BULLETIN DE LA S. M. F.

MICHAEL F. ATIYAH

On the Krull-Schmidt theorem with application to sheaves

Bulletin de la S. M. F., tome 84 (1956), p. 307-317
<http://www.numdam.org/item?id=BSMF_1956__84__307_0>

Bull. Soc. Math. France,
84, 1956, p. 307 à 317.

ON THE KRULL-SCHMIDT THEOREM WITH APPLICATION TO SHEAVES;

BY

M. ATIYAH.

1. Introduction. — It is well-known that many standard algebraic results in the theory of groups, rings, modules, etc., can be proved more generally for suitable categories, in the sense of EILENBERG-MACLANE [5]. This has the usual advantages of abstraction. It singles out those features of a given algebraic structure which are essential to the results in question, and by so doing it extends the validity of these results to other domains. In this Note we shall be concerned with the Krull-Schmidt theorem for modules, which asserts under suitable conditions the existence and essential uniqueness of a direct decomposition into indecomposable factors. It is clear that if such a theorem is to have a meaning in some general category, then such notions as kernel, image and direct sum must be defined in the category, and must possess the usual properties. Such a category, called an exact category, has been considered by BUCHSBAUM [1]. Basing ourselves on his paper we then have at our disposal all the necessary notions with the usual properties. Our purpose will be to investigate conditions under which the Krull-Schmidt theorem holds in an exact category. This categorical formulation will then enable us to obtain a Krull-Schmidt theorem for suitable categories of sheaves. This is of special interest in algebraic geometry, and it was this case of the theorem which provided our motivation.

2. Exact categories. — We recall briefly the definition and elementary properties of an exact category, but for full details we refer to [1].

An exact category $\mathfrak{A}$ consists of :

(i) a collection of objects A ;

(ii) a distinguished object o called the zero object;

(iii) an abelian group $H(A, B)$ given for each pair of objects $A, B \in \mathfrak{A}$; the elements of $H(A, B)$ are called maps, and we write $\varphi : A \to B$ instead of $\varphi \in H(A, B)$;

(iv) a homomorphism $H(B, C) \otimes H(A, B) \to H(A, C)$ given for each triple of objects $A, B, C \in \mathfrak{A}$; we write $\psi\varphi$ for the image of $\psi \otimes \varphi$ in $H(A, C)$.

The primitive terms (i)-(iv) are subjected to certain axioms (I-V) which ensure that the usual properties are satisfied. Specifically we note the following :

1. There exists a unique identity map $e_A : A \to A$; a map $\varphi : A \to B$ is called an equivalence if there exists a map $\psi : B \to A$ such that

$$\psi\varphi = e_A, \qquad \varphi\psi = e_B;$$

ψ is then unique and is denoted by φ^{-1}.

2. If $\varphi : A \to B$, $\psi : B' \to A'$, then we define the induced homomorphism (of abelian groups)

$$H(\varphi, \psi) : H(B, B') \to H(A, A'),$$

by

$$H(\varphi, \psi)\beta = \psi\beta\varphi.$$

Then exactness in $\mathfrak{A}$ is defined in such a way that the following holds :

(i) $$0 \to A \to B \to C$$

is exact in $\mathfrak{A}$ if and only if the induced sequence

$$0 \to H(F, A) \to H(F, B) \to H(F, C)$$

is exact for all $F \in \mathfrak{A}$;

(ii) $$A \to B \to C \to 0$$

is exact in $\mathfrak{A}$ if and only if the induced sequence

$$H(A, F) \leftarrow H(B, F) \leftarrow H(C, F) \leftarrow 0$$

is exact for all $F \in \mathfrak{A}$.

3. The kernel of $\varphi : A \to B$ consists of a *pair* (K, σ) with $K \in \mathfrak{A}$, $\sigma : K \to A$, such that $0 \to K \xrightarrow{\sigma} A \xrightarrow{\varphi} B$ is exact; the pair (K, σ) is uniquely defined by φ, up to equivalence. Similar remarks hold for the cokernel, image and coimage of a map.

