James Glimm

Two cartesian products which are euclidean spaces


<http://www.numdam.org/item?id=BSMF_1960__88__131_0>
TWO CARTESIAN PRODUCTS WHICH ARE EUCLIDEAN SPACES

by

JAMES GLIMM

(Princeton) (1).

Whitehead has given an example of a three-dimensional manifold \( W \) which is not (homeomorphic to) \( E^3 \), Euclidean 3-space [3]. We prove the following theorem about \( W \), the first statement of which is due to A. Shapiro.

**Theorem.** — If \( W \) is the manifold described below then \( W \times E^1 \) is homeomorphic to \( E^4 \). Also \( W \times W \) is homeomorphic to \( E^3 \times W \) (which is homeomorphic to \( E^6 \)).

That \( W \) is not homeomorphic to \( E^3 \) was proved in [1], [2]. In [1] it is shown that no cube in \( W \) contains \( W_0 \) (defined below), which implies \( W \) is not \( E^3 \). The homeomorphism \( W \times E^1 \approx E^4 \) can be used to show the existence of a two element (and so compact) group of homeomorphisms of \( E^3 \) onto itself whose fixed point set is \( W \). The problem of showing that \( W \times W \) is homeomorphic to \( E^6 \) was suggested to the author by L. Zippin.

Let \( W_0, W_1, R_o, R_1 \) be solid tori with \( W_0 \) simply self-linked in the interior of \( W_1 \) (see fig. 1) and \( R_0 \) trivially imbedded in the interior of \( R_1 \). Let \( I_0 \) and \( I_1 \) be closed bounded intervals of \( E^1 \) with \( I_0 \) contained in the interior of \( I_1 \). Let \( w \) (resp. \( r \)) be a 3-cell in the interior of \( W_0 \) (resp. \( R_0 \)), let \( e \) (resp. \( f, g \)) be a homeomorphism of \( E^3 \) (resp. \( E^3, E^1 \)) onto itself with \( e(W_0) = W_1 \) (resp. \( f(R_0) = R_1, g(I_0) = I_1 \)) and \( e \mid w \) (resp. \( f \mid r \)) the identity. Let

\[
W_n = e^n(W_0), \quad R_n = f^n(R_0), \quad I_n = g^n(I_0).
\]

Let \( W = \bigcup_{n=1}^{\infty} W_n \), we suppose that

\[
E^2 = \bigcup_{n=1}^{\infty} R_n, \quad E^3 = \bigcup_{n=1}^{\infty} I_n.
\]

(1) Fellow of the National Science Foundation (U. S. A.).
Let $S = \{ h | A : A \subseteq E^3, h \text{ is a homeomorphism of } E^3 \text{ onto itself which is the identity outside a compact set } \}$; we further suppose $e \in S$, $f | R_\lambda \in S$ and $\lambda'(R_\lambda) = W_0$ for some $\lambda'$ in $S$.

**Proof.** We prove both statements simultaneously. Let $V_n$ denote $I_n$ (resp. $W_n$), $V$ denote $E^3$ (resp. $W$). For each positive integer $n$, we construct a homeomorphism $h_n : W_n \times V_n \rightarrow R_n \times V_n$ with the properties

1. $h_n(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1}$;
2. $h_n | W_{n-2} \times V_{n-2} = h_{n-1} | W_{n-2} \times V_{n-2}$ ($n \geq 2$).

Suppose we have constructed all the $h_n$'s. Then we define

$$\Phi : W \times V \rightarrow E^3 \times V$$

as follows. If $(x, y) \in W \times V$, then for some $n$, $(x, y) \in W_n \times V_n$. Let $\Phi(x, y) = h_{n+1}(x, y)$. By (2) we see that $\Phi$ is well-defined, by (1) we see that $\Phi$ is onto. Since $h_n$ is a homeomorphism, $\Phi$ is also.

Suppose the following lemma is true. Using the lemma, we will construct the $h_n$.

