes the proof of the lemma.

We shall need in Section 6 the following result of Reiter's.

LEMMA 4.14. — *Spectral synthesis holds for each closed subgroup of Y*

PROOF. — REITER [7].

5. Symmetric algebras. Zéro-dimensional groups. — There are two important kinds of symmetry a subspace of $C_0(X)$ may have, symmetry with respect to reflection (induced by sending each group element into its negative), and symmetry with respect to complex conjugation. In general these two kinds of symmetry are independent, but we shall prove that for closed invariant subalgebras of $C_0(X)$ either kind of symmetry implies the other. And since symmetric subalgebras of $C_0(X)$ are in natural one-one correspondence with closed subgroups of X, the structure theory for symmetric algebras is essentially trivial. We shall also prove that when X is totally disconnected, all the closed invariant subalgebras of $C_0(X)$ are symmetric.

THEOREM 5.1. — *Let A be a closed invariant subalgebra of $C_0(X)$. Then the following conditions on A are equivalent.*

(1) *A contains with every f its complex conjugate function $\bar{f}$.*

(2) *A contains with every f its reflected function f', defined by $f'(x) = f(-x)$.*

(3) *$\sigma(A)$ is a group.*

(4) *A is the whole algebra $C_0(X/H)$ lifted to X from some quotient group X/H.*

Furthermore the subgroups $\sigma(A)$ and H are Pontrjagin annihilators of each other, $\sigma(A)$ open and H compact.

PROOF. — Suppose (1) and consider the partition of X induced by the equivalence relation

$$\{ x_1 \equiv x_2 \text{ if and only if } f(x_1) = f(x_2) \text{ for all } f \in A \}.$$

Because A is invariant, the partition cell H that contains o is a subgroup. For if x_1 and $x_2 \in H$, then for all $f \in A$,

$$f(x_1 - x_2) = (T_{-x_2}f)(x_1) = (T_{-x_2}f)(0) = f(x_2) = f(0).$$

And in fact whenever $x_1 \equiv x_2$, then $x_1 - x_2 \in H$. For

$$f(x_1 - x_2) = (T_{-x_2}f)(x_1) = (T_{-x_2}f)(x_2) = f(0).$$

Hence the partition cells are the cosets of H. By the Weierstrass-Stone approximation theorem, A must include all C_0 functions compatible with this partition. We have thus proved that (1) implies (4).

The functions lifted from $C_0(X/H)$ can also be described as those f admitting all $y \in H$ as periods. Clearly the function $f(-x)$ will admit H as periods whenever $f(x)$ does. Hence (4) implies (2).

The automorphism of X sending x into $-x$ induces the same kind of automorphism on the dual group Y. If A is stable under this automorphism, then so also is $\sigma(A)$, and the semigroup is actually a group. Hence (2) implies (3).

If $\sigma(A)$ is a group, then 4.13 and 4.14 force A to contain all f in $C_0(X)$ whose spectra lie in $\sigma(A)$. In particular, A is closed under complex conjugation, since $\sigma(\bar{f}) = -\sigma(f)$. Hence (3) implies (1), and the circle of implications is complete.

Now assume that A is defined as in (4), and let H_0 be the Pontrjagin orthogonal of $\sigma(A)$. That $H_0 \subseteq H$ is Theorem 2 of REITER [7]. Conversely suppose $x \notin H_0$. Then there is some $f \in A$ and some $y \in \sigma(f)$ such that $(x, y) \neq 1$. To show $x \notin H$ we show that $T_x f \neq f$. There is some compact neighborhood V of y such that the function x never equals 1 on V. Because $y \in \sigma(f)$ there is some $g_0 \in L^1(X)$ with $\sigma(g_0) \subseteq V$ and $\int_X g_0 f' \neq 0$. Furthermore, the Fourier transform ψ_0 of g_0 can be factored $\psi_0 = \psi(1 - x)$, with ψ the Fourier transform of some $g \in L^1(X)$. That is,

$$g_0 = g - T_{-x}g \qquad \text{and} \qquad \int g_0 f' = \int g f' - \int (T_{-x}g) f' \neq 0.$$

Hence

$$\int g f' \neq \int (T_{-x}g) f' = \int g (T_x f)'$$

Hence $f \neq T_x f$, and $H = H_0$ is established. The compactness of H is a consequence of the fact that all $f \in A$ vanish at infinity. This completes the proof of Theorem 5.1.