Other elementary properties of an exact category we shall use without explicit comment. However we must take care that all our propositions and proofs are formulated in terms which are defined in an exact category.

Primarily this means that we must avoid such notions as « element » or « sub-object » of an object. For this reason some of the proofs which we give will appear more involved then the corresponding proofs in, say, the category of modules, but no essential difficulty arises.

There is one result which we shall require later, and which is not found explicitly in [1]. This we shall now prove.

Lemma 1. — *Let*

$$\mathcal{L}\left\{\begin{array}{l} 0 \rightarrow B \xrightarrow{\beta} A \xrightarrow{\beta'} B', \\ 0 \rightarrow C \xrightarrow{\gamma} A \xrightarrow{\gamma'} C' \end{array}\right.$$

be a pair of exact sequences in $\mathcal{A}$. *Then* $\gamma'\beta$ *is a monomorphism* (1) *if and only if* $\beta'\gamma$ *is a monomorphism.*

Proof. — First we observe that the lemma holds in the category of abelian groups; for in that case $\gamma'\beta$ is a monomorphism if and only if $\beta(B) \cap \gamma(C) = 0$. Thus for all $F \in \mathcal{A}$, the lemma holds for the induced pair of sequences $H(F, \mathcal{L})$. But $\gamma'\beta : B \rightarrow C'$ is a monomorphism if and only $H(e_F, \gamma'\beta) : H(F, B) \rightarrow H(F, C')$ is a monomorphism for all $F \in \mathcal{A}$. Combining these facts together the lemma follows.

Lemma 1*. — *Let*

$$\mathcal{R}\left\{\begin{array}{l} B \xrightarrow{\beta} A \xrightarrow{\beta'} B' \rightarrow 0, \\ B \xrightarrow{\gamma} A \xrightarrow{\gamma'} C' \rightarrow 0 \end{array}\right.$$

be a pair of exact sequences in $\mathcal{A}$. *Then* $\gamma'\beta$ *is an epimorphism if and only if* $\beta'\gamma$ *is an epimorphism.*

Proof. — This follows from lemma 1 by duality, or directly by considering $H(\mathcal{R}, F)$ for all $F \in \mathcal{A}$.

Combining lemma 1 and 1*, we obtain :

Lemma 2. — *Let*

$$\begin{array}{l} 0 \rightarrow B \xrightarrow{\beta} A \xrightarrow{\beta'} B' \rightarrow 0, \\ 0 \rightarrow C \xrightarrow{\gamma} A \xrightarrow{\gamma'} C' \rightarrow 0 \end{array}$$

be a pair of exact sequences in $\mathcal{A}$. *Then* $\gamma'\beta$ *is an equivalence if and only if* $\beta'\gamma$ *is an equivalence.*

3. Chain conditions. — The simplest proof of the Krull-Schmidt theorem for modules assumes that both the ascending and descending chain conditions hold. These however are known to be unnecessarily restrictive conditions,

(1) $\varphi : A \rightarrow B$ is a monomorphism if $0 \rightarrow A \xrightarrow{\varphi} B$ is exact, and dually for an epimorphism.

and in particular these conditions are not satisfied for the categories of sheaves which we consider later. We shall consider therefore a weaker chain condition.

DEFINITION. — *A bi-chain* of $\mathfrak{A}$ is a sequence of triples $\{A_n, i_n, p_n\}$ with the following properties :

(i) $$A_n \in \mathfrak{A} \ (n \geqq 0),$$

(ii) $$i_n : A_n \to A_{n-1} \quad \text{is a monomorphism } (n \geqq 1),$$

(ii)* $$p_n : A_{n-1} \to A_n \quad \text{is an epimorphism } (n \geqq 1).$$

DEFINITION. — A bi-chain $\{A_n, i_n, p_n\}$ is said to *terminate* if there exists an integer N such that, for all $n \geqq N$, i_n and p_n are equivalences.

DEFINITION. — The *bi-chain condition* holds in $\mathfrak{A}$ if every bi-chain of $\mathfrak{A}$ terminates.