**Lemma.** If we are given a homeomorphism $\beta' : w \times V_0 \rightarrow R_0 \times V_0$ (into), and if $\beta'$ has the form $\lambda' | w \times I$ where $\lambda'$ is a homeomorphism in $S$ of $W_0$ onto $R_0$, then there is a homeomorphic extension $\beta$ of $\beta'$,

$$\beta : W_1 \times V_1 \rightarrow R_1 \times V_1, \quad \beta(W_0 \times V_0) = R_0 \times V_0,$$
and \( \beta \mid \text{Bdry}(W_1 \times V_1) = \lambda \times I \) for \( \lambda \) some homeomorphism in \( S \) of \( W_1 \) onto \( R_1 \).

Let \( \lambda' \) be a homeomorphism in \( S \) mapping \( W_0 \) onto \( R_0 \). Let \( h_1 = \beta \), the extension of \( \beta' = (\lambda' \mid w) \times I \) given by the lemma. We suppose inductively that for \( n \) a positive integer greater or equal to 2, \( h_{n-1} \) has been constructed, and \( h_{n-1} \mid \text{Bdry}(W_{n-1} \times V_{n-1}) = \gamma \times I \), for \( \gamma \) some homeomorphism in \( S \) of \( W_{n-1} \) onto \( R_{n-1} \). We note that \( h_1 \) has this property. Observe that \((\gamma^{-1} \times I) h_{n-1} \) is a homeomorphism of \( W_{n-1} \times V_{n-1} \) onto itself leaving the boundary pointwise fixed. Let \( h \) be the extension of this map to \( W_n \times V_n \) which is the identity on \( W_n \times V_n \)-Interior \((W_{n-1} \times V_{n-1})\). Let \( r' \) be a 3-cell with Interior \( R_{n-1} \supset r' \supset R_{n-2} \). Let \( w' = \gamma^{-1}(r') \). Let \( k : W_n \rightarrow W_n \) be a homeomorphism in \( S \), \( k \mid (W_n\text{-Interior } W_{n-1}) = \text{identity}, k(w') \subset w \). Let \( \bar{\beta} \) be the extension of \( \gamma k^{-1} \times I \mid w \times V_{n-1} \) to a homeomorphism of \( W_n \times V_n \) onto \( R_n \times V_n \) as given by the lemma. Let \( h_n = \bar{\beta}(k \times I) h \). We check that \( h_n \) satisfies (1) and (2),

\[
h_n(W_{n-1} \times V_{n-1}) = \bar{\beta}(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1}.
\]

If \( z \in W_{n-2} \times V_{n-2} \), then \((k \times I) h(z) \in w \times V_{n-1} \) and

\[
h_n(z) = \bar{\beta}(k \times I) h(z) = (\gamma^{-1} \times I)(k \times I)(\gamma^{-1} \times I) h_{n-1}(z) = h_{n-1}(z)
\]
as asserted. Also

\[
h_n \mid \text{Bdry}(W_n \times V_n) = \bar{\beta}(k \times I) h \mid \text{Bdry}(W_n \times V_n)
\]

\[
= \lambda k \times I \mid \text{Bdry}(W_n \times V_n),
\]

where the last equality arises from the form of \( \bar{\beta} \) on \( \text{Bdry}(W_n \times V_n) \) and the fact that \((k \times I)(\text{Bdry}(W_n \times V_n)) = \text{Bdry}(W_n \times V_n) \). Thus \( h_n \) satisfies the induction hypothesis and all the \( h_n \) can be defined, if we prove the lemma.

**Proof of Lemma.** — Given \( \beta' = \lambda' \mid w \times I : w \times V_0 \rightarrow R_0 \times V_0 \), we can extend \( \lambda' \mid w \) to a homeomorphism in \( S \lambda \) of \( W_1 \) onto \( R_1 \). In fact let \( j \) be a homeomorphism in \( S \lambda \) of \( R_0 \) onto itself which maps \( R_0 \) onto \( R_0 \) and \( \lambda'(w) \) into \( R \). Let

\[
\lambda = j^{-1} f j \lambda' e^{-1}.
\]