COROLLARY 5.2. — *Let S be a closed subgroup of the locally compact abelian group Y. Then S is locally a set of multiplicity if and only if S is open.*

PROOF. — If S is an open subgroup of Y, then by Lemma 4.1 for every open subset V of S there is some $f \in C_0(X)$ such that $\sigma(f) \subseteq V$. Hence S is locally a set of multiplicity. Conversely, if S is locally a set of multiplicity then by Theorem 5.1, $S = \sigma(A)$ for some unique closed invariant A and $\sigma(A)$ is open.

We have thus proved that there is natural one-one correspondence between the compact subgroups of X and the symmetric closed invariant subalgebras of $C_0(X)$.

When X is discrete, then all closed subalgebras of $C_0(X)$ are symmetric, wheter or not they are invariant. *See* RUDIN [9]. When X is only zero-dimensional, for instance when X is a p-adic group, then $C_0(X)$ contains non-symmetric closed subalgebras. *See* RUDIN [8]. The following theorem asserts, however, that the invariant subalgebras must still be symmetric.

THEOREM 5.3. — *On a totally disconnected group X, every closed invariant subalgebra of $C_0(X)$ is symmetric.*

PROOF. — Let $\{H_\alpha\}$ be the open compact subgroups of X. Since X is totally disconnected, then $\cap H_\alpha = \{0\}$. The orthogonal subgroups $H_\alpha^\perp$ in Y will also be open and compact. The group $\cup H_\alpha^\perp$ is closed because it is open, hence

$$\cup H_\alpha^\perp = (\cap H_\alpha)^\perp = \{0\}^\perp = Y.$$

Now since A is a closed invariant subalgebra of $C_0(X)$, then $\sigma(A)$ is a closed subsemigroup of Y. Hence $\sigma(A) \cap H_\alpha^\perp$ is a closed subsemigroup of $H_\alpha^\perp$, hence (because $H_\alpha^\perp$ is compact) it is a subgroup of $H_\alpha^\perp$, hence

$$\sigma(A) = \bigcup_\alpha (\sigma(\mathrm{A}) \cap H_\alpha^\perp)$$

is itself a group. This completes the proof of Theorem 5.3.

6. Angular semigroups, approximate units. — In this section a subsemigroup S of Y will be called *angular* if it contains the zero element of Y and is the closure of its interior S^0. We are thus in mild disagreement with the HILLE-PHILLIPS terminology in [4], p. 265. But it is clear that our angular semigroup is the closure of theirs, and theirs the interior of ours. (In fact, for S to equal the closure of its interior it is enough that 0 belong to the closure of this interior.) The angular subsemigroups of the plane are classified in their Theorem 8.7.7, and an inductive extension of this classification method to Y^n (= the n-dimensional vector group) is sketched on page 269 of their book. Using these Y^n results, WRIGHT [12] is able to describe all the angular subsemigroups of arbitrary locally compact abelian groups.

In this section we characterize the closed invariant subalgebras A of $C_0(X)$ for which $\sigma(A) \subseteq Y$ is angular, X an arbitrary locally compact abelian group and Y its dual. We also establish spectral synthesis for such $\sigma(A)$, so that each angular semigroup appears as the spectrum of exactly one closed invariant subalgebra.

An approximate unit in a (commutative, metrisable) topological ring A is a sequence e_n of elements of A such that for each element f of A, $e_n f \to f$.

In Lemmas 6.1 through 6.4 the algebra A will always be a closed invariant subalgebra of $C_0(X)$.

LEMMA 6.1. — *The following four conditions on A are equivalent :*

(1) *A has an approximate unit.*

(2) *There exist $e_\nu \in A$ converging weak* to the constant* 1.