We note that the bi-chain condition is self-dual. Moreover, if $\{A_n, i_n, p_n\}$ is a bi-chain of $\mathfrak{A}$, Im (i_n) is a descending chain and Ker(p_n) is an ascending chain. Hence the bi-chain condition holds in $\mathfrak{A}$ if the ascending and descending chain conditions both hold.

Let $\{A_n, i_n, p_n\}$ be a bi-chain of $\mathfrak{A}$. Then $0 \to A_n \xrightarrow{i_n} A_{n-1}$ and $A_{n-1} \xrightarrow{p_n} A_n \to 0$ are exact, and so by the properties of $\mathfrak{A}$ mentioned in (**2**) of paragraph **2**, $H(p_n, i_n) : H(A_n, A_n) \to H(A_{n-1}, A_{n-1})$ is a monomorphism. Thus $\{H(A_n, A_n), H(p_n, i_n)\}$ is a descending chain of abelian groups. We say that such a descending chain *terminates* if $H(p_n, i_n)$ is an isomorphism for all sufficiently large n.

LEMMA 3. — *Let $\{A_n, i_n, p_n\}$ be a bi-chain of $\mathfrak{A}$. Then $\{A_n, i_n, p_n\}$ terminates if and only if the descending chain $\{H(A_n, A_n), H(p_n, i_n)\}$ terminates.*

Proof. — If, for all $n \geqq N$, i_n and p_n are equivalences, then $H(p_n, i_n)$ is an isomorphism; in fact we have

$$H(p_n, i_n)\, H(p_n^{-1}, i_n^{-1}) = 1,$$

where 1 denotes the identity automorphism of $H(A_{n-1}, A_{n-1})$.

Conversely suppose that, for $n \geqq N$, $H(p_n, i_n)$ is an isomorphism. Then, for some $\varphi : A_n \to A_n$, we must have $e_{A_{n-1}} = i_n \varphi p_n$; but this implies that i_n is an epimorphism and p_n a monomorphism. Hence both are equivalences.

COROLLARY. — *Let $\mathfrak{A}$ be an exact category with the further structure :*

a. For all pairs $A, B \in \mathfrak{A}$, $H(A, B)$ is a finite-dimensional vector space over a field k;

b. For all pairs $\varphi, \psi \in \mathfrak{A}$, $H(\varphi, \psi)$ is a k-homomorphism.

Then the bi-chain condition holds in $\mathfrak{A}$.

Proof. — This follows immediately from lemma 3, since every descending chain of finite-dimensional vector spaces necessarily terminates.

4. The Krull-Schmidt theorem. — In this section we show how the bi-chain condition implies the Krull-Schmidt theorem.

DEFINITION. — The maps $i_r : A_r \to A$, $p_r : A \to A_r$ $(r = 1, \ldots, n)$ give a *direct decomposition* of A, written $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$, if

(i) $$p_r i_r = e_{A_r}, \qquad p_r i_s = 0 \qquad (r \neq s),$$

(ii) $$\sum_r i_r p_r = e_A.$$

If for some r, i_r (and hence also p_r) is an equivalence, the decomposition is *trivial;* in this case the remaining maps i_s and p_s $(r \neq s)$ are necessarily zero. Each A_r occurring in a direct decomposition of A is said to be a *direct factor* of A. If $i : B \to A$, $p : A \to B$ with $pi = e_B$, then B is a direct factor of A; for let $\mathrm{Ker}(p) = (C, j)$ and define $q : A \to C$ by $jq = e_A - ip$, then i, j, p, q give the direct decomposition $A = B \oplus C$. Finally, if $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$, then it is easy to show that there exists $B \in \mathfrak{A}$, unique up to equivalence, such that $B = A_2 \oplus \ldots \oplus A_n$; moreover we then have exact sequences

$$0 \to A_1 \xrightarrow{i_1} A \xrightarrow{\beta} B \to 0, \qquad 0 \leftarrow A_1 \xleftarrow{p_1} A \xleftarrow{\beta'} B \leftarrow 0.$$

DEFINITION. — A is *indecomposable* if every direct decomposition of A is trivial.

DEFINITION. — $A = A_1 \oplus \ldots \oplus A_n$ is a *Remak decomposition* of A if each A_r is indecomposable and non-zero [2].