Then \( \lambda \) is a homeomorphism in \( S \lambda \) of \( W_1 \) onto \( R_1 \) and \( \lambda \mid w = j^{-1} f j \lambda' \mid w = \lambda'(w) \mid w \) so \( \lambda \) is the desired extension of \( \lambda' \mid w \). It is now sufficient to construct a homeomorphism \( h \) of \( W_1 \times V_1 \) onto itself which leaves \( w \times V_1 \) pointwise fixed with \( h \mid \text{Bdry}(W_1 \times V_1) = \mu \times I \) for some \( \mu \) in \( S \lambda \) which maps \( W_1 \) onto \( W_1 \), and with \( h(W_0 \times V_0) = \lambda^{-1}(R_0) \times V_0 \). In fact \((\lambda \times I) h = \beta \) is a homeomorphism of \( W_1 \times V_1 \) onto \( R_1 \times V_1 \), \( \beta \) extends \( \beta' \), and

\[
\beta(W_0 \times V_0) = \lambda \lambda^{-1}(R_0) \times V_0 = R_0 \times V_0,
\]

\[
\beta \mid \text{Bdry}(W_1 \times V_1) = \lambda \mu \times I \mid \text{Bdry}(W_1 \times V_1).
\]
The homeomorphism $h$ will be given as the product of four homeomorphisms $A$, $\Sigma$, $\Delta$ and $P$ of $W_1 \times V_1$ onto itself. $A$, $\Sigma$ and $\Delta$ will each leave $\text{Bdry } (W_1 \times V_1) \cup (w \times V_0)$ pointwise fixed. $A$ will lift the dark portion of $W_0$, $\Sigma$ will slide this lifted part away from the link, and $\Delta$ will drop the image under $\Sigma A$ of the dark part of $W_0$ back into its original plane. We suppose $W_1$ is $D \times C$ where $D$ is the square $\{ (u, v) : 0 \leq u, v \leq 20 \}$ and $C$ is the circle $\{ \theta : 0 \leq \theta < 2\pi \}$. We suppose that

$$W_0 \subset \{ (u, v) : 0 \leq u, v \leq 10 \} \times C, \quad w \subset D \times [0 : 6 \leq \theta < 2\pi],$$

the link in $W_0 \subset D \times [0 : .5 \leq \theta < 1]$. Let $\alpha, \beta, \gamma, \delta$ be functions on $C$, let $\alpha, \beta, \gamma, \delta$ be functions on $[0, 20]$, defined as follows. Let

$$\alpha([0, 2]) = 1, \quad \alpha([4, 2\pi]) = 0, \quad \beta(0) = 0,$$

$$\beta([1.5, 1.4]) = 1, \quad \beta([0, 2\pi]) = 0,$$

$$\gamma([0, 1]) = 0, \quad \gamma([2, 2\pi]) = 1, \quad \delta([0, 1]) = 0,$$

$$\delta([1.5, 3]) = 1, \quad \delta([5, 2\pi]) = 0,$$

and let $\alpha, \beta, \gamma, \delta$ be linear on intervals for which they are not defined above. Let

$$\alpha(0) = 0, \quad a([9, 10]) = 1, \quad a(20) = 0,$$

$$b([0, 10]) = 0, \quad b([11, 12]) = 1, \quad b(20) = 0,$$

$$c(0) = 0, \quad c([9, 12]) = 1, \quad c(20) = 0,$$

and let $\alpha, \beta, \gamma, \delta$ be linear on intervals for which they are not defined above. Let $\varepsilon$ be a continuous map of $W_1$ into $[0, 1]$ such that $\varepsilon(u, v, \theta) = \alpha(\theta)$ for $(u, v, \theta)$ in the dark part of $W_0$, $\varepsilon = 0$ on the rest of $W_0$ and on Bdry $W_1$. If $(u, v), (x, y) \in D$, $\theta, \psi \in C$, let

$$\Lambda(u, \theta, x, y, \psi) = (u, v, \theta, x, y + 2\varepsilon(u, v, \theta) a(x) a(y), \psi),$$

$$\Sigma(u, v, \theta, x, y, \psi) = (u, v, \theta + \beta(\theta) a(x)) \times [(1 - \gamma(\theta)) b(y) + \gamma(\theta) c(y)] a(u) a(v), x, y),$$

$$\Delta(u, v, \theta, x, y, \psi) = (u, v, \theta, x, y - 2\delta(\theta) c(y) a(x) a(u) a(v), \psi).$$

