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Two cartesian products which are euclidean spaces


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TWO CARTESIAN PRODUCTS WHICH ARE EUCLIDEAN SPACES

BY

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WHITEHEAD has given an example of a three-dimensional manifold $W$ which is not (homeomorphic to) $E^3$, Euclidean 3-space [3]. We prove the following theorem about $W$, the first statement of which is due to A. SHAPIRO.

**Theorem.** — If $W$ is the manifold described below then $W \times E^1$ is homeomorphic to $E^4$. Also $W \times W$ is homeomorphic to $E^3 \times W$ (which is homeomorphic to $E^6$).

That $W$ is not homeomorphic to $E^3$ was proved in [1], [2]. In [1] it is shown that no cube in $W$ contains $W_0$ (defined below), which implies $W$ is not $E^3$. The homeomorphism $W \times E^1 \approx E^4$ can be used to show the existence of a two element (and so compact) group of homeomorphisms of $E^4$ onto itself whose fixed point set is $W$. The problem of showing that $W \times W$ is homeomorphic to $E^6$ was suggested to the author by L. ZIPPIN.

Let $W_0$, $W_1$, $R_0$, $R_1$ be solid tori with $W_0$ simply self-linked in the interior of $W_1$ (see fig. 1) and $R_0$ trivially imbedded in the interior of $R_1$. Let $I_0$ and $I_1$ be closed bounded intervals of $E^1$ with $I_0$ contained in the interior of $I_1$. Let $w$ (resp. $r$) be a 3-cell in the interior of $W_0$ (resp. $R_0$), let $e$ (resp. $f$, $g$) be a homeomorphism of $E^3$ (resp. $E^3$, $E^1$) onto itself with $e(W_0) = W_1$ [resp. $f(R_0) = R_1$, $g(I_0) = I_1$] and $e \mid w$ (resp. $f \mid r$) the identity. Let

\[ W_n = e^n(W_0), \quad R_n = f^n(R_0), \quad I_n = g^n(I_0). \]

Let $W = \bigcup_{n=1}^{\infty} W_n$, we suppose that

\[ E^3 = \bigcup_{n=1}^{\infty} R_n, \quad E^4 = \bigcup_{n=1}^{\infty} I_n. \]

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Let $S = \{ h \mid A: A \subset \mathbb{R}^3, h \text{ is a homeomorphism of } \mathbb{R}^3 \text{ onto itself which is the}
\text{identity outside a compact set } \};$ we further suppose $e \in S$, $f \subset S$ and $\lambda(R_e) = W_0$ for some $\lambda'$ in $S$.

**Proof.** — We prove both statements simultaneously. Let $V_n$ denote $I_n$ (resp. $W_n$), $V$ denote $E^1$ (resp. $W$). For each positive integer $n$, we construct a homeomorphism $h_n: W_n \times V_n \to R_n \times V_n$ with the properties

1. $h_n(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1};$
2. $h_n|W_{n-2} \times V_{n-2} = h_{n-1}|W_{n-2} \times V_{n-2} \quad (n \geq 2).$

Suppose we have constructed all the $h_n$. Then we define

$$\Phi: W \times V \to \mathbb{R}^3 \times V$$

as follows. If $(x, y) \in W \times V$, then for some $n$, $(x, y) \in W_n \times V_n$. Let $\Phi(x, y) = h_{n+1}(x, y)$. By (2) we see that $\Phi$ is well-defined, by (1) we see that $\Phi$ is onto. Since $h_n$ is a homeomorphism, $\Phi$ is also.

Suppose the following lemma is true. Using the lemma, we will construct the $h_n$.

**Lemma.** — If we are given a homeomorphism $\beta': \omega \times V_0 \to R_0 \times V_0$ (into), and if $\beta'$ has the form $\lambda' \mid \omega \times I$ where $\lambda'$ is a homeomorphism in $S$ of $W_0$ onto $R_0$, then there is a homeomorphic extension $\beta$ of $\beta'$,

$$\beta: W_1 \times V_1 \to R_1 \times V_1, \quad \beta(W_0 \times V_0) = R_0 \times V_0,$$
and $\beta | \text{Bdry}(W_1 \times V_1) = \lambda \times I$ for $\lambda$ some homeomorphism in $S$ of $W_1$ onto $R_1$.