(3) *There exist $e_n \in A$ with $\| e_n \|_X = 1$ and e_n converging to the constant* 1 *uniformly on compact subsets of X.*

(4) *$\sigma(A)$ contains $0 \in Y$.*

Proof that (1) implies (2). Multiplication by e_n defines an operator E_n on A. For each f, $E_n f$ converges. Hence E_n is bounded in the strong operator topology. Hence by the uniform boundedness theorem (Banach-Steinhaus) $\| E_n \|$ is bounded. But

$$\| E_n \| = \| e_n \|_X, \qquad \text{since} \quad \| E_n e_n \|_X = \| e_n \|_X \| e_n \|_X.$$

Hence $\| e_n \|_X$ is bounded. Now let Z be any compact subset of X. Since A is translation-invariant, then each $z \in Z$ has some neighborhood V_z and some $f_z \in A$ such that $\inf\{ |f_z(x)| : x \in V_z \} > 0$. Since $\| e_n f_z - f_z \|_X \to 0$, e_n must converge to 1 uniformly on V_z. But Z can be covered by finitely many V_z, hence $e_n \to 1$ uniformly on all of Z. But $\| e_n \|_X$ bounded and $e_n \to 1$ on compact subsets together imply that $e_n \to 1$ weak*.

The equivalence of (2) and (3) and (4) is a consequence of Theorem 4.2 and Lemma 3.7. For, by Theorem 4.2, to say $0 \in \sigma(A)$ is to say $0 \in \sigma(f)$

for some $f \in A$. And by Lemma 3.7, $0 \in \sigma(f)$ if and only if the constant 1 is a narrow (or weak*) limit of linear combinations of translates of f. Narrow convergence is what is described in condition (2), except that we have strengthened $\| e_n \|_X \to 1$ to read $\| e_n \|_X = 1$. This strengthening can be accomplished by a trivial modification of the strictly converging sequence. The fact that a sequence (instead of a net) suffices for narrow convergence is a consequence of the σ-compactness of X, which in turn follows from the metrisability of Y.

Proof that (3) implies (1). Given $f \in C_0(X)$, with (say) $\|f\|_X = 1$, choose the compact subset Z so that $|f(x)| < \varepsilon$ for $x \notin Z$. Then if $| e_n(x) - 1 | < \varepsilon$ for $x \in Z$, we will have

$$| e_n(x) f(x) - f(x) | < \varepsilon \|f\|_X = \varepsilon \qquad \text{for} \quad x \in Z.$$

Thus e_n is an approximate unit for A.

Lemma 6.2. — *If A has an approximate unit, and if (for some $q < \infty$) the subalgebra $L^q \cap A$ is uniformly dense in A, then $\sigma(A)$ is angular.*

Proof. — Without loss of generality we can assume q an integer. Let V be any neighborhood of 0 in Y, with $\overline{V}$ compact, and let W be a neighborhood of 0 with closure $(W + \ldots (q \text{ times}) \ldots + W) \subseteq V$. Since $0 \in \sigma(A)$ by Lemma 4.1, there is some $f_0 \in A$ with $\sigma(f_0) \subseteq W$. And since f_0 is a uniform limit of $f \in L^q(\mathrm{X}) \cap A$, then by Lemma 3.6 one of these has $\sigma(f) \cap U$ non-empty. Convoluting by suitable $g \in L^1(X)$, we can even make $\sigma(f) \subseteq U$, with f still in $L^q \cap A$ and non-zero. Then the q-th power f^q belongs to $L^1(X) \cap \mathrm{A}$, and $\sigma(f^q) \subseteq V$ since $\sigma(g_1 g_2) \subseteq$ closure $(\sigma(g_1) + \sigma(g_2))$. Furthermore $\sigma(f^q)$ is the support of the continuous fonction obtained by convoluting $\hat{f}$ with itself q times, hence has an interior, and the lemma is proved.