If $V_1 = I_1$, we identify $I_0$ with $[10] \times [9, 10] \times [0] \subset W_1$ and $I_1$ with $[10] \times [0, 20] \times [0] \subset W_1$. Then $\Lambda$, $\Sigma$, and $\Delta$ map $W_1 \times I_1$ onto itself and $h' = \Delta \Sigma \Lambda | W_1 \times I_1$ (resp. $h' = \Delta \Sigma \Lambda$) is a homeomorphism of $W_1 \times V_1$ onto itself which leaves $\text{Bdry } (W_1 \times V_1) \cup (w \times V_0)$ pointwise fixed. For $(x, y, \psi) \in V_0$, $\Delta \Sigma \Lambda (W_0 \times (x, y, \psi))$ is trivially imbedded in $W_1 \times (x, y, \psi)$ and the projection $\text{proj}$ on $W_1$ of $\Delta \Sigma \Lambda (W_0 \times (x, y, \psi))$ is independent of $x, y, \psi$ in $V_0$. To see this it is sufficient to compute $\Delta \Sigma \Lambda (u, v, \theta, x, y, \psi)$ for $(u, v, \theta)$ in $W_0$, $x, y \in [9, 10]$ and $\theta$ a point of non-linearity of $\alpha, \beta, \gamma$ or $\delta$. Suppose we have a homeomorphism $\rho'$ of $W_1$ onto $W_1$ which leaves $\text{Bdry } W_1 \cup w$ pointwise fixed, and with $\rho'(W_0) = I_0^{-1}(A_0)$. Define $P = \rho' \times I : W_1 \times V_1 \to W_1 \times V_1$, define $h = P h'$. Then $h$ has the necessary properties.
Since $\lambda^{-1}(R_0)$ is trivially imbedded in $W_1$, it is in a 3-cell in the interior of $W_1$. There is a homeomorphism $g'$ of $E^3$ onto itself leaving $E^3 - W_1$ pointwise fixed and such that $g'(W_0)$ and $\lambda^{-1}(R_0)$ both lie in a 3-cell $u$ in the interior of $W_1$. It is evident that there is a homeomorphism in $S$ mapping $W_0$ onto $W_0'$ and so there is a homeomorphism $g''$ in $S$ of $E^3$ onto itself mapping $g'(W_0)$ onto $\lambda^{-1}(R_0)$. We can find a 3-cell $U$ outside of which $g''$ is the identity and a homeomorphism $\varphi$ mapping $U$ onto $u$ which is the identity on $\lambda^{-1}(R_0) \cup g'(W_0')$. Define $g = \text{identity outside } u, g = \varphi g'' \varphi^{-1}$ on $u$. Then $h = gg'$ is a homeomorphism leaving boundary $W_1$ fixed and mapping $W_0$ onto $\lambda^{-1}(R_0)$. Since $w \subset \text{Interior } W_0', h(w) \subset \text{Interior } \lambda^{-1}(R_0)$ and since $w \subset \text{Interior } \lambda^{-1}(R_0)$ there is a homeomorphism $i$ of $E^3$ onto itself leaving $E^3 - \lambda^{-1}(R_0)$ fixed and mapping $h(w)$ into $w$. Let $U_0, u_0$ be 3-cells, with $U_0 \supset W_1, \lambda^{-1}(R_0) \supset u_0, \text{Interior } u_0 \supset w$ and let $\varphi_0$ be a homeomorphism of $U_0$ onto $u_0$ leaving $w$ pointwise fixed. Let $j = \varphi_0(ih)^{-1} \varphi_0^{-1}$ on $u_0, j = \text{identity on } W_1 - u_0$. Then $\rho' = jih$ is a homeomorphism of $W_1$ onto $W_1$,

$$\rho'(W_0') = jih \lambda^{-1}(R_0) = \lambda^{-1}(R_0),$$

$\rho' | \text{Bdry } W_1 = \text{identity}$ and $\rho' | w = \varphi_0(ih)^{-1} \varphi_0^{-1} iih | w = \varphi_0 | w = \text{identity}$. This completes the proof.

BIBLIOGRAPHIE.


(Manuscrit reçu le 30 novembre 1959.)

James Glimm,
Institute for advanced Study,
Princeton (Etats-Unis).