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An application of the Morse theory to the topology of Lie-groups

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AN APPLICATION OF THE MORSE THEORY TO THE TOPOLOGY OF LIE-GROUPS (1);

BY

M. Raoul BOTT.

(Princeton).

1. INTRODUCTION.

This paper is primarily devoted to a detailed account of the results announced and sketched in [5]. It will be followed by a joint paper with H. SAMELSON on the loop-space of symmetric spaces in general.

We recall the principal results of [5]. Let G be a compact Lie group; $T' \subset G$ shall denote a torus in G and $C(T')$ the centralizer of T' in G. The space of loops on G is denoted by $\Omega(G)$. (For detailed definitions, *see* § 9.)

THEOREM A. — *If G is a connected, simply connected, compact Lie group, then the spaces $\Omega(G)$ aud $G/C(T')$ have the following properties :*

a. They are free of torsion;

b. Their odd Betti numbers vanish;

c. Their Betti numbers can be read off from the diagram of G.

The manner in which the diagram of G determines these Betti numbers is the following one. Let D denote this diagram on the tangent space, $\mathfrak{t}$, to a maximal torus $T \subset G$. We let $\mathcal{F}$ be a fundamental *chamber* of D and denote by Δ the fundamental *cells* of D. If $X \in \mathfrak{t}$, $\lambda(X)$ shall be defined as the number of planes of D *crossed* by the straight line joining X to the origin of $\mathfrak{t}$. The function λ is constant in each fundamental cell and this constant value is denoted by $\lambda(\Delta)$. In general if s is any line segment in $\mathfrak{t}$, $\lambda(s)$ shall equal the number of planes of D crossed by s. We denote by Γ the lattice of G in D.

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THEOREM B. — 1° *The Poincaré series of the loop-space, $\Omega(G)$, is given by*

$$P(\Omega(G);\, t) = \sum t^{2\lambda(\Delta)},$$

where Δ runs over the cells of the fundamental chamber $\mathcal{F}$.

2° *If X is a lattice point of Γ in* $\mathfrak{t}$, *and if $C(X)$ is the centralizer of the direction X, then the Poincaré series of $G/C(X)$ is given by*

$$P(G/C(X);\, t) = \left\{ \sum t^{2\lambda(\Delta)} \right\} t^{-2\lambda(X)}$$

where now the summation is extended over the cells of $\mathcal{F}$ which contain X in their closure.

As every $C(T')$ is conjugate to some $C(X)$, $X \in \Gamma$, part 2° of this theorem describes the homology of all the spaces $G/C(T')$. It also relates their Poincaré series to that of $\Omega(G)$.

These formulae can of course be expressed entirely in terms of the root-forms of G on $\mathfrak{t}$:

Let $\{\theta_i\}_{\mathcal{F}}$ $(i = 1, \ldots, m)$, denote the root-forms on $\mathfrak{t}$ which take positive values in $\mathcal{F}$. Also let W denote the Weyl group of G in $\mathfrak{t}$. Finally, if $X \in \mathfrak{t}$, define a function X^*, on W by :

$X^*(w)$ = number of the root-forms $\{\theta_i\}_{\mathcal{F}}$ whose values at X and $w \cdot X$ differ in sign. Thus,

$X^*(w)$ = number of planes of the infinitesimal diagram D', crossed by the line from X to $w \cdot X$. Finally let $\rho(X)$ denote the order of the subgroup of W which leaves X fixed. In terms of these notions, theorem B is equivalent to :

THEOREM B'. — 1° *The Poincaré series of $\Omega(G)$ is given by*

$$P(\Omega(G);\, t) = \frac{1}{|\Delta|} \int_{\mathcal{F}} t^{2\sum_{i=1}^{m} [\theta_i(X)]}\, dv,$$

where the integral is taken over $\mathcal{F}$, $|\Delta|$ denotes the volume of a fundamental cell of D, and $[a]$ denotes the greatest integer less than a;

2° *The Poincaré series of $G/C(X)$, X arbitrary in* $\mathfrak{t}$, *is given by*

$$P\{(G/C(X));\, t\} = \frac{1}{\rho(X)} \sum_{w \in W} t^{X^*(w)}.$$

In particular (for X in $\mathcal{F}$) the $2q^{\text{th}}$ Betti number of G/T is equal to the *number of elements in W which change the sign of precisely q of the root-forms $\{\theta_i\}_{\mathcal{F}}$.*

An immediate corollary of these theorems is that $\pi_3(G)$ is a free-group. If G is simple, then $\pi_3(G) = Z$.

At the time of publication of [5], these results contributed to the subject involved in two ways.

a. They gave a general proof that the spaces $G/C(T')$ and $\Omega(G)$ are torsion-free.

b. They gave new formulas for the Betti numbers of the spaces $G/C(T')$ and $\Omega(G)$.

With regard to *b*, it should be recalled that Borel [4 *a*] had already described the rational cohomology ring of $G/C(T')$ (as of all homogeneous spaces G/H, with H of maximal rang in G) in terms of the Weyl groups of G and $C(T')$. His construction therefore yields the Betti numbers of the $G/C(T')$ as a byproduct. In the case of G/T, for instance, Borel proves that over the rationals $H^*(G/T)$ is isomorphic to a quotient ring, $P(x_1, \ldots, x_l)/I^+$, of the polynomial ring in l variables modulo the ideal, I^+, generated by the invariants of positive degree of a group of reflections (the Weyl group of G) in $(x_1, \ldots, x_l)$.

The rational cohomology ring of $\Omega(G)$ can also be described in an explicit manner. Following [11 *b*] there exists a map, f, of an appropriate product of odd spheres

$$S^{2m_1-1} \times S^{2m_2-1} \times \ldots \times S^{2m_l-1}$$

into G, which induces an isomorphism onto in the rational cohomology groups. Hence, by Serre's $\mathcal{C}$-theory, f induces an isomorphism of the homotopy groups modulo finite groups. It follows that the map $\tilde{f}$ of the loop-spaces, which is determined by f, has the same property. Therefore $\tilde{f}^*$ will again be an isomorphism onto over the rationals. On the other hand, the Poincaré series of the space of loops on this product of spheres is

$$\{1 - t^{2(m_1-1)}\}^{-1} \times \ldots \times \{1 - t^{2(m_l-1)}\}^{-1}.$$

It will be seen in Appendix II how a comparison of these two expressions yields a new relation between the m_i's.

Concerning the results summarized under a, it should be remarked that Borel had already at the time of publication of [5] shown that G/T is free of torsion in all cases except E_6, E_7, E_8. Since then both he and Chevalley have announced proofs of this general fact, using a cell-subdivision of these spaces found independently by Chevalley and Harish-Chandra [4 *b*]. By means of this cell-subdivision Chevalley also obtained the formula for the Betti numbers of G/T given in theorem B'. Finally this same formula for the

Betti numbers of G/T was derived independently by BOREL and HIRZEBRUCH, using the general Riemann Roch theorem [6]. On the other hand, there is at this time no alternate proof that $\Omega(G)$ is free of torsion.

In this paper a generalization of the result concerning $\Omega(G)$ is proved. Theorems A, B, and B′ follow from it easily. For $X \in \mathfrak{t}$, let $N(X)$ be the orbit of $\exp X$ under the adjoint action of G on itself. Thus

$$N(X) = \bigcup_{\sigma \in G} \sigma \{ \exp X \} \sigma^{-1}.$$

Let $\tilde{N}(X)$ be the orbit of X under the Weyl group of G, and set P equal to the orbit of a general point, $\tilde{P} \in \mathfrak{t}$, under the lattice Γ. Finally $\tilde{S}(\mathfrak{t}, \tilde{N}(X), P)$ shall denote the set of straight lines joining a point of $\tilde{N}(X)$ to a point of P.

We study the space, $\Omega(G, N(X), P)$, of paths in G starting on $N(X)$ and ending at $P = \exp \tilde{P}$, and prove :

THEOREM C. — *The space $\Omega(G, N(X), P)$ is free of torsion for any $X \in \mathfrak{t}$. Its odd Betti numbers vanish. The Poincaré series of this space is given by*

$$P(\Omega(G, N(X), P); t) = \sum t^{2\lambda(s)},$$

where s runs over the straight-line segments of $\tilde{S}(\mathfrak{t}, \tilde{N}(X), \tilde{P})$.

As mentionned in [5], our procedure amounts to a straightforward application of the Morse theory to the Riemannian geometry of G. This theory, in its most geometrical and classical form, applies well to the study of a Lie group for two reasons. Firstly, G admits so many global isometries that focal points which usually are a local phenomena along a given geodesic, can be described in terms of global motions. This mobility of G is expressed here in the two theorems that the adjoint action of G, both on G and on the Lie algebra of G, is « variationally complete ».