Lemma 6.3. — *Spectral synthesis holds on each angular subsemigroup S of the group Y.*

Proof. — We must show that when $f \in C(X)$ has $\sigma(f) \subseteq S$, then f is a weak* limit of trigonometric polynomials e with $\sigma(e) \subseteq S$. Choose points $y \in X$ approaching 0 through the interior S^0 of S. Then the functions y approach the constant 1 narrowly on X, and by Lemma 2.4, yf approaches f weak*. Furthermore,

$$\sigma(yf) = y + \sigma(f) \subseteq S^0 + S \subseteq S^0,$$

the last inclusion because $S^0 + S$ is open and $S + S \subseteq S$. Finally, by Lemma 3.8, yf is a weak* limit of trigonometric polynomials e with $\sigma(e) \subseteq S^0$. Hence spectral synthesis is established on S.

LEMMA 6.4. — *Suppose $\sigma(A)$ angular. Let B consist of all $g \in L^1(X)$ that have compact spectra interior to $\sigma(A)$. Then B is a uniformly dense subalgebra of A.*

PROOF. — Because the $g \in B$ have compact spectra, then $B \subseteq C_0(X)$. And by spectral synthesis, $B \subseteq A$. Because

$$\sigma(f_1 f_2) \subseteq \text{closure of } (\sigma(f_1) + \sigma(f_2)),$$

and because $\sigma(A)$ is a closed semigroup, B is an algebra under pointwise multiplication. Because translations on X become on Y multiplications by characters, B is translation-invariant. Finally, if ψ is the Fourier transform of $g \in B$, then its complex conjugate $\overline{\psi}$ is the Fourier transform of $g^{\sim}$, defined by $g^{\sim}(x) = \overline{g(-x)}$.

Let N consist of all $g \in L^1(x)$ that are orthogonal to B. If B is not dense in A, then Lemma 4.5 will find us some $g \in N$ that is not orthogonal to A. Again by spectral synthesis, we must have $\hat{g}(y) \neq 0$ for some y_0 in $\sigma(A)$ and even, if we wish, interior to $\sigma(A)$. Because B is invariant, then so is N. Thus $g_2 = g \star g^{\sim}$ belongs to N, with

$$\hat{g}_2(y) = |g(y)|^2 \geqq 0 \quad \text{for all } y \in Y \qquad \text{and} \qquad \hat{g}_2(y_0) = |g(y_0)|^2 > 0.$$

By Lemma 2.2, there is some $g_3 \in L^1(X)$ with $\sigma(g_3)$ interior to S, and

$$\hat{g}_3(y_0) > 0 \qquad \text{and} \qquad \hat{g}_3(y) \geqq 0 \quad \text{for all } y.$$

Then

$$\int \hat{g}_3(y)\,\hat{g}_2(y)\,dy = \int g_3(-x) g_2(x)\,dx > 0.$$

Hence g_2 is not orthogonal to B after all. Contradiction.

We collect the above lemmas in the following theorem.

THEOREM 6.5. — *There is natural one-one correspondence between the angular subsemigroups S of Y and the closed invariant subalgebras A of $C_0(X)$ that contain approximate units and dense integrable subalgebras.*

The condition that the subalgebra of *integrable* functions be dense is of course stronger than the sufficient condition of Corollary 6.2 and weaker than the necessary condition of Lemma 6.4. In stating Theorem 6.5 we have omitted the refinements partly for the sake of simplicity, and partly because the condition (weak or strong) may turn out to be redundant. At least, there are two important cases in which we can dispense with it. We present one of these below and the other in the next section.

THEOREM 6.6. — *Let X have a compact open subgroup H. Let A be a closed invariant subalgebra of $C_0(X)$ with approximate unit. Then $\sigma(A)$ is an open subsemigroup of Y.*

PROOF. — The Pontrjagin orthogonal $H^{\perp}$ of H is also compact and open. Let $K = \sigma(A) \cap H^{\perp}$. This is a closed subsemigroup of a compact group, and is thus a group. It is not empty, because by Lemma 6.1 it contains at least o. Because $H^{\perp}$ is open Lemma 2.1 there is some non-zero $f \in A$ with $\sigma(f) \subseteq K$. By Theorem 5.1, f is periodic with respect to the Pontrjagin orthogonal $K^{\perp}$, wich must therefore be compact. Hence K itself is open. And the semigroup $\sigma(A)$, wich contains K, is also open, because

$$K + \sigma(A) \subseteq (A).$$

Hence the theorem.