Let $\lambda'$ be a homeomorphism in $S$ mapping $W_0$ onto $R_0$. Let $h_1 = \beta$, the extension of $\beta' = (\lambda' | w) \times I$ given by the lemma. We suppose inductively that for $n$ a positive integer greater or equal to 2, $h_{n-1}$ has been constructed, and $h_{n-1} | \text{Bdry}(W_{n-1} \times V_{n-1}) = \gamma \times I$, for $\gamma$ some homeomorphism in $S$ of $W_{n-1}$ onto $R_{n-1}$. We note that $h_1$ has this property. Observe that $(\gamma^{-1} \times I) h_{n-1}$ is a homeomorphism of $W_{n-1} \times V_{n-1}$ onto itself leaving the boundary pointwise fixed. Let $h$ be the extension of this map to $W_n \times V_n$ which is the identity on $W_n \times V_n - \text{Interior}(W_{n-1} \times V_{n-1})$. Let $r'$ be a 3-cell with Interior $R_{n-1} \supset r' \supset R_{n-2}$. Let $w' = \gamma^{-1}(r')$. Let $k : W_n \to W_n$ be a homeomorphism in $S$, $k | (W_n - \text{Interior } W_{n-1}) = \text{identity}$, $k(w') \subset w$. Let $\beta$ be the extension of $\gamma^{-1} \times I | w \times V_{n-1}$ to a homeomorphism of $W_n \times V_n$ onto $R_n \times V_n$ as given by the lemma. Let $h_n = \beta(k \times I) h$. We check that $h_n$ satisfies (1) and (2),

$$h_n(W_{n-1} \times V_{n-1}) = \beta(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1}.$$

If $z \in W_{n-2} \times V_{n-2}$, then $(k \times I) h(z) \in w \times V_{n-1}$ and

$$h_n(z) = \beta(k \times I) h(z) = (\gamma^{-1} \times I) (k \times I) (\gamma^{-1} \times I) h_{n-1}(z) = (k \times I) h_{n-1}(z)$$

as asserted. Also

$$h_n | \text{Bdry}(W_n \times V_n) = \beta(k \times I) h | \text{Bdry}(W_n \times V_n) = (k \times I) | \text{Bdry}(W_n \times V_n),$$

where the last equality arises from the form of $\beta$ on $\text{Bdry}(W_n \times V_n)$ and the fact that $(k \times I)(\text{Bdry}(W_n \times V_n)) = \text{Bdry}(W_n \times V_n)$. Thus $h_n$ satisfies the induction hypothesis and all the $h_n$ can be defined, if we prove the lemma.

**Proof of Lemma. —** Given $\beta' = \lambda' | w \times I : w \times V_0 \to R_0 \times V_0$, we can extend $\lambda' | w$ to a homeomorphism in $S \lambda$ of $W_1$ onto $R_1$. In fact let $j$ be a homeomorphism in $S$ of $R_1$ onto itself which maps $R_0$ onto $R_0$ and $\lambda'(w)$ into $r$. Let

$$\lambda = j^{-1} f j \lambda' e^{-1}.$$

Then $\lambda$ is a homeomorphism in $S$ of $W_1$ onto $R_1$ and $\lambda | w = j^{-1} f j \lambda' | w = \lambda' | w$ so $\lambda$ is the desired extension of $\lambda' | w$. It is now sufficient to construct a homeomorphism $h$ of $W_1 \times V_1$ onto itself which leaves $w \times V_1$ pointwise fixed with $h | \text{Bdry}(W_1 \times V_1) = \mu \times I$ for some $\mu$ in $S$ which maps $W_1$ onto $W_1$, and with $h(W_0 \times V_0) = \lambda^{-1}(R_0) \times V_0$. In fact $(\lambda \times I) h = \beta$ is a homeomorphism of $W_1 \times V_1$ onto $R_1 \times V_1$, $\beta$ extends $\beta'$, and