Secondly, the indices which occur in the Morse theory, here all turn out to be even. Because of this very fortuitous circumstance the Morse inequalities must be actual equalities for purely « dimensional » reasons.

In a subsequent paper SAMELSON and I will study the general situation when a group, K, acts on a Riemann manifold in a variationally complete way. (This turns out to be the case for instance in the symmetric spaces.) In the general case the Morse inequalities do not become equalities for dimensional reasons. Nevertheless we can show that the equalities hold, at least mod 2, and so compute the mod 2 homology of certain loop-spaces and certain homogeneous spaces.

This paper, then, is mainly a « Zusammenstellung » of well known propositions in two disciplines : the Morse theory, and the geometry of Lie groups due largely to CARTAN. We start with a review of the pertinent facts and definitions in the theory of Morse.

2. J-FIELDS.

Throughout this paper, M shall denote a C^∞, paracompact manifold in a fixed complete Riemannian structure. Instead of the term, C^∞, we often use " smooth ". Let M_p, $p \in M$, stand for the tangent space [2] to M at p, and if X, $Y \in M_p$, (X, Y) is their inner product.

A map

$$g : \mathbf{R} \to M$$

of the real numbers, $\mathbf{R}$, into M is called a geodesic, if :

(2.1) g satisfies the differential equations of a geodesic [10], for all $t \in \mathbf{R}$,

(2.2) t is proportional to arc-length from $g(0)$.

The restriction of g to a nontrivial interval of the type $[0, a]$, $0 < a$, in $\mathbf{R}$, will be called a segment of g. Segments will always be denoted by the symbol s.

A smooth function : $t \to Y_t$, which assigns to every $t \in \mathbf{R}$, a tangent vector Y_t in $M_{g(t)}$, is by definition a " vector field along g ".

The tangent field along g is by definition the assignement $t \to X_t$, where

$$X_t f = \lim_{h \to 0} [f\{g(t+h)\} - f\{g(t)\}]/h$$

for smooth functions, f, on M.

If $h \in \mathbf{R}$, let $h_* : M_{g(t)} \to M_{g(t+h)}$ be the isomorphism, of the two spaces in question, which is defined by parallel translating the vectors of the first space, along g, into the second one. Now if $t \to Y_t$ is a field along g, the formula

$$Y'_t = \lim_{h \to 0} \{ Y_t - h_* Y_{t-h} \}/h$$

defines a new field : $t \to Y'_t$ along g — the covariant derivative of Y along g.

In the theory of Morse the vector space of " infinitesimal isometries of g " plays a crucial role. We refer to it as the space of Jacobi (or just J)-fields along g, and denote it by J_g. This vector space can be defined in terms of the notion of a « *variation* » of g. Such a variation shall be a family of maps

$$V_\alpha : \mathbf{R} \to M$$

indexed by α in some vicinity of 0 on $\mathbf{R}$ satisfying the following conditions :

(2.3) $V_\alpha(t)$ depends smoothly on α and t;

(2.4) V_α is a geodesic for each α;

(2.5) $V_0 = g$.

Definition 2.1. — *A vector field : $t \to Y_t$ is a J-field along g, if and only if there exists a variation V_α of g, so that*

$$(2.6)\qquad Y_t = \frac{\partial}{\partial\alpha} V_\alpha(t)\Big]_{\alpha=0} \qquad (t \in \mathbf{R}).$$

Remarks. — (1) The formula (2.6) is meant as a shorthand for

$$Y_t f = \frac{\partial}{\partial\alpha} f(V_\alpha(t))\Big|_{\alpha=0};$$

(2) The factor space J_g can be alternately characterized as the space of solutions of the " Jacobi equation " along g. In our notation this question takes the form

$$(2.7)\qquad Y_t'' + R(X_t, Y_t)X_t = 0$$

where $t \to Y_t''$ is the second covariant derivative of Y along g, $t \to X_t$ is the tangent field along g, and $R(X_t, Y_t)$ is the linear map $M_{g(t)} \to M_{g(t)}$ defined by the curvature form evaluated at X_t, Y_t.

We will be interested only in two facts which follow trivially from this definition.

(2.8) If s is a segment of g, and J_s is defined as the restriction of fields in J_g to s, then the map

$$s_* : J_g \to J_s$$

defined as this restriction, is an isomorphism onto.

(2.9) $Y \in J_g$ is determined uniquely by the initial values : Y_0 and Y_0'.
Hence

$$(2.10)\qquad \dim J_g = 2 \dim M.$$

3. Focal points.

Let N be a proper, smooth, submanifold of M. A geodesic, g, will be said to be a geodesic of (M, N) — or of $M \bmod N$ — If :

(3.1) g starts on N;

(3.2) The initial direction of g is perpendicular to N.

Similarly we speak of geodesic segments, s, on (M, N). For these objects the infinitesimal variations which preserve (3.1) and (3.2) are of special interest. They constitute a subspace, J_g^N, of J_g which we call the focal subspace of J_g relative to N. Precisely :

Definition 3.1. — *The field $t \to Y_t$ is contained in J_g^N if and only if* *there exists a variation V_α of g such that*

$$(3.3)\qquad Y_t = \frac{\partial}{\partial\alpha} V_\alpha(t)\Big|_{\alpha=0} \qquad (t\in\mathbf{R}),$$

$$(3.4)\qquad V_\alpha(0)\in N,$$

$$(3.5)\qquad \frac{\partial}{\partial t} V_\alpha(t)\Big|_{t=0} \in N_q^{\perp} \qquad [q = V_\alpha(0)],$$

where $N_q^{\perp}$ denotes the orthogonal complement of N_q in M_q.

For a segment s of g, J_s^N is again defined as the image of J_g^N in J_s under the restriction map : $s_*: J_g \to J_s$.

Definition 3.2. — *Let s be a geodesic segment M mod N. The subspace of J_s^N consisting of the J-fields which vanish at the end-point of s shall be referred to as the focal kernel of s mod N, and will be denoted by $\Lambda^N(s)$. Also,*

a. If $\dim\Lambda^N(s) > 0$, then s is called a focal segment of N in M;

b. The end-point of a focal segment is called a focal point of N in M;

c. The set of all focal points of N in M is called the focal set of N in M.

Remarks. — (1) Focal points are clearly the generalization of conjugate points along a geodesic (the case : $N=p$). At the same time they generalize the focal points of elementary optics. For instance, if M is Euclidean 3-space E_3, and N is the circle : $x_1^2 + x_2^2 = 1$, then the focal set of N in M is precisely the x_3-axis.

(2) Clearly J_g^N can be characterized in terms of certain initial conditions on the elements of J_g. These classical initial conditions [9] take the following form :

Proposition 3.1. — *An element $Y\in J_g$ is in J_g^N if and only if*

$$(3.6)\qquad Y_0\in N_p,$$

$$(3.7)\qquad Y'_0 + T_g . Y_0 \in N_p^{\perp},$$

where : p is the initial point, $g(0)$, of g, and T_g is a self-adjoint linear transformation of N_p, determined completely by g. The form $(T_g X, Y)$ on N_p is the second fundamental form at p relative to g.

An immediate corollary is

$$(3.8)\qquad \dim J_g^N = \dim N.$$

T_g can be defined in several ways. We recall the following geometrical definition which follows readily from the formulae in [9; p. 26]. Suppose

g starts at $p \in N$, with tangent-vector $X \in N_p^{\perp}$. It is the fact that X is perpendicular to N at p which gives rise to the transformation T_g on N_p. To see what $T_g \cdot Y$, $Y \in N_p$, is, let $y(t)$ be a curve on N, starting at p in direc tion Y. By parallel translation X defines a field, $\tilde{X}$, along $y(t)$. Let $t \to X_t^*$ be the field along $y(t)$ which assigns to t the orthogonal projection of $\tilde{X}_t$ on $N_{y(t)}$. Now, because X_0^* vanishes, the limit

$$\lim_{t \to 0} X_t^*/t$$

exists, and defines the vector $T_g \cdot Y$ in N_p.

A point of $M - N$ which is not in the focal set of (M, N) is called a regular point of (M, N). According to a theorem of Morse [9], the regular points of (M, N) are plentiful. Namely,

Proposition 3.2. — *Let s be a geodesic segment of (M, N). Then the index of s, relative to N, defined by*

$$\lambda^N(s) = \sum_{s' \subset s} \dim \Lambda^N(s') \tag{3 9}$$

is always a finite integer.

(Here the summation is to be extended over all subsegments of s.)