Finally, let us observe that for an arbitrary angular semigroup S, our algebra $\alpha(S)$ coïncides with the Arens-Singer algebra A_0 described in [1]. In particular, these authors can identify the maximal ideal space of $\alpha(S)$.

THEOREM 6.7. — *For an angular semigroup S, there is natural one-one correspondence between the multiplicative functionals on $\alpha(S)$ and the continuous multiplicative mappings of S into the closed complex unit disk.*

7. The Phragmen-Lindelöf algebra. — The presence of an approximate unit in A, when X is the real line, leads to the following complete description of A.

THEOREM 7.1. — *Let X be the real line, and let $A \subsetneq C_0(X)$ be a closed invariant algebra with approximate unit. Then A must contain exactly one of the two functions $\frac{1}{x \pm i}$. Suppose A does not contain $\frac{1}{x - i}$. Then every function f in A extends to be analytic in the upper half-plane Z and continuous on its closure, including the point at infinity. And conversely, if a function F is analytic in the upper half-plane Z and continuous on the closure of Z in the Riemann sphere, then the restriction of F to the real line differs from some $f \in A$ by a constant.*

PROOF. — The spectrum $\sigma(A)$ is a perfect subset of the line and contains o. In particular, o is a limit point. HILLE-PHILLIPS [4] (Theorem 8.6.1, p. 264) states that there are only three closed subsemigroups of the line containing o as a non-isolated point : the right half-line, the left half-line, and the whole line. If $A \subsetneq C(X)$, then by Lemma 6.4, A must consist of all f that have spectrum in the right half line, or else of all f that have spectrum in the left half-line. Hence if we suppose $\sigma(A) =$ right half-line, then $\frac{1}{x + i} \in A$ and $\frac{1}{x - i} \notin A$. Then for every continuous φ, the Fourier transform

$$f(x) = \frac{1}{2\pi} \int_0^b e^{itx} \varphi(t)\, dt \qquad (0 < b < \infty)$$

belongs to A. If we replace x in the above integral by $z = x + iy$, then

$$F(z) = \frac{1}{2\pi}\int_0^b e^{itz}\,\varphi(t)\,dt$$

defines an entire function (of exponential type). This extended function is bounded in the upper half-plane Z because

$$|F(z)| \leq \frac{1}{2\pi}\int_0^b e^{-ty}\,|\varphi(t)|\,dt \leq \frac{1}{2\pi}\int_0^b |\varphi(t)|\,dt.$$

On the real axis, F coincide with the original Fourier transform f and vanishes at $\pm\infty$. Now we can use Phragmen-Lindelöf to prove that F even vanishes at infinity in the upper half-plane Z. The version in Titchmarsh [11] (Theorem 5.63, p. 179) is exactly what is needed.

Finally, let us take the uniform closure of the algebra B of all $f \in A$ that have continuous Fourier transforms φ of compact support. By Lemma 6.5, this uniform closure is all of A. That is, the most general $f \in A$ is the limit of f_n that are boundary values of F_n analytic in the open upper half-plane, continuous on its closure, and vanishing at infinity. This limit is uniform on the boundary of the half-plane hence by the maximum modulus principle the differences $F_m - F_n$ go to zero uniformly on the whole upper half-plane. Thus the F_n converge uniformly to an F that is analytic on the open half-plane Z, continuous on its closure, and zero at infinity. And f is the boundary value of F. Since f is any function in A, we have now proved the first half of our theorem.

Suppose conversely that F is analytic on the open half-plane Z, continuous on its closure, and zero at infinity. Define

$$F_n(z) = \frac{-n^2}{(z + in)^2}\,F(z).$$

Since F_n converges to F uniformly on the real axis, it is enough to prove that $\sigma(F_n)$ lies in the right half-line. The Fourier transform φ_n is a function, and because F_n is analytic in the upper half-plane, φ_n ban be computed by integrating along any horizontal line at height $R > 0$ above the real axis,

$$\varphi_n(t) = \int_{-\infty}^{\infty} F_n(x + i\,\mathrm{R})\,e^{(\mathrm{R}+ix)t}\,dx.$$

Now if $t < 0$, we let R approach ∞ and get $\varphi_n(t) = 0$. The proof of Theorem 7.1 is thus complete.