$$\beta(W_0 \times V_0) = \mu^{-1}(R_0) \times V_0 = R_0 \times V_0,$$

$$\beta | \text{Bdry}(W_1 \times V_1) = \mu \times I | \text{Bdry}(W_1 \times V_1).$$
The homeomorphism \( h \) will be given as the product of four homeomorphism \( \Lambda, \Sigma, \Delta \) and \( P \) of \( W_1 \times V_1 \) onto itself. \( \Lambda, \Sigma, \Delta \) will each leave \( \text{Bdry} (W_1 \times V_1) \cup (w \times V_0) \) pointwise fixed. \( \Lambda \) will lift the dark portion of \( W_0 \), \( \Sigma \) will slide this lifted part away from the link, and \( \Delta \) will drop the image under \( \Sigma \Lambda \) of the dark part of \( W_0 \) back into its original plane. We suppose \( W_1 \) is \( D \times C \) where \( D \) is the square \( \{(u, v) : 0 \leq u, v \leq \pi \} \) and \( C \) is the circle \( \{ \theta : 0 \leq \theta < 2\pi \} \). We suppose that

\[
W_0 \subset \{(u, v) : 9 \leq u, v \leq 10 \} \times C, \quad w \subset D \times \{ \theta : 6 \leq \theta < 2\pi \},
\]

the link in \( W_0 \subset D \times \{ \theta : 9 \leq \theta < 1 \} \). Let \( \alpha, \beta, \gamma, \delta \) be functions on \( C \), let \( a, b, c \) be functions on \([0, 20]\), defined as follows. Let

\[
\begin{align*}
\alpha([0, 2]) &= 1, & \alpha([4, 2\pi]) &= 0, & \beta(0) &= 0,
\alpha([5, 4]) &= 1, & \beta([6, 2\pi]) &= 0, \\
\gamma([0, 1]) &= 0, & \gamma([2, 2\pi]) &= 1, & \delta([0, 1]) &= 0,
\delta([1.5, 3]) &= 1, & \delta([5, 2\pi]) &= 0,
\end{align*}
\]

and let \( \alpha, \beta, \gamma, \delta \) be linear on intervals for which they are not defined above. Let

\[
\begin{align*}
\alpha(0) &= 0, & a([9, 10]) &= 1, & a(20) &= 0, \\
b([0, 10]) &= 0, & b([11, 12]) &= 1, & b(20) &= 0, \\
c(0) &= 0, & c([9, 12]) &= 1, & c(20) &= 0,
\end{align*}
\]

and let \( \alpha, b, c \) be linear on intervals for which they are not defined above. Let \( \varepsilon \) be a continuous map of \( W_0 \) into \([0, 1]\) such that \( \varepsilon(u, v, \theta) = \alpha(\theta) \) for \((u, v, \theta)\) in the dark part of \( W_0 \), \( \varepsilon = 0 \) on the rest of \( W_0 \) and on \( \text{Bdry} W_1 \). If \((u, v), (x, y) \in D, \theta, \psi \in C \), let

\[
\begin{align*}
\Lambda(u, v, \theta, x, y, \psi) &= (u, v, \theta, x, y + \varepsilon(u, v, \theta)) a(x) a(y, \psi), \\
\Sigma(u, v, \theta, x, y, \psi) &= (u, v, \theta + \beta(\theta)) a(x) a(y, \psi) \\
x (1 - \gamma(\theta)) b(y) + \gamma(\theta) c(y)) a(u) a(v, x, y), \\
\Delta(u, v, \theta, x, y, \psi) &= (u, v, \theta, x, y + 2\delta(\theta)) c(y) a(x) a(u) a(v, \psi).
\end{align*}
\]