LEMMA 4. — *If the bi-chain condition holds in $\mathfrak{A}$, then every non-zero $A \in \mathfrak{A}$ has a Remak decomposition.*

Proof. — Suppose that A is non-zero and has no Remak decomposition. Then A must have a non-trivial decomposition $A = A_1 \oplus B_1$ where at least one of the factors, say A_1, has no Remak decomposition. Repeating this process we obtain at the n-th stage $A_{n-1} = A_n \oplus B_n$. Put $A_0 = A$, and let

$$i_n : A_n \to A_{n-1}, \qquad p_n : A_{n-1} \to A_n$$

be the maps defining the decompositions. Then $\{A_n, i_n, p_n\}$ is a bi-chain of $\mathfrak{A}$ which does not terminate. This is a contradiction, and the lemma is therefore proved.

---

(2) We say A is *non-zero* if it is not equivalent to 0.

Lemma 5. — *Let $A \in \mathfrak{A}$, and let $\theta \in H(A, A)$. Then θ defines a bi-chain $\{A_n, i_n, p_n\}$ of $\mathfrak{A}$. Moreover, if this bi-chain terminates, then for sufficiently large n, A_n is a direct factor of A.*

Proof. — Roughly speaking A_n is $\theta^n A$, i_n is the natural inclusion $\theta^n A \to \theta^{n-1} A$, and p_n is $\theta : \theta^{n-1} A \to \theta^n A$. Precisely, let $\mathrm{Im}(\theta^n) = (A_n, j_n)$ where $j_n : A_n \to A$ is a monomorphism. Then there is a monomorphism $i_n : A_n \to A_{n-1}$ such that

$$j_n = i_1 i_2 \ldots i_n.$$

Also there exist epimorphisms $p_n : A_{n-1} \to A_n$, defined by

$$(1) \qquad j_n p_n = \theta j_{n-1}.$$

We define the epimorphism $q_n : A \to A_n$ by $q_n = p_n p_{n-1} \ldots p_1$. Then $\{A_n, i_n, p_n\}$ is a bi-chain of $\mathfrak{A}$. Suppose now that this terminates, then for sufficiently large n, i_n and p_n are equivalences. We assert that $q_n j_n : A_n \to A_n$ is then an equivalence. In fact we have

$$q_n j_n = i_{n+1} i_{n+2} \ldots i_{2n} p_{2n} \ldots p_{n+1};$$

to show this it is sufficient to show equality after premultiplying by the monomorphism j_n, but

$$\begin{aligned} j_n q_n j_n &= \theta j_{n-1} q_{n-1} j_n && \text{by (1)}, \\ &= \theta^n j_n && \text{by repeated application of (1)}, \end{aligned}$$

and

$$\begin{aligned} j_n(i_{n+1} \ldots i_{2n} p_{2n} \ldots p_{n+1}) &= j_{2n}(p_{2n} \ldots p_{n+1}) \\ &= \theta^n j_n && \text{by repeated application of (1)}. \end{aligned}$$

Put $q_n j_n = \sigma_n$, then σ_n^{-1} exists and we have

$$j_n : A_n \to A, \qquad \sigma_n^{-1} q_n : A \to A_n$$

with

$$(\sigma_n^{-1} q_n) j_n = e_{A_n}.$$

Hence A_n is a direct factor of A, as required.

Lemma 6. — *Let the bi-chain condition hold in $\mathfrak{A}$, and let $A \in \mathfrak{A}$ be indecomposable. Then every $\theta \in H(A, A)$ is either an equivalence or is nilpotent.*

Proof. — Let $\{A_n, i_n, p_n\}$ be the bi-chain defined by θ as in lemma 5. By hypothesis this terminates, and hence from lemma 5 A_n is a direct factor of A for sufficiently large n. But A is indecomposable; hence either j_n and q_n (defined as in lemma 5) are equivalences or they are zero. In the former case this means that θ^n and so θ is an equivalence, while in the latter $\theta^n = 0$, i. e. θ is nilpotent.