8. Vector groups. Half-spaces, maximal subalgebras. Cones, conformal subalgebras. — This section is devoted to special results for the

n-dimensional real vector group X^n. We write $x = (\xi_1, \ldots, \xi_n)$ for a point in X^n and $y = (\eta_1, \ldots, \eta_n)$ for a point in the dual space Y^n. And we write $x.y$ for the inner product $\xi_1\eta_1 + \ldots + \xi_n\eta_n$, so that the Pontrjagin coupling $y(x)$ or $x(y)$ can here be written $\exp(ix.y)$.

The first theorem of this section can be regarded as a generalization of the classical theorem that a convex cone lies in a half-space. With the aid of this fact about subsemigroups we can describe the maximal closed invariant subalgebras of $C_0(X^n)$; they are determined by the half-spaces (i. e. maximal subsemigroups) of Y^n. This fact might be used to define a natural « half-space » in an arbitrary locally compact abelian group. We can also describe the most general intersection of such maximal subalgebras. As a by-product we see that either of the following is sufficient to make a closed invariant algebra be all of $C_0(X^n)$: (1) that it separate points from compact sets, (2) that it be invariant under rotations (if $n \geqq 2$).

LEMMA 8.1. — *Let $y_0, \ldots, y_n$ be $n+1$ points in the vector space Y^n. In order that the convex hull K of $y_0, \ldots, y_n$ have* 0 *as an interior point it is necessary and sufficient that $y_1, \ldots, y_n$ be a basis for Y^n and that y_0 have all coordinates with respect to this basis strictly negative.*

(Notice that the choice of y_0 is arbitrary.)

PROOF. — If $y_1, \ldots, y_n$ do not form a basis, then they all lie in some hyperplane through 0. Even if y_0 does not lie in this hyperplane, 0 must belong to the face spanned by $y_1, \ldots, y_n$ or else not belong to K at all. If on the other hand $y_1, \ldots, y_n$ do form a basis, but y_0 has (say) its first coordinate $\geqq 0$, then every convex combination of $y_0, \ldots, y_n$ will have first coordinate $\geqq 0$, and again 0 cannot be an interior point of K.

Conversely suppose that $y_1, \ldots, y_n$ form a basis and that y_0 has all coordinates strictly negative, say $y_0 = (-\alpha_1, \ldots, -\alpha_n)$. The convex hull K is in fact a simplex. And if we set $\beta = 1 + \alpha_1 + \ldots + \alpha_n$, we see that 0 belongs to its interior because none of the coefficients in the convex combination

$$0 = \frac{1}{\beta} y_0 + \frac{\alpha_1}{\beta} y_1 + \ldots + \frac{\alpha_n}{\beta} y_n \quad \text{is zéro.}$$

LEMMA 8.2. — *Let S be a subsemigroup of the coordinatized vector group Y^n, and suppose S contains all the natural basis vectors* $(1, 0, \ldots, 0)$ *and* $(0, 1, 0, \ldots, 0)$ *and...and* $(0, \ldots, 0, 1)$. *Then either S lies in some half-space or else S contains some vector having all coordinates strictly negative.*

PROOF. — Assume S lies in no half-space. Let C be the closed convex cone generated by S. Since every such cone is the intersection of half-spaces, then C must be all of Y^n. (Actually the convex cone spanned by S is itself Y^n, It is unnecessary to take limits. But we shall not need this sharper fact.)

Hence we can find real $\beta_1, \ldots, \beta_m$ all > 0 and $z_1, \ldots, z_m \in S$ such that

$$\beta_1 z_1 + \ldots + \beta_m z_m = (-1, \ldots, -1).$$

Replacing the β_j by nearby rationnals γ_j we have

$$\gamma_1 z_1 + \ldots + \gamma_m z_m = (-\alpha_1, \ldots, -\alpha_n)$$

with all α_i strictly negative. And if we then multiply through by the product of the denominators of the β_j, we produce some point actually in S that has all coordinates strictly negative, proving Lemma 8.2.