Proposition 3.3 — *The regular points of (M, N) are everywhere dense in M.*

4. The Morse series of M mod N.

Suppose that $M \supset N$ as before, and that P is a fixed regular point of (M, N). To this situation Morse assigns a formal power series, $\mathfrak{M}(M, N, P; t)$, [or more shortly $\mathfrak{M}(t)$] which we now describe. Its interest lies in the fact that although this series is determined entirely by geometric considerations, namely by the numbers $\dim \Lambda^N(s)$, described in the last section, it nevertheless has topological implications.

Let $S(M, N, P)$ be the set of geodesic segments of (M, N) which end at P, and along which the parameter is precisely arc-length. Because P is regular, no segment of $S(M, N, P)$ is focal relative to N.

Definition 4.1. — *The Morse series of (M, N, P) is defined by*

$$\mathfrak{M}(M, N, P; t) = \sum t^{\lambda^N(s)} \qquad [s \in S(M, N, P)], \tag{4.1}$$

where $\lambda^N(s)$ is the index of s relative to N, as defined by (3.9).

Remarks. — Clearly one can consider this series only if its coefficients are finite numbers. If this fails to happen, as it may, we say : the Morse series of (M, N, P) does not exist. For instance, if M is the flat torus, N a point Q, $S(M, Q, P)$ will contain an infinite number of elements, all of index 0. It seems likely that the Morse series exists whenever $\pi_1(M, N)$ is finite; however, I know of no proof of this fact, and it will not be needed. In any case the series 4.1 has only a countable number of terms. This is a direct consequence of the regularity of P and the paracompactness of M. For, as can also be found in MORSE [9] :

PROPOSITION 4.1. — *The set $S(M, N, P)$, (P regular !) contains only a finite number of segments of length less than a given number.*

5. THE MORSE INEQUALITIES.

The topological implications of $\mathfrak{M}(t)$ are contained in the " Morse inequalities " which relate $\mathfrak{M}(t)$ to the Poincaré series of a function space, $\Omega(M, N, P)$, constructed over M. Following SEIFERT and THRELLFALL [10] rather than MORSE for the time being, this space $\Omega(M, N, P)$ is defined in the following manner :

DEFINITION 5.1. — *$\Omega = \Omega(M, N, P)$ shall denote the space of all piecewise regular maps of the unit interval $[0, 1]$ into M which are parametrized proportionaly to arc-length, take 0 into a point of N and map 1 onto P.*

A topology is introduced into Ω by making Ω into a metric space

$$\rho(u_1, u_2) = \max_{t \in [0,1]} d(u_1(t), u_2(t)) + | L(u_1) - L(u_2) |$$

for two maps $u_i \in \Omega$, where $L(u_i)$ is the length of u_i and $d(p, q)$, $p, q \in M$, is the metric on M.

Let $H(\Omega; k)$ denote the singular homology [10], of Ω with respect to coefficients k. If k is a field,

$$P(\Omega; k; t) = \sum_{n \geq 0} \dim H_n(\Omega; k) t^n \tag{5.1}$$

is the Poincaré series of Ω relative to k.

REMARK. — In homotopy theory one usually treats the space, Ω', of continuous paths from N to P in M endowed with the compact open topology Though not of the same homotopy-type, the spaces Ω and Ω' have isomorphic singular homology groups, as is shown in [10; p. 77]. The groups $H(\Omega)$ are therefore topological invariants of (M, N, P).

The Morse inequalities. — If the Morse series of (M, N, P) exists, then the Poincaré series of $\Omega = \Omega(M, N, P)$ exists relative to any field k.

Further

$$(5.2)\qquad \mathfrak{M}(M, N, P; t) - P(\Omega; k; t) = (1+t)\, B(k; t),$$

where $B(k; t)$ is a formal series with non-negative coefficients.

For a discussion of these inequalities, *see* paragraphe **12**. Here we will only derive an immediate corollary to this theorem on which all our applications are based.

COROLLARY 5.1. — *If the Morse series of (M, N, P) exists and contains no odd powers of t, then*

$$\mathfrak{M}(M, N, P; t) = P(\Omega; k; t) \quad \text{for any } k.$$

In particular $H(\Omega)$ is then free of torsion.

PROOF. — I.

$$\mathfrak{M}(t) = \sum_{i \geqq 0} m_i t^i; \qquad P(\Omega; k; t) = \sum_{i \geqq 0} p_i t^i; \qquad B(t) = \sum_{i \geqq 0} b_i t^i,$$

the inequalities of Morse imply that

$$m_i - p_i = b_i + b_{i-1} \qquad (i = 1, 2, \ldots),$$
$$m_0 - p_0 = b_0$$

with $b_i \geqq 0$.

Hence $m_{2i+1} = 0$ $(i = 0, 1, 2, \ldots)$ implies

$$b_{2i+1} + b_{2i} = 0 \qquad \text{whence} \quad b_{2i+1} = 0,\ b_{2i} = 0,$$

i. e. $B(t) \equiv 0$. The rest is clear.

6. VARIATIONAL COMPLETENESS.

Suppose that K is a compact group of isometries of the Riemannian manifold M, and that N is an orbit of a point in M under the action of K. In this case the infinitesimal motions of M determined by K, are universal J-fields mod N, i. e. they restrict to elements of J_s^N along any geodesic segment, s, of (M, N). When the action of K on M is sufficiently rich, it may happen that $\Lambda^N(s)$ consists entirely of fields which can be extended to the global infinitesimal motions of K. In this very special situation, which however is the one we meet in both subsequent applications, the Morse series of $(M, N. P)$ can be computed very simply. Here we derive this new formula for the Morse series, under this extension condition, which we call variational completeness. Let $\pi : K \times M \to M$ be the left representation of K on M under consideration. We also use

$$(6.1)\qquad \pi(\sigma) : M \to M$$

for the transformation determined by $\sigma \in K$. We write $o(p)$ for the orbit of p in M under K.

$$(6.2) \qquad o(p) = \pi(K)p.$$

The following definition enables one to treat all orbits at the same time.

DEFINITION 6.1. — *A geodesic segment, s, will be called transversal (propertly π-transversal) if its initial direction is perpendicular to the orbit of its initial point. If p is the initial point of such a segment, J_s^N $[\Lambda^N(s)]$, $N = o(p)$, will be denoted by $J_s^\pi[\Lambda^\pi(s)]$ respectively.*

The Lie algebra [2] of K shall be denoted by $\mathbf{k}$, and is identified with the tangent space to K at its neutral element. The mapping π determines a representation, $\dot{\pi}$, of $\mathbf{k}$ by vector fields on M. These are the infinitesimal motions of K on M. By definition :

$$(6.4) \qquad \dot{\pi}(X)_p = \frac{\partial}{\partial t}\{\pi(\exp tX)p\}\Big|_{t=0} \qquad (X \in \mathbf{k}, p \in M).$$

Here, as throughout, $\exp : \mathbf{k} \to K$, is the usual exponential map [2], which takes $X \in \mathbf{k}$ into the corresponding point on the one-parameter subgroup generated by X.

If s is a transversal geodesic segment on M, the variation

$$(6.4\,bis) \qquad V_\alpha(t) = \pi(\exp \alpha X)s(t)$$

clearly has the properties enumerated in paragraph 2. Formulae (6.3) and (6.4) therefore justify the earlier remark that the restriction of $\dot{\pi}(X)$ to s, is in J_s^π. We let $\dot{\pi}_s : \mathbf{k} \to J_s^\pi$ stand for $\dot{\pi}$ followed by restriction to s.

DEFINITION 6.2. — *The action of K on M via π is called variationally complete, if for any π-transversal geodesic segment, s,*

$$(6.5) \qquad \Lambda^\pi(s) \subset \dot{\pi}_s(\mathbf{k}).$$

Let $\mathbf{c}(p)$, $p \in M$, be the subspace of $\mathbf{k}$, whose $\dot{\pi}$-image vanishes at p. Then $\mathbf{c}(s) \subset \mathbf{k}$, shall be the kernel of $\dot{\pi}_s$. Clearly then,

$$(6.6) \qquad \dim o(p) = \dim \mathbf{k} - \dim \mathbf{c}(p).$$

Further, if s is transversal, with endpoint, q, then (6.5) implies that the sequence

$$(6.7) \qquad 0 \to \mathbf{c}(s) \to \mathbf{c}(q) \xrightarrow{\dot{\pi}_s} \Lambda^\pi(s) \to 0$$

is exact. From (6.6) and (6.7), it follows that

$$(6.8) \qquad \dim \Lambda^\pi(s) = \{\dim \mathbf{k} - \dim \mathbf{c}(s)\} - \dim o(q).$$

Now suppose that s is a segment of a geodesic which passes through a point, P, on an orbit of maximal dimension. Then

$$\mathbf{c}(s) = \mathbf{c}(P) \tag{6.9}$$

because the identity component of the stability group of P leaves M_P fixed. In that case therefore, (6.8) takes the form

$$\dim \Lambda^\pi(s) = \dim \mathfrak{o}(P) - \dim \mathfrak{o}(q). \tag{6.10}$$

PROPOSITION 6.1. — *Let the action of K on M be variationally complete, and let N be the orbit of any point of M under K. If a regular point, P, of $M - N$ is chosen on an orbit of maximal dimension, the index, $\lambda^N(s)$, of any segment $s \in S(M, N, P)$ is given by*

$$\lambda^N(s) = \sum_{0 < t \leq a} \delta\{s(t)\},$$

where $s = g \mid [0, a]$, $a > 0$ and $\delta(p) = \dim \mathfrak{o}(P) - \dim \mathfrak{o}(p)$, $p \in M$.