From lemma 4 and 6, the Krull-Schmidt theorem now follows by a standard argument (*see*, for instance, JACOBSON [6] or ZASSENHAUS [9]), but for completeness we shall give the proof here. It should be noticed that it is at this stage that we shall require lemma 2.

LEMMA 7. — *Let the bi-chain condition hold in* $\mathfrak{A}$, *and let* $A \in \mathfrak{A}$ *be indecomposable. If* $\theta + \varphi = \omega$ *where* $\theta, \varphi, \omega \in H(A, A)$ *and* ω *is an equivalence, then at least one of* θ, φ *is an equivalence.*

Proof. — Put $\theta' = \omega^{-1}\theta$, $\varphi' = \omega^{-1}\varphi$. It is sufficient to show that θ' or φ' is an equivalence. But we have $\theta' + \varphi' = e_A$, so that θ' and φ' commute. Hence at least one, say θ', is not nilpotent; for if $\theta'^n = \varphi'^n = 0$, then

$$e_A = (\theta' + \varphi')^{2n} = 0,$$

a contradiction. Then, by lemma 6, θ' is an equivalence.

LEMMA 8. — *Let the bi-chain condition hold in* $\mathfrak{A}$, *and let* $A \in \mathfrak{A}$ *be indecomposable. If* $\sum_{s=1}^{m} \theta_s = e_A$, *where* $\theta_s \in H(A, A)$, *then at least one of the* θ_s *is an equivalence.*

Proof. — We proceed by induction on $m \geqq 2$. For $m = 2$ we have lemma 7. For $m > 2$, suppose the result true for $m - 1$, and put $\theta = \theta_m$, $\varphi = \sum_{s=1}^{m-1} \theta_s$, $\omega = e_A$; then by lemma 7 and the inductive hypothesis at least one of the θ_s is an equivalence.

THEOREM 1. — *Let* $\mathfrak{A}$ *be an exact category in which the bi-chain condition holds. Then the Krull-Schmidt theorem holds in* $\mathfrak{A}$. *More precisely, every non-zero* $A \in \mathfrak{A}$ *has a Remak decomposition, and if*

$$A = A_1 \oplus \ldots \oplus A_n, \qquad A = A'_1 \oplus \ldots \oplus A'_m$$

are two Remak decompositions of A, *then* $m = n$ *and after re-ordering the suffixes* $A_i \cong A'_i$ (*i. e.* A_i *and* A'_i *are equivalent*).

Proof. — Let A be non-zero. By lemma 4, A has a Remak decomposition. Choose one in which the number of factors is a minimum, and let this number be $n(A)$. We prove the theorem by induction on $n(A)$. For $n = 1$ it is trivial; suppose it is true for $n - 1$. Then we have to prove the uniqueness of decomposition for any A with $n(A) = n$. Let $A = A_1 \oplus \ldots \oplus A_n$ be a Remak decomposition with n factors, and let $A = A'_1 \oplus \ldots \oplus A'_m$ be any other Remak decomposition. We denote by i_r, p_r, i'_s, p'_s the maps defining the decompositions. Put $\theta_s = p_1 i'_s p'_s i_1 : A_1 \to A_1$, then we have

$$\sum_{s=1}^{m} \theta_s = p_1 \left(\sum_{s=1}^{m} i'_s p'_s \right) i_1 = p_1 i_1 = e_{A_1}.$$

Since A_1 is indecomposable, we apply lemma 6 and deduce that, for some s, θ_s is an equivalence. Re-order the suffixes so that $s = 1$. Then we have the maps

$$A_1 \xrightarrow{p'_1 i_1} A'_1 \xrightarrow{p_1 i'_1} A_1,$$

in which the composition $\theta_1 = p_1 i'_1 p'_1 i_1$ is an equivalence. Hence A_1 is a direct factor of A'_1, and since A'_1 is also indecomposable this implies that $p'_1 i_1 : A_1 \to A'_1$ is an equivalence. Now we have exact sequences

$$0 \to A_1 \xrightarrow{i_1} A \xrightarrow{\alpha} B \to 0,$$
$$0 \to B' \xrightarrow{\alpha'} A \xrightarrow{p'_1} A'_1 \to 0,$$

where

$$B = A_2 \oplus \ldots \oplus A_n, \qquad B' = A'_2 \oplus \ldots \oplus A'_m.$$

By lemma 2, since $p'_1 i_1$ is an equivalence, $\alpha\alpha'$ is also an equivalence. Hence by inductive hypothesis $n - 1 = m - 1$, and after re-ordering the suffixes $A_i \cong A'_i (i > 2)$. As we already have $A_1 \cong A'_1$, the induction is established, and the theorem proved.