THEOREM 8.3. — *Let S be a closed subsemigroup of an n-dimensional real vector group Y^n, and suppose S is contained in no half-space. Then S is a group (and hence is the direct sum of a lattice-point subgroup and a vector subspace).*

PROOF. — It is enough to prove that $-y_0 \in S$ whenever $y_0 \in S$. Since S lies in no hyperplane we can find $y_1, \ldots, y_{n-1}$ such that $y_0, \ldots, y_{n-1}$ form a basis for Y^n. And then by Lemma 8.2 we can find

$$y_n = -\alpha_0 y_0 - \ldots - \alpha_{n-1} y_{n-1} \quad \text{with all } -\alpha_i < 0.$$

Hence by Lemma 8.1, 0 is an interior point of the simplex spanned by $y_0, \ldots, y_n$ and y_0 must in turn have all its coordinates strictly negative with respect to the basis $y_1, \ldots, y_n$.

From now until the end of the proof we shall use the basis $y_1, \ldots, y_n$. The integral linear combinations of these vectors form a lattice-point subgroup L of Y^n, and the quotient Y^n/L is an n-dimensional torus. In particular Y^n/L is compact, hence some subsequence $\pi(m_k y_0)$ of the images $\pi(my_0)$ of the positive integral multiples of y_0 must converge to the identity element of the torus. [Proof : Some subsequence $\pi(n_k y_0)$ must converge to something. Define $m_k = n_{k+1} - n_k$]. Hence for large m_k, we can find integers $-h_1, \ldots, -h_n$ (necessarily large negative integers) such that $m_k y_0 - (-h_1 y_1 - \ldots - h_n y_n)$ is arbitrarily small. Hence

$$(m_k - 1) y_0 + h_1 y_1 + \ldots + h_n y_n$$

will be arbitrarily close to $-y_0$, and since S is closed Theorem 8.3 is proved.

COROLLARY 8.4. — *Let S be a closed subsemigroup of Y^n, and locally a set of multiplicity. Then either $S = Y^n$ or S lies in some half-space.*

PROOF. — By Theorem 8.3, if S does not lie in a half-space, then S is the direct sum of a lattice-point subgroup and a vector subspace. And by Lemma 5.2, then S cannot be a set of multiplicity unless $S = Y^n$.

Let H be a closed half-space in Y^n. As in Theorem 4.10 above, we define $\alpha(H)$ to be the closed invariant subalgebra of $C_0(X^n)$ consisting of all f that have spectrum in H.

THEOREM 8.5. — *The maximal closed invariant subalgebras of $C_0(X^n)$ are exactly the algebras $\alpha(H)$ defined by the various half-spaces H in the dual space Y^n. Every closed invariant subalgebra of $C_0(X^n)$ is contained in a maximal one. In particular every closed invariant subalgebra is anti-symmetric, i. e. never contains a pair f and $\bar{f}$ with $f \not\equiv 0$.*

PROOF. — Let A be a closed invariant proper subalgebra of $C_0(X^n)$, with spectrum $\sigma(A)$. Then $\sigma(A)$ is contained in some half-space H. Hence $\alpha(H)$ contains A. And the same argument applied to $A = \alpha(H)$ shows that $\alpha(H)$ is maximal, for H cannot be contained in any other half-space.

COROLLARY 8.6. — *Let $f_1 \in C_0(X^n)$ have a spectrum that lies in no half space of Y^n. Then every $f \in C_0(X^n)$ is a uniform limit of linear combinations of pointwise products of translates of f_1.*

PROOF. — No $\alpha(H)$ contains f_1.

[Notice that if pointwise products are not allowed then we must demand that $\sigma(f)$ be all of Y^n.]

LEMMA 8.7. — *For some fixed non-zero $x \in X^n$, let the half-space H be defined as $[y : y.x \geq 0]$. For each $f \in C_0(X^n)$ define f_x to be the restriction of f to the one-dimensional subspace T of X^n spanned by x. Then $\alpha(H)$ consists of those f such that f_x has spectrum in the right half-line* (or equivalently, by Theorem 7.1, *f_x extends to be analytic and zero at infinity in the upper half-plane of the complexification of T*).