This proposition follows immediately from (3.9) and (6.10). In this situation, then, the Morse series of (M, N, P) can be read off if one knows the segments of $S(M, N, P)$, and the places where they intersect orbits of lower dimension. Remark also that because the maximal orbits make up an open set in M, P can always be chosen on such an orbit.

7. THE VARIATIONAL COMPLETENESS OF THE ADJOINT ACTION.

G shall stand for a fixed compact connected Lie group, in a fixed left-and-right-invariant Riemannian structure. Under the adjoint action of G on G we mean the map $\pi : G \times G \to G$ defined by $\pi(\sigma)\tau = \sigma\tau\sigma^{-1}$.

PROPOSITION 7.1. — *The adjoint action of G on G is variationally complete.*

We will use the following notation :

$\mathfrak{g}$ the Lie algebra of G, which is identified with G_ε, the tangent space to G at the identity ε;

$l_\sigma[r_\sigma]$, left [right] translation by $\sigma \in G$;

$\dot{l}_\sigma[\dot{r}_\sigma]$, the linear maps induced by $l_\sigma[r_\sigma]$ on the respective tangent spaces.

$\mathrm{Ad}\,\sigma$, $\dot{l}_\sigma \dot{r}_{\sigma^{-1}}$ applied to $\mathfrak{g}$.

Also recall that

(7.1) $$\pi(\sigma)\exp X = \exp(\mathrm{Ad}\,\sigma\cdot X) \qquad (X\in\mathfrak{g},\ \sigma\in G),$$

(7.2) $$\mathrm{Ad}\,X\cdot Y = [X, Y] = \lim_{t\to 0}\left\{\frac{\mathrm{Ad}(\exp tX) - I}{t}\right\} Y,$$

(7.3) $$(\mathrm{Ad}\,\sigma\cdot X,\ \mathrm{Ad}\,\sigma\cdot Y) = (X, Y) \qquad (X, Y\in\mathfrak{g}),$$

(7,4) $$([X, Y], Z) + (Y, [X, Z]) = 0 \qquad (X, Y, Z\in\mathfrak{g}),$$

(7.5) The geodesics of G coincide with the translates of one-parameter subgroups of G.

By definition 6.2, proposition 7.1 is equivalent to the following one :

Proposition 7.2. — *If s is a π-transversal geodesic segment, then* $\Lambda^{\pi}(s)\subset\dot{\pi}_s(\mathfrak{g})$.

Lemma 7.1. — *For $X\in\mathfrak{g}$, $\dot{\pi}(X)$ is the assignment.*

(7.6) $$\sigma\to\dot{r}_{\sigma}X - \dot{l}_{\sigma}X.$$

Proof. — This follows immediately if we rewrite $\exp(tX)\sigma\exp(-tX)$ in the form $r_{\sigma}\{\exp(tX)\exp(-\mathrm{Ad}\,\sigma(tX))\}$.

Lemma 7.2. — *The segment $s : t\to\sigma\exp(tX)$, $X\in\mathfrak{g}$, $0\leqq t\leqq a$, $a>0$; is transversal if and only if*

(7.7) $$\mathrm{Ad}\,\sigma X = X.$$

Proof. — By lemma 7.1 the tangent space to the orbit through σ is spanned by

$$\dot{r}_{\sigma}Y - \dot{l}_{\sigma}Y \qquad (Y\in\mathfrak{g}).$$

The initial direction of s at σ is clearly $\dot{l}_{\sigma}X$. Now

$$\big(\dot{l}_{\sigma}X,\ (\dot{r}_{\sigma}-\dot{l}_{\sigma})\mathfrak{g}\big) = 0$$

is equivalent to $(X, \{I-\mathrm{Ad}\,\sigma^{-1}\}\mathfrak{g}) = 0$. By (7.3) this expression transforms to

$$(\{\mathrm{Ad}\,\sigma - I\}X,\ \mathfrak{g}) = 0.$$

In other words : $\mathrm{Ad}\,\sigma X = X$.

Lemma 7.3. — *Let s be a π-transversal geodesic segment, with initial point σ, and initial direction $\dot{l}_{\sigma}X$, $X\in\mathfrak{g}$. Then $\mathbf{c}(s)\subset\mathfrak{g}$ is characterized by the equations*

(7.8) $$[X, Y] = 0, \qquad \mathrm{Ad}\,\sigma Y = Y.$$

Proof. — Recall that by definition, $\mathbf{c}(s)$ is the kernel of $\dot{\pi}_s:\mathfrak{g}\to J^{\pi}_s$.

If $\dot{\pi}_s(Y)=0$, then $\dot{r}_{\sigma\exp(tX)}Y=\dot{l}_{\sigma\exp(tX)}Y$ for $0\leq t\leq a$ $(a>0)$. Equivalently, $\mathrm{Ad}\{\exp(tX)\}Y=\mathrm{Ad}\sigma^{-1}\cdot Y$, $0\leq t\leq a$. Because the righthand side is independent of t, $[X, Y]$ must vanish. But in that case $\exp tX$ commutes with Y, and therefore $\mathrm{Ad}\sigma^{-1}\cdot Y=Y$. The converse is no harder.

From (3.8) it is known that $\dim J_s^{\pi}=\dim \mathfrak{g}$. Hence $\dot{\pi}(\mathfrak{g})$ does not make up all of J_s^{π}. There is missing a subspace, complementary to $\dot{\pi}_s(\mathfrak{g})$, of the dimension of $\mathbf{c}(s)$. To construct it, consider the map

$$(7.9)\qquad \omega_s:\mathbf{c}(s)\to J_s^{\pi}$$

defined by

$$(7.10)\qquad \omega_s(Z)_t=t\,\dot{l}_{\sigma\exp tX}Z\qquad [Z\in\mathbf{c}(s),\ 0\leq t\leq a].$$

Notice that $\omega_s(Z)$ is not the restriction of a vector field on G. It is, however, a properly defined field along s. To show that $\omega_s(Z)\subset J_s^{\pi}$, consider the following variation of s.

For $Z\in\mathbf{c}(s)$, $\alpha\in\mathbf{R}$, let $V_\alpha:\mathbf{R}\to G$ be defined by

$$(7.11)\qquad V_\alpha(t)=\sigma\exp t\{X+\alpha Z\}.$$

Because $V_\alpha(0)=\sigma$, this variation will satisfy the conditions (3.4), (3.5), if and only if V_α starts perpendicularly to $o(\sigma)$. This initial direction is clearly $\dot{l}_\sigma\{X+\alpha Z\}$. By lemma 7.2, this is the case because $Z\in\mathbf{c}(s)$, and hence $\mathrm{Ad}\sigma Z=Z$.

On the other and, $\left.\dfrac{\partial}{\partial\alpha}V_\alpha(t)\right|_{\alpha=0}$ is easily seen to agree with $\omega_s(Z)$ if one uses the fact that X and Z commute.

LEMMA 7.4. — *Let s be a transversal segment on G. Then :*

(*a*) *Every element $V\in J_s^{\pi}$ is of the form*

$$Y=\dot{\pi}_s(Y)+\omega_s(Z),\qquad [Y\in\mathfrak{g},\ Z\in\mathbf{c}(s)];$$

(*b*) *If for $Y\in\mathfrak{g}$, $Z\in\mathbf{c}(s)$,*

$$\dot{\pi}_s(Y)+\omega_s(Z)=0.$$

Then $Z=0$ and $Y\in\mathbf{c}(s)$.

This lemma can be summarized in the statement that the sequence

$$(7.12)\qquad 0\to\mathbf{c}(s)\to\mathfrak{g}\oplus\mathbf{c}(s)\to J_s^{\pi}\to 0$$

is exact. Here the first homomorphism includes $\mathbf{c}(s)$ in $\mathfrak{g}$, while the second one is $\dot{\pi}_s\oplus\omega_s$.