5. **Applications.** — We shall apply the preceding results in the following two cases :

(i) Let X be a complete algebraic variety over an algebraically closed field k, $\mathcal{O}$ the sheaf of local rings on X; we take for $\mathfrak{A}$ the category of coherent algebraic sheaves (the maps of $\mathfrak{A}$ being the $\mathcal{O}$-homomorphisms);

(ii) Let X be a compact complex manifold, $\mathcal{O}$ the sheaf of germs of holomorphic functions on X; we take for $\mathfrak{A}$ the category of coherent analytic sheaves (the maps of $\mathfrak{A}$ being the $\mathcal{O}$-homomorphisms).

For the definition of coherent sheaves, and for the proof that they form an exact category, we refer to Serre [8].

In both (i) and (ii) we have the basic theorem that, if $A \in \mathfrak{A}$, the cohomology groups $H^0(X, A)$ are finite-dimensional vector spaces over k, where in case (ii) k denotes the complex field (*see* [7] and [4]). If A, $B \in \mathfrak{A}$, then we can define a sheaf

$$\mathrm{Hom}_{\mathcal{O}}(A, B) \in \mathfrak{A},$$

and we have

$$H(A, B) = H^0(X, \mathrm{Hom}_{\mathcal{O}}(A, B)) \quad (\textit{see}\ [8]).$$

Thus condition (a) in the corollary of lemma 3 holds, and it is a trivial matter to verify (b). Hence we deduce that *the bi-chain condition holds in* $\mathcal{A}$, *in both cases* (i) *and* (ii). Applying theorem 1, we obtain :

THEOREM 2. — *Let* $\mathcal{A}$ *be the exact category of coherent sheaves on* (i) *a complete algebraic variety*, *or* (ii) *a compact complex manifold*, *then the Krull-Schmidt theorem holds in* $\mathcal{A}$.

Contained in the category of coherent sheaves there are some of special interest, namely the *locally free* sheaves (*see* [8]). These are important because they correspond in a one-to-one way (at least when X is connected) with vector bundles over X [algebraic bundles in case (i), analytic bundles in case (ii)] (*see* [8]). We should warn however that the locally free sheaves cannot, in any non-trivial way be considered as the objects of an exact category; this arises from the fact that the image of a locally free sheaf under an $\mathcal{O}$-homomorphism need not be locally free, and even if this holds for two homomorphisms φ, ψ it need not hold for $\varphi\psi$. However we have the following, which is sufficient for our purposes :

LEMMA 9. — *Let* $\mathcal{A}$ *be one of categories* (i) *or* (ii) *above, and let* $A \in \mathcal{A}$ *be locally free. Then every direct factor of* A *is locally free.*

Proof. — A sheaf $A \in \mathcal{A}$ is locally free if and only if A_x is a free $\mathcal{O}_x$-module for each $x \in X$ (*see* [8]). But, since $\mathcal{O}_x$ is a local ring, every direct factor of a free $\mathcal{O}_x$-module is itself free (*see* [3], chap. VIII, th. 6.1′). This proves the lemma.

COROLLARY. — *Let* E *be a vector bundle over* X (*assumed connected*), *and let* A *be the sheaf of germs of sections* [*regular algebraic in* (i), *holomorphic in* (ii)] *of* E. *Then* E *is indecomposable if and only if* A *is indecomposable.*

Proof. — If $E = E_1 \oplus E_2$, then $A = A_1 \oplus A_2$. Conversely, if $A = A_1 \oplus A_2$, then by lemma 9, A_1 and A_2 are locally free and so correspond to vector bundles E_1 and E_2, and $E = E_1 \oplus E_2$.