PROOF. — Write $\alpha^+(H)$ for the f such that f_x has spectrum in the right half-line. The mapping $f \to f_x$ is a homomorphism of $C_0(X^n)$ onto $C_0(T)$. Our $\alpha^+(H)$ is the complete inverse image of the subalgebra $A^+(T) =$ all $f \in C_0(T)$ that have spectrum in the right half-line. Hence $\alpha^+(H)$ is closed and invariant.

To prove the lemma, it will be enough to show that $\alpha(H) \subseteq \alpha^+(H)$. For clearly $\alpha^+(H) \subsetneq C_0(X^n)$. On the other hand, by Lemma 8.5 above, $\alpha(H)$ is maximal. Suppose $f \in \alpha(H)$. By REITER [5], Theorem 1, the spectrum of f_x lies in the closure of the image of $\sigma(f)$ under the quotient $Y^n \to Y^n/T^\perp$. Hence $\sigma(f_x)$ lies in the image of H, which is exactly the right half-line. And the lemma is proved.

Notice that since $\alpha(H)$ is invariant under translation, then the restriction of $f \in \alpha(H)$ to any line parallel to T, which can be written $g(\tau) = f(x_0 + \tau x)$, will also have spectrum in the right half-line.

COROLLARY 8.8. — *Let A be a closed invariant subalgebra of $C_0(X^n)$ that separates points from compact sets* ($f = 0$ at the point, $f \equiv 1$ on the compact set). *Then A is all of $C_0(X^n)$.*

PROOF. — By Theorem 8.6, if $A \subsetneq C_0(X)$ then $A \subseteq$ some $\alpha(H)$. Let H be defined by $x \in X^n$, and let T be the line spanned by x. (*See* Lemma 8.7.) Then since each f_x on T is the boundary values of an analytic function, it is impossible to separate a point of T from a closed subinterval of T.

[Notice, however, that we do not claim any sort of « quasi-analyticity » for $\alpha(H)$. For $\alpha(H)$ will always contain non-zero f that vanish on infinite open cylinders parallel to T.]

COROLLARY 8.9. — *For $n \geqq 2$, let A be a closed subalgebra of $C_0(X^n)$ invariant under proper euclidean motions of X^n. Then A is all of $C_0(X^n)$.*

PROOF. — $\sigma(A)$ is invariant under proper rotations, and hence $= X^n$.

(Notice that the above argument fails for $n = 1$, since the real line has no non-trivial proper euclidean motions.)

Finally, let us characterize the intersections of maximal invariant subalgebras. If $H_1, H_2, \ldots$ are half-spaces in Y^n, then clearly $\cap\, \alpha(H_k) = \alpha(\cap\, H_k)$. If the cone $C = \cap\, H_k$ fails to have an interior then it will lie in some hyperplane, and hence (*see* Lemma 8.4) $\alpha(C)$ will consist of the zero function alone. If, on the other hand, C does have an interior, then C is an angular subsemigroup of Y^n and hence $\alpha(C)$ is the unique closed invariant subalgebra of $C_0(X^n)$ attached to C. Since C is invariant under homotheties ($y \to \lambda y$, $\lambda > 0$) of Y^n, then $\alpha(C)$ is invariant under the transformations of $C_0(X^n)$ induced by homotheties of X^n. Conversely, it is clear that cones are the only closed subsemigroups of X^n invariant under homotheties. Hence we have proved :

THEOREM 8.10. — *The intersection of maximal closed invariant subalgebras of $C_0(X^n)$ is a subalgebra A invariant under all proper conformal linear transformations of X^n. Conversely, any closed subalgebra A invariant under proper conformal linear transformations is the intersection of maximal subalgebras.*

We may omit the word "invariant" in describing A above because a linear angle-preserving transformation is the sum of a homothety and a translation. The converse can be strengthened by requiring in addition to translation only the one homothety $x \to 2x$ of X^n. For then $\sigma(A)$ will be invariant under $y \to \frac{1}{2} y$, and must still be a cone.

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