If \( V_i = I_i \), we identify \( I_0 \) with \([10] \times [9, 10] \times \{0\} \subset W_1 \) and \( I_1 \) with \([10] \times [0, 20] \times \{0\} \subset W_1 \). Then \( \Lambda, \Sigma, \Delta \) and \( \Lambda \) map \( W_1 \times I_1 \) onto itself and \( h' = \Delta \Sigma \Lambda \mid W_1 \times I_1 \) (resp. \( h' = \Delta \Sigma \Lambda \) ) is a homeomorphism of \( W_1 \times V_1 \) onto itself which leaves \( \text{Bdry} (W_1 \times V_1) \cup (w \times V_0) \) pointwise fixed. For \((x, y, \psi) \in V_0, \Delta \Sigma \Lambda (W_0 \times (x, y, \psi)) \) is trivially imbedded in \( W_1 \times (x, y, \psi) \) and the projection \( \pi_0 \) on \( W_1 \) of \( \Delta \Sigma \Lambda (W_0 \times (x, y, \psi)) \) is independent of \( x, y, \psi \) in \( V_0 \). To see this it is sufficient to compute \( \Delta \Sigma \Lambda (u, v, \theta, x, y, \psi) \) for \((u, v, \theta) \) in \( W_0 \), \( x, y \) in \([9, 10]\) and \( \theta \) a point of non-linearity of \( \alpha, \beta, \gamma \) or \( \delta \). Suppose we have a homeomorphism \( \rho' \) of \( W_i \) onto \( W_1 \) which leaves \( \text{Bdry} W_i \cup W \) pointwise fixed, and with \( \rho'(W_0) = \pi^{-1}(R_0) \). Define \( P = \rho' \mid I : W_1 \times V_1 \to W_1 \times V_1 \), define \( h = Ph' \). Then \( h \) has the necessary properties.
Since $\lambda^{-1}(R_o)$ is trivially imbedded in $W_1$, it is in a 3-cell in the interior of $W_1$. There is a homeomorphism $g'$ of $E^3$ onto itself leaving $E^3 - W_1$ pointwise fixed and such that $g'(W_o)$ and $\lambda^{-1}(R_o)$ both lie in a 3-cell $u$ in the interior of $W_1$. It is evident that there is a homeomorphism in $S$ mapping $W_0$ onto $W'_0$ and so there is a homeomorphism $g''$ in $S$ of $E^3$ onto itself mapping $g'(W'_0)$ onto $\lambda^{-1}(R_o)$. We can find a 3-cell $U$ outside of which $g''$ is the identity and a homeomorphism $\varphi$ mapping $U$ onto $u$ which is the identity on $\lambda^{-1}(R_o) \cup g'(W'_0)$. Define $g = \text{id} \text{outside } U, g = g'' \varphi^{-1}$ on $u$.

Then $h = gg'$ is a homeomorphism leaving boundary $W_1$ fixed and mapping $W_0$ onto $\lambda^{-1}(R_o)$. Since $w \subset \text{Interior } W'_0, h(w) \subset \text{Interior } \lambda^{-1}(R_o)$ and since $w \subset \text{Interior } \lambda^{-1}(R_o)$ there is a homeomorphism $i$ of $E^3$ onto itself leaving $E^3 - \lambda^{-1}(R_o)$ fixed and mapping $h(w)$ into $w$. Let $U_o, u_o$ be 3-cells, with $U_o \supset W_1, \lambda^{-1}(R_o) \supset u_o, \text{Interior } u_o \supset w$ and let $\varphi_o$ be a homeomorphism of $U_o$ onto $u_o$ leaving $w$ pointwise fixed. Let $j = \varphi_o(ih)^{-1} \varphi^{-1}$ on $u_o, j = \text{id}$ on $W_1 - u_o$. Then $\rho' = jih$ is a homeomorphism of $W_1$ onto $W_1$.

$$\rho' | \text{Bdry } W_1 = \text{id} \quad \text{and} \quad \rho' | w = \varphi_o(ih)^{-1} \varphi^{-1} \varphi_0 \text{id} | w = \varphi_0 | w = \text{id}. \quad \text{This completes the proof.}$$

**BIBLIOGRAPHIE.**


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