Combining this corollary with theorem 2, we obtain :

THEOREM 3. — *Let* $\mathcal{E}$ *be the class of vector bundles over* (i) *a connected complete algebraic variety*, *or* (ii) *a connected compact complex manifold*, *then the Krull-Schmidt theorem holds in* $\mathcal{E}$.

Remark. — Theorems 2 and 3 become false if we remove the completeness or compactness conditions. For instance, if X is an affine algebraic variety it is known that every vector bundle over X is a direct factor of a trivial bundle (*see* [8]); on the other hand there certainly exist non-trivial bundles over affine varieties, and this clearly contradicts theorem 3 (and so theorem 2). In particular if X is an affine algebraic curve, then this fact is already well-known in a different formulation; by results of [8] the failure of theorem 2 in this case reduces to the fact that the Krull-Schmidt theorem does not hold for finitely-generated modules over a Dedekind ring.

We conclude with a few comments on the bi-chain condition. Let $\mathfrak{A}$ be an exact category, and consider the following conditions :

(P) If $A \in \mathfrak{A}$, and $\theta \in H(A, A)$ is an epimorphism, then θ is an equivalence;

(P*) If $A \in \mathfrak{A}$, and $\theta \in H(A, A)$ is a monomorphism, then θ in an equivalence.

(P) holds if the ascending chain condition holds, and dually (P*) holds if the descending chain condition holds.

Lemma 10. — *If the bi-chain condition holds in $\mathfrak{A}$, then* (P) *and* (P*) *both hold in $\mathfrak{A}$.*

Proof. — Since the bi-chain condition is self-dual it is sufficient to show that (P) holds. Let $\theta : A \rightarrow A$ be an epimorphism ; constructing the corresponding bi-chain as in lemma 5, we take

$$A_n = A, \qquad i_n = e_A, \qquad p_n = \theta.$$

By hypothesis this bi-chain terminates, and so for large n, p_n is an equivalence, i. e. θ is an equivalence.

Corollary. — *Let $\mathfrak{A}$ be the exact category of coherent sheaves on* (i) *a complete algebraic variety, or* (ii) *a compact complex manifold, then conditions* (P) *and* (P*) *hold in $\mathfrak{A}$.*

We may remark finally that, although the descending chain condition doesnot hold in the category of coherent sheaves, the ascending chain condition does hold. In the algebraic case this is elementary (and does not require completeness), while in the compact complex case it is a consequence of a theorem of Cartan [2] ([3]).

The Institute for Advanced Study Princeton, N. J.

BIBLIOGRAPHY.

[1] D. A. Buchsbaum, *Exact categories and duality* (*Trans. Amer. math., Soc.*, vol. 80, 1955, p. 1-34).

[2] H. Cartan, *Idéaux de fonctions analytiques de n variables complexes* (*Ann. sc. Éc. Norm. Sup.*, t. 61, 1944, p. 149-198).

([3]) I am indebted to J.-P. Serre for this remark.

[3] H. CARTAN, and S. EILENBERG, *Homological algebra* (*Princeton math. Ser.*, n° 19, 1956).
[4] H. CARTAN and J.-P. SERRE, *Un théorème de finitude concernant les variétés analytiques compactes* (*C. R. Acad. Sc.*, t. 237, 1953, p. 128-130).
[5] S. EILENBERG and S. MACLANE, *General theory of natural equivalences* (*Trans. Amer. math. Soc.*, vol. 58, 1945, p. 231-294).
[6] N. JACOBSON, *Theory of Rings* (*Amer. math. Soc.*, Math. Surveys, 1942).
[7] J.-P. SERRE, *Sur la cohomologie des variétés algébriques.* (*J. Math. pures et appl.*, t. 35, 1956).
[8] J.-P. SERRE, *Faisceaux algébriques cohérents* (*Ann. Math.*, vol. 61, 1955, p. 197-278).
[9] H. ZASSENHAUS, *The Theory of Groups*, Chelsea, 1949.

(Manuscrit reçu en avril 1956).