PROOF. — Part (*b*) implies part (*a*) by a dimension argument. To prove (*b*) consider what it means that the expression $\dot{\pi}_s(Y)+\omega_s(Z)$ represent o in J_s^π. If s is of length $a > 0$, this condition translates into

$$(7.13) \qquad Y - \mathrm{Ad}\,\{\sigma \exp(tX)\}\, Y + tZ = 0 \qquad (0 \leqq t \leqq a).$$

Therefore, for $t = a$

$$(7.14) \qquad (Y, Z) - (\mathrm{Ad}\,\{\sigma \exp(aX)\}\, Y, Z) + a(Z, Z) = 0.$$

By (7.3), and because $Z \in \mathbf{c}(s)$, the first two terms cancel. Hence $Z = 0$. But then (7.13) expresses the fact that $\dot{\pi}_s(Y) = 0$, i.e $Y \in \mathbf{c}(s)$.

The proof of proposition (7.2) is now immediate. By virtue of (7.12), it is sufficient to show that the inverse image of Λ_s^π under $\dot{\pi}_s \oplus \omega_s$ is in $\mathfrak{g}$. The pair Y, Z, $Y \in \mathfrak{g}$, $Z \in \mathbf{c}(s)$ will be in this inverse image if and only if

$$\dot{\pi} s(Y) + \omega_s(Z) \quad \text{vanishes at the end-point}$$

of s, say at $\sigma \exp aX$. But then (7.13) still holds with $t = a$, and one concludes, again via (7.14), that $Z = 0$.

8. The geodesics of (G, N, P).

Recall the following well-known facts about a compact connected group G [1], [7].

(8.1) G contains a maximal torus;

(8.2) Two maximal tori are conjugate;

(8.3) Every element of G lies on at least one maximal torus;

(8.4) The component of the identity, $C'(\sigma)$, of the centralizer of an element $\sigma \in G$, is the union of the maximal tori containing σ.

It follows immediately from (8.4), that if P is a point on an orbit of maximal dimension, then the centralizer of P is precisely a single maximal torus T. Let P be a fixed such general point of G. Let N be any orbit of π on G which does not contain P.

Proposition 8.1. — *Let P a general point of G and T the maximal torus it determines. If $N = \mathrm{o}(\sigma)$ is any orbit of G under π, which does not intersect P, then*

$$S(G, N, P) = S(T, N \cap T, P).$$

Proof. — It $\sigma \in G$, let $C(\sigma)$ be the centralizer of σ in G. Lemma 7.2 obviously yields the following result :

Lemma 8.1. — *For any $\sigma \in G$, the tangent spaces to $C(\sigma)$ and $\mathrm{o}(\sigma)$, at σ, are complementary and orthogonal.*

In particular, therefore, at P, the space T_P makes up the orthogonal complement of the tangent space to the orbit through P.

It is well known that if a geodesic, g, is perpendicular to an infinitesimal isometry, X, at one point, then it always remains at right angles to X. Hence every segment $s \in S(G, N, P)$ (which must be perpendicular to the orbit N at its start) is perpendicular to $o(P)$ at its end point. The segment s is therefore tangent to T at P. Because T is a subgroup, s will have to lie in T. Conversely a geodesic segment on T joining $\tau \in N \cap T$ to P, is perpendicular to $o(\tau)$ at its initial point and therefore is in $S(M, N, P)$.

The set of segments $S(T, N \cap T, P)$ is conveniently described on $\mathbf{t}$, the universal covering space of T, which is here identified whith $T\varepsilon$, so that $\exp | \mathbf{t}$ becomes the covering projection.

$N \cap T$ will be a finite set of points because the orbits are compact and meet T at right angles. Let $\tilde{N} \in \mathbf{t}$ be the image of any cross section of $N \cap T$ in $\mathbf{t}$, and let $\tilde{S}(\mathbf{t}, \tilde{N}, P)$ denote the set of all straight lines in $\mathbf{t}$ which join a point of $\tilde{N}$ to a point $P \in \mathbf{t}$ which covers P. Because paths in T lift uniquely to $\mathbf{t}$, once their starting point is lifted, $\tilde{S}(\mathbf{t}, \tilde{N}, P)$ gives a faithful representation of $S(T, N \cap T, P) = S(G, N, P)$. We record this fact in the following proposition.

PROPOSITION 8.2. — *The exponential map* $\exp | \mathbf{t}$, *maps the straight lines of* $\tilde{S}(\tilde{\mathbf{t}}, \tilde{N}, P)$ *in a one-to-one fashion onto the segments of* $S(G, N, P)$.

To read off the Morse series of (G, N, P) it is now sufficient to find the intersections of the segments of $S(T, N \cap T, P)$ with the intersection of the exceptional orbits with T. These points, the so-called singular points of T, form a set $\bar{D}(G)$ in T whose inverse image under $\exp | \mathbf{t}$ is called the diagram of G on $\mathbf{t}$. It will be denoted by $D(G)$; and its main properties are reviewed in the next section.

9. The diagram of G.

As before let $T \subset G$ be a maximal torus of the compact connected group G; and let $\mathbf{t} \subset \mathfrak{g}$ be its tangent space at ε. The dimension, l, of T is the rank of G. A point, σ, of T will be singular if the dimension of its centralizer, $C(\sigma)$, is greater than l. On the other hand the tangent space of $C(\sigma)$ at ε is clearly spanned by the subspace of $\mathfrak{g}$ which is left pointwise fixed by $\mathrm{Ad}\sigma$. The study of singular points is therefore reduced to the study of the adjoint representation restricted to T. This action of T on $\mathfrak{g}$ decomposes $\mathfrak{g}$ into orthogonal invariant subspaces,

$$\mathfrak{g} = \mathbf{t} \oplus \mathbf{e}_1 \oplus \mathbf{e}_2 \oplus \cdots \oplus \mathbf{e}_m \tag{9.1}$$

with $\mathbf{t}$ pointwise fixed under $\mathrm{Ad} | T$, while each $\mathbf{e}_i$ is a two-plane on which T is represented nontrivially by rotations.

Let $U_i \subset T$ be the subgroup of T which leaves the plane $\mathbf{e}_i$ pointwise fixed under $\mathrm{Ad} \mid T$. Clearly the union of the U_i constitutes the set of singular points of T. Further if $\sigma \in T$ is on precisely f of the sets $U_i (i = 1, \ldots, m)$ then dim $C(\sigma) = l + 2f$, and hence

$$(9.2) \qquad \delta(\sigma) = 2f$$

(δ is here the function defined earlier in Section **6**).

Let $\{\mathbf{u}_i\}$ $(i = 1, \ldots, m)$ be the tangent space to U_i at ε. The planes $\{\mathbf{u}_i\}$ in $\mathbf{t}$ form the infinitesimal digram of G in $\mathbf{t}$. It shall be denoted by $D'(G)$. The diagram proper, $D(G)$, consists of the complete inverse image in $\mathbf{t}$ of the singular points in T. It will therefore consist of families of equi-spaced planes, each family parallel to a certain $\mathbf{u}_i \in D'(G)$.

The kernels U_i are by no means arbitrary. Among others they have the following properties [**1**], [**7**], [**12**].

$$(9.3) \qquad \dim U_i \cap U_j = l - 2 \qquad \text{if} \quad i \neq j.$$

Hence $U_i \neq U_j$ for $i \neq j$.

(9.4) $\qquad U_i$ has at most two components,

$$(9.5) \qquad \bigcap_{j=1}^{i=m} U_i = Z \quad \text{is the center of } G.$$

The diagram, $D'(G)$, can also be described in terms of the « root forms of G » : the quotient group T/U_i is one-dimensional and therefore isomorphic to the circle group, $\mathbf{R}/\mathbf{Z}$, of the real numbers modulo the integers. Further such an isomorphism $T/U_i \approx \mathbf{R}/\mathbf{Z}$ is well determined up to sign. The canonical map $T \to T/U_i$ therefore lifts to a linear function

$$\mathbf{t} \to \mathbf{R}$$

which is well defined up to sign. We denote it by $\pm \theta_i$. The totality of the $\pm \theta_i$ are the root forms of G on $\mathbf{t}$. They are met more directly in the infinitesimal theory of Lie groups. In any case $D'(G)$ consists of the points which are in the kernel of at least one of the θ_i. Similarly

$$D(G) = \bigcup_{i=1}^{m} \{ X \in \mathbf{t} \mid \theta_i(X) \equiv 0 \bmod 1 \}.$$

A connected piece of $\mathbf{t} - D'(G)$ is called a fundamental chamber of G on $\mathbf{t}$. We denote such a set by the symbol $\mathcal{F}$. Similarly, Δ, shall be reserved for the components of $\mathbf{t} - D(G)$. The Δ are the fundamental cells of G in $\mathbf{t}$.

All these notions are intimately connected with a certain group of automorphisms which G induced on T. This is the group W-{normalizer of T in G}/T. Clearly W acts on T and therefore on $\mathbf{t}$ as a group of automorphisms. Further it must leave the singular set invariant. W is the Weyl group of G.

It is known that :

(9.6) The faithful representation of W on $\mathbf{t}$ is generated by the reflections of $\mathbf{t}$ in the planes $\mathbf{u}_i$ of $D'(G)$;

(9.7) Every fundamental chamber, $\mathcal{F}$, of $D'(G)$ is simultaneously a fundamental domain of W in $\mathbf{t}$.

The elements of W permute the totality of root forms $\{\pm\theta_i\}$. A fundamental chamber $\mathcal{F}$, singles out precisely m-root forms which take on positive values in $\mathcal{F}$. Such a system of root forms is referred to as a positive set of root forms with respect to $\mathcal{F}$, and is denoted by $\{\theta_i\}_{\mathcal{F}}$. On the set $\{\theta_i\}_{\mathcal{F}}$ the Weyl group is therefore represented by signed permutations.

W and $D'(G)$ are essentially infinitesimal invariants of G. Thus if G and G_1 are locally isomorphic, such an isomorphism induces corresponding isomorphisms of W_1 onto W_2. The global properties of G are described by the extended Weyl group on G. This is the transformation group $\tilde{W}$ of $\mathbf{t}$, generated by W and the covering transformations of $\mathbf{t}\to T$. This latter group, Γ, is of course abstractly isomorphic to $\pi_1(T)=\mathbf{Z}+\mathbf{Z}+\ldots+\mathbf{Z}$ (l factors).

The orbit of a point $P\in\mathbf{t}$ under Γ is a lattice of points in $\mathbf{t}$. The lattice corresponding in this manner to the origin of $\mathbf{t}$ shall also be denoted by Γ. The lattice Γ is left invariant by the action of W.

If G is connected, simply connected and compact, $\tilde{W}$ is described by $D(G)$. In that case it is known that :

(9.8) $\tilde{W}$ is generated by the reflections in the planes of $D(G)$. From this it follows that;

(9.9) Each fundamental cell of $D(G)$ is a fundamendal domain of $\tilde{W}$ on $\mathbf{t}$;

(9.10) The closure of every fundamental cell is compact and contains precisely one point of the lattice Γ.

In this case then the effect of $w\in\tilde{W}$ on $\mathbf{t}$ is completely described by the effect of w on one cell of $D(G)$. The following lemma is a consequence of this fact :

LEMMA 9.1. — *Let G be connected, simply connected and compact. Let $\mathcal{F}$ be a fundamental chamber of W on $\mathbf{t}$, and let $\Delta_{\mathcal{F}}$ be the unique cell of $D(G)$, which is contained in $\mathcal{F}$, and whose closure contains the origin of $\mathbf{t}$. Then there is a one-to-one correspondence between the cells of the* *form*

a. $$\{\gamma\Delta_{\mathcal{F}}\} \qquad (\gamma\in\Gamma)$$

and the cells of the form

b. $$\Delta\in\mathcal{F}.$$

This correspondence is established by elements of W.

Proof. — Set

$$A=\{\gamma\Delta_{\mathcal{F}}\mid\Delta\in\mathcal{F}\}, \qquad B=\{\Delta\mid\Delta\in\mathcal{F}\}.$$

An element $\gamma\Delta_{\mathcal{F}}$ is in some fundamental chamber $\mathcal{F}'$. By (9.8) there exists a unique element $w'\in W$ with $w'\mathcal{F}'=\mathcal{F}$. We assign to $\gamma\Delta_{\mathcal{F}}$, the element $w'\gamma\Delta_{\mathcal{F}}\subset\mathcal{F}$. This defines a function, f, from A to B.

(1) The function f is one-to-one. For if

$$w'_1\gamma_1\Delta_{\mathcal{F}}=w'_2\gamma_2\Delta_{\mathcal{F}}, \qquad \gamma_i\in\Gamma, \qquad w'_i\in W,$$

then $\gamma_1^{-1}\{w'_1\}^{-1}w'_2\gamma_2$ is the identity by (9.9). Hence

$$\{w'_1\}^{-1}w'_2=\gamma_1\gamma_2^{-1}.$$

This element must lie in $W\cap\Gamma$, which clearly is the identity. Hence

$$\gamma_1=\gamma_2, \qquad w'_1=w'_2.$$

(2) f is onto.

Let the closure of $\Delta\in\mathcal{F}$, contain the lattice point γ. Then the closure of $\gamma^{-1}\Delta$ contains the origin of $\mathbf{t}$, and therefore differs from $\Delta_{\mathcal{F}}$ by an element of W. That is,

$$\Delta=\gamma w'\Delta_{\mathcal{F}}, \qquad \gamma\in\Gamma, \qquad w'\in W'.$$

But then $\Delta'=w'^{-1}\gamma w\Delta_{\mathcal{F}}$ is an element of A which maps onto under f.

10. First application : $\Omega(G, N, P)$.

Let G be a compact connected and simply-connected Lie group, $T\subset G$ a maximal torus of G, N any orbit of the adjoint action of G on G. Let P be a general point of T not on N. Finally $\tilde{S}(\mathbf{t}, \tilde{N}, P)$ shall be the set of straight lines in $\mathbf{t}=T\varepsilon$, already defined in Section 8.

Theorem I. — *The space of paths, $\Omega(G, N, P)$, is free of torsion. Its odd Betti numbers vanish, and the Poincaré series of $\Omega(G, N, P)$ coincides with the Morse series of (G, N, P).*

Theorem II. — *The Morse series of* (G, N, P) *is given by the formula*

$$(10.1)\qquad \mathfrak{M}(G, N, P; t) = \sum t^{2\lambda(s)} \qquad [s \in \tilde{S}(\mathfrak{t}, \tilde{N}, P)],$$

where $\lambda(s) =$ *number of planes of* $D(G)$ *crossed by the line* s.

The proof of theorem II emerges from propositions 6.2, 7.1, 8.2 and the expression (9.2) for the function δ. The series (10.1) has finite coefficients because the cells of $D(G)$ have compact closures. Because this series has no odd powers of t, corollary 5.1 applies. This proves theorem I.

Theorem C of the introduction is also immediate. For if N is the orbit of $\exp X$, $X \in \mathfrak{t}$, then a possible cross section over $N \cap T$ is the orbit of X under the Weyl group W. This orbit can therefore play the role of $\tilde{N}$ in theorems I and II.

In the special case $P = \varepsilon$, the space $\Omega(G, \varepsilon, P)$ is by definition the « loop space » $\Omega(G)$ of theorems A and B. In this case $\tilde{S}(\mathfrak{t}, \tilde{O}, P)$ consists of the segments $s = \overrightarrow{OQ}$ where Q runs over the points γP; $\gamma \in \Gamma$. If one now applies lemma 9.1 to this situation, one obtains the formula of theorem B, Part 1, directly from theorem C. To obtain the formula of theorem B', Part 1, observe that in $\mathcal{F}$,

$$\lambda(\Delta) = \sum_{i=1}^{m} [\theta_i(X)] \quad \text{for any } X \text{ in } \Delta,\ \theta_i \in \{\theta_i\}_{\mathcal{F}}.$$

Therefore when the integral $\int_{\mathcal{F}}$ is replaced by $\sum_{\Delta \in \mathcal{F}} \int_{\Delta}$ one obtains the series $\sum_{\Delta \in \mathcal{F}} t^{2\lambda(\Delta)}$ of theorem B.

11. Second application $G/C(X)$.

Let $\tilde{M}$ be an open, spherical disc about the origin in $\mathfrak{g}$, which is so small that $\exp | \tilde{M}$ maps $\tilde{M}$ homeomorphically into G. The image of $\tilde{M}$ under exp shall be denoted by M. The adjoint action, π, clearly maps M into itself.

Proposition 11.1. — *The adjoint action of G on M is variationally complete.*

It is clear that this property is inherited by M from G, because variational completeness is a local property along geodesics. In the same way, if $P \in M$ is on a maximal orbit, T the maximal torus which is the centralizer of P, and N any orbit of π in M which does not meet P, then propositions 8.1 and 8.2 transfer obviously to this situation. In order, they translate into :

Proposition 11.2. — *The set* $S(M, N, P)$ *coincides with the set* $S(M \cap T, N \cap T, P)$.

The inverse image of N in $\tilde{M}$ is now unique. It shall be denoted by $\tilde{N}$. The unique inverse image of P in $\tilde{M}$ shall be $\tilde{P}$, and $\tilde{S}(\tilde{M} \cap \mathbf{t}, \tilde{N} \cap \mathbf{t}, \tilde{P})$ shall stand for the set of straight lines joining points of $\tilde{N}$ to $\tilde{P}$ in $\tilde{M}$.

PROPOSITION 11.3. — *The map* $\exp | \tilde{M}$, *maps the lines of* $\tilde{S}(\tilde{M} \cap \mathbf{t}, \tilde{N} \cap \mathbf{t}, \tilde{P})$, *onto the segments* $S(M, N, P)$ *in a one-to-one fashion.*

Finally the formula (6.10) implies that :

PROPOSITION 11.4. — *The index of* $s \in \tilde{S}(\tilde{M} \cap \mathbf{t}, \tilde{N} \cap \mathbf{t}, \tilde{P})$ *(properly, of its image in* G*) is given by*

(11.1) $\lambda^N(s) = 2$ *times the number of planes of* $D'(G)$ *crossed by* s.

Here one could replace $D(G)$ by the infinitesimal diagram because

$$\tilde{M} \cap D(G) = \tilde{M} \cap D'(G).$$

Just as in theorem C, these propositions prove that $H(\Omega(M, N, P))$ is free of torsion, has vanishing odd Betti numbers, and gives a recipe for computing their Poincaré series. However, we also have :

PROPOSITION. 11.5. — *The function space* $\Omega(M, N, P)$ *is of the same homotopy type as* N. *Therefore*

$$H(\Omega(M, N, P)) \approx H(N). \tag{11.2}$$

This follows from the fact that $(\tilde{M}, \tilde{N}, \tilde{P})$ is mapped homeomorphically onto (M, N, P) by $\exp | \tilde{M}$. Therefore it is sufficient to study $\Omega(\tilde{M}, \tilde{N}, \tilde{P})$. Because M is a euclidean sphere this space clearly contains the space, $\hat{\Omega}(\tilde{M}, \tilde{N}, \tilde{P})$, consisting of the straight lines from $\tilde{N}$ to $\tilde{P}$, as a deformation retract. But $\hat{\Omega}(\tilde{M}, \tilde{N}, \tilde{P})$ is clearly homeomorphic to $\tilde{N}$.

THEOREM III. — *Let* $X \in \mathbf{t}$, *and denote by* $C(X)$ *the centralizer of* X *in* G. *Then*

a. $G/C(X)$ *is free of torsion;*

b. The Poincaré series of $G/C(X)$ *is given by*

$$P(G/C(X); t) = \sum t^{2\lambda(s)}, \tag{11.3}$$

where the summation extends over the straight lines from a general point P *in* $\mathbf{t}$, *to the points* $w \cdot X$; $w \in W$; *and* $\lambda(s) =$ *number of planes of* $D'(G)$ *crossed by* s.

PROOF. — Notice first that all quantities in this formula remain unchanged when X is changed to ρX; $\rho \in \mathbf{R} (\rho \neq 0)$. Therefore we can assume that $X \in M$. But then $G/C(X)$ is homeomorphic to N, the orbit of X under the adjoint representation. Hence, by proposition 11.5, the formula (11.3) has to be proved for $\Omega(M, N, P)$. Via proposition 11.1 to 11.4, the argument of the first application achieves this, for we have only replaced $\tilde{N} \cap \mathfrak{t}$ by the orbit of X under W.

REMARK. — Theorem III could also have been proved by first showing that :

PROPOSITION 11.6. — *The adjoint representation :* $\sigma \to \operatorname{Ad}\sigma$ *of a compact Lie group on its Lie algebra is variationally complete.*

Because the proof of this fact is the complete « infinitesimal » analogue of what was done in Section 7, we preferred to use the foregoing argument to obtain theorem III.

Theorem III immediately yields Part 2 of theorems A and B'. In the latter formulation the index of s has just been evaluated as the number of root forms of $\mathfrak{F}$ which have different signs at the end points of s. The formulation of Part 2, in theorem B, proceeds from theorem III by a straightforward counting argument, which we leave to the reader.

12. APPENDIX I : THE MORSE INEQUALITIES.

If two formal series $\mathfrak{M}(t)$ and $\Omega(t)$ satisfy the condition

$$\mathfrak{M}(t) - \Omega(t) = (1 + t)\, B(t),$$

where $B(t)$ has non-negative coefficients, we write $\mathfrak{M}(t) > \Omega(t)$ and say $\mathfrak{M}(t)$ dominates $\Omega(t)$. It is clear that domination is a transitive relation between formal power series

If

$$\mathfrak{M}(t) = \sum_{i \geq 0} m_i t^i, \qquad \Omega(t) = \sum_{i \geq 0} p_i t^i,$$

then the condition $M(t) > \Omega(t)$ is equivalently expressed by the set of inequalities

$$(12.1) \qquad \begin{cases} m_0 \geq p_0, \\ m_1 - m_0 \geq p_1 - p_0, \\ m_2 - m_1 + m_0 \geq p_2 - p_1 + p_0, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

and these are the Morse inequalities in their best known form.

The simplest instance in which they occur is the case of a finite complex K. For if we set : $m_i =$ number of cells of dimension i; $p_i = i$-th Betti number of K, then these integers satisfy the Morse inequalities.

To see this, let C, Z, B, H be the space of chains, cycles, boundaries, and homology classes of a finite complex K, over a field k. Let

$$C(t)=\sum \{\operatorname{rank} C_n\} t^n,$$

and define $Z(t)$, ... similarly. Then by definition of the quantities involved, we have the exact sequences

$$0 \to Z \to C \to B \to 0, \tag{12.2}$$

$$0 \to B \to Z \to H \to 0. \tag{12.3}$$

Remembering that the boundary homomorphism $d : C \to B$ reduces dimension by one these sequences imply that

$$C(t)=Z(t)+t\,B(t),$$
$$Z(t)=B(t)+H(t).$$

Adding we obtain $C(t)-H(t)=(1+t)\,B(t)$, i.e. just the Morse inequalities.

An obvious abstraction of this construction gives the following proposition.

PROPOSITION 12.1. — *Let* $A=\sum A_n$ *be a graded vector space of finite type, with differential operator* $d: A_n \to A_{n-1}$, *and let* $A(t)=\sum_{n>0} \dim(A_n)\,t^n$. *If* $H(A)$ *is defined in the usual way as the kernel of* d *modulo the image of* d, *then the series* $H(A)\,(t)=\sum \dim H_n(A)\,t^n$, *and* $A(t)$ *satisfy the Morse inequalities*

$$A(t) > H(A)(t). \tag{12.4}$$

With the aid of this proposition, the proof of the Morse inequalities can be brought back to one of Morse's basic deformation theorems by the use of Leray's spectral sequence [8]. To do this one proceeds in the following manner (*see* for instance [3]). Suppose then that the Morse series of (M, N, P) exists (P a regular point of $M-N$). Let $S=S(M, N, P)$, $\Omega=\Omega(M, N, P)$. Also, $|S|$, shall denote the set of lengths of the segments of S. By proposition 4.1, $|S|$ will be a discrete set of real numbers a_i which we assume to be indexed by $i=1, 2, 3, \ldots$ in a monotone increasing fashion, $a_{i+1} > a_i$. And l_i ($i=0, 1, 2, \ldots$), shall be a separating set of $|S|$,

$$0 < l_0 < a_1 < l_1 < a_2 < \ldots.$$

For any real number l, define

$$\Omega\ (l)=\{u\in\Omega \mid L(u) \leqq l\},$$
$$\Omega^-(l)=\{u\in\Omega \mid L(u) < l\}.$$

Finally,

$$\Omega_i = \begin{cases} \Omega(l_i), & (i = 0, 1, 2), \\ \emptyset \quad \text{the vacuous set}, & i < 0. \end{cases}$$

The singular chains, $C(\Omega)$ of Ω are filtered by the subgroups

$$C(\Omega_i) \qquad (i \in \mathbf{Z}).$$

For we clearly have $C(\Omega_{i+1}) \supset C(\Omega_i)$, while $C(\Omega) = \bigcup_{i \in \mathbf{Z}} C(\Omega_i)$ as L is continuous on Ω. In the resulting spectral sequence

$$E_1 = \sum_{i \in \mathbf{Z}} H(\Omega_{i+1}, \Omega_i).$$

Recall now the basic proposition concerning spectral sequence [**8**], [**11** *a*]:

PROPOSITION **12.2**. — *If in the grading of $C(\Omega)$ (which is bounded from below!) E_1 is of finite type, then :*

a. $H(\Omega_i)$ is of finite type for each $i \in \mathbf{Z}$;

b. The images D_i of $H(\Omega_i)$ in $H(\Omega)$ constitute a filtering of $H(\Omega)$, i.e.

$$\bigcap_{i \in Z} D_i = 0, \qquad D_{i+1} \supset D_i, \qquad \bigcup D_i = H(\Omega);$$

c. If $H(\Omega) = \sum_{i \in Z} D_{i+1}/D_i$ then $\mathcal{G}H$ is naturally isomorphic to E_∞, in a dimension-preserving fashion, where E_∞ is the limit of a sequence of graded groups E_r $(r = 1, 2, \ldots)$ with differential operators d_r (which decrease the dimension index by one) such that E_{r+1} is the homology group, HE_r, of E_r with respect to d_r.

Here E_∞ is the limit of the E_r in the following sense : given an integer n, there exists an integer r_n such that $E_{r_n} = E_{r_n+1} = \ldots$ in dimensions $\leqq n$. Then E_∞ is by definition isomorphic to this « stable » group in dimensions $\leqq n$.

Suppose now that E_1 is of finite type so that the above proposition applies. For a fixed field k as coefficients, let

$$E_r(t), \quad E_\infty(t), \quad \mathcal{G}H(\Omega)(t), \quad H(\Omega)(t)$$

be the Poincaré series of the corresponding graded groups. Then by property b, $\mathcal{G}H(\Omega)(t) = H(\Omega)(t)$ while by property c

$$E_1(t) > E_r(t) > E_\infty(t) = \mathcal{G}H(\Omega)(t),$$

whence $E_1(t) > H(\Omega)(t)$. We therefore have the following corollary :

COROLLARY 12.1. — *Under the conditions of this proposition the Poincaré series of E_1 dominates the Poincaré series of $H(\Omega)$ for any coefficient field k.*

This corollary disposes then of the algebraic part of the nondegenerate theory of Morse. The second, much harder and geometric part, is given by the subsequent proposition.

For each $a_i \in |S|$, let ω_i denote the segments of S of length precisely a_i. The symbol ω_i will also stand for the union of the points in Ω which these segments represent. Here is a form of Morse's basic theorem [**9**, p. 229], [**10**, p. 57].

THEOREM. — *a. The inclusion*

$$(\Omega^-(a_i) \cup \omega_i,\ \Omega^-(a_i)) \to (\Omega(l_i),\ \Omega(l_{i-1}))$$

induces an isomorphism onto in the singular homology;

b. $$H_n(\Omega^-(a_i) \cup s,\ \Omega^-(a_i)) = H_{n-\lambda(s)}(P).$$

Here s is any segment of S of length a_i, $\lambda(s)$ is its index, and P is a point.

This proposition in effect evalutes the term E_1 of the earlier spectra sequence. In particular, if the Morse series of (M, N, P) exists, then E_1 is of finite type and

$$\mathfrak{M}(M, N, P)(t) = E_1(t)$$

for any field of coefficients. By our earlier remarks this proposition therefore yields the Morse inequalities.

We should draw attention to the fact that our definition of the index of a geodesic segment s has consistently differed from the one given in SEIFERT-TRELFALL. That the two definitions are actually the same is the content of Morse's highly nontrivial focal point theorem [**9**, p. 58]. The definition given here is the more geometric one. All our applications depend on it heavily.

13. APPENDIX II : THE BETTI-NUMBERS OF E_6, E_7 AND E_8.

Theorem B of the introduction gives the Poincaré series of $\Omega(G)$ in the form

$$P(\Omega(G);\, t) = \sum_{\Delta \in \mathcal{J}} t^{2\lambda(\Delta)}, \tag{13.1}$$

where $\lambda(\Delta) = \sum [\theta^k(x)]$ $(i = 1, \ldots, m,$ θ^k the positive roots of $\mathcal{F}$, and x any interior point of $\Delta)$.

On the other hand, $P(\Omega(G); t)$ is also given by the formula

$$(13.2)\qquad \{1-t^{2(m_1-1)}\}^{-1}\ldots\{1-t^{2(m_l-1)}\}^{-1}.$$

Further these m_i determine the Poincaré series of the group itself, namely

$$(13.3)\qquad P(G; t)=\prod_{i=1}^{l}\{1+t^{(2m_i-1)}\}.$$

During a recent conversation, J. P. Serre heuristically obtained a relation between the numbers m_i, by comparing these two expressions near $t=1$. Here we will derive it. Very fortuitously this formula determines the numbers, m_i, for the exceptional Lie groups E_6, E_7 and E_8.

The relation in question is the following one : Let $\varphi_1, \ldots, \varphi_l$ be the subset of the $\{\theta^k\}_{\mathcal{F}}$ which describe the bounding planes of $\mathcal{F}$. Let $d=\sum_1^l d_i\varphi_i$ be the dominant root of $\mathcal{F}$, and set $a=\sum_1^l a_i\varphi_i$ equal to the sum $\sum_1^{2m}\theta^k$ of all the root forms in $\{\theta^k\}_{\mathcal{F}}$. The $2l$ integers d_i and a_i are uniquely determined up to order by the group. We refer to the d_i as the coefficients of the dominant root, and to the a_i as the coefficients of the sum of the positive roots. The m_i are called the exponents of G.

PROPOSITION 13.1. — *Let G be simple, connected, simply connected and compact. Then the following relation holds*

$$(13.4)\qquad l!\prod_1^l d_i\prod_1^l(m_i-1)=\prod_1^l a_i.$$

Here l is the rank of G, the d_i are the coefficients of the dominant root of G, the a_i are the coefficients of the sum of the positive roots of G and the m_i are the exponents of G.

PROOF. — Let $\varphi_1, \ldots, \varphi_l$ be the bounding planes of $\mathcal{F}$ as before. Let $e_1, \ldots, e_l$ be the dual basis to the φ_i[i.e. $\varphi_i(e_j)=\delta_{ij}$]. The points of $\mathcal{F}$ are then precisely the points of the form

$$X=\sum x_i e_i \qquad (x_i>0).$$

Let I be the unit cube of the closure of $\mathcal{F}$

$$I=\left\{X=\sum x_i e_i \mid 0\leq x_i\leq 1\right\}.$$

Let $T_i : \mathcal{F} \to \mathcal{F}$, be the translation

$$T_i(X) = (X + e_i).$$

Finally, if $(n_1, \ldots, n_l)$ is an n-tuple of non-negative integers, define $I(n_1, \ldots, n_l)$ by

$$I(n_1, \ldots, n_l) = T_1^{n_1} \circ T_2^{n_2} \circ \ldots \circ T_l^{n_l} \circ I.$$

It is easily checked that these translated cubes form a closed covering, U, of the closure of $\mathcal{F}$. Further, distinct cubes have disjoint interiors. Hence

$$\sum_{\Delta \in \mathcal{F}} t^{\lambda(\Delta)} = \sum_{I' \in U} \left\{ \sum_{\Delta \in I'} t^{\lambda(\Delta)} \right\}.$$

Because all root forms take integral values on any e_i, the function λ transforms according to the law

$$\lambda(x + e_i) = \lambda(x) + a_i \qquad [a_i = a(e_i)].$$

Hence $\lambda\{T_i(\Delta)\} = \lambda(\Delta) + a_i$. Therefore, if $I' = I(n_1, \ldots, n_e)$, then

$$\sum_{\Delta \in I'} t^{2\lambda(\Delta)} = t^{2n_1 a_1} t^{2n_2 a_2} \ldots t^{2n_l a_l} \sum_{\Delta \in I} t^{2\lambda(\Delta)}.$$

It follows that

$$\sum_{\Delta \in \mathcal{F}} t^{2\lambda(\Delta)} = \prod_1^l (1 - t^{2a_i})^{-1} Q(t^2),$$

where

$$Q(t^2) = \sum_{\Delta \in I} t^{2\lambda(\Delta)}.$$

Finally $(1 - t^{2a_i})$ admits a factor $(1 - t^2)$, as $a_i > 0$. Bringing this in evidence we obtain

$$(13.5) \qquad \sum_{\Delta \in \mathcal{F}} t^{2\lambda(\Delta)} = (1 - t^2)^{-l} \prod_1^l \left(\sum_0^{(a_i - 1)} t^{2k} \right) Q(t^2).$$

Similarly $(1 - t^2)$ can be factored from each term of (13.2), yielding

$$(13.6) \qquad P(\Omega(G); t) = (1 - t^2)^{-l} \prod_{i=1}^l \left(\sum_0^{(m_i - 1)} t^{2k} \right)^{-1}.$$

If we equate the last two equations, cancel $(1 - t^2)^{-l}$ and then set $t = 1$, we obtain

$$(13.7) \qquad \prod_1^l a_i = \prod_{i=1}^l (m_i - 1) Q(1).$$

It remains to compute $Q(1)$. Clearly this number counts the number of cells, Δ, in I. By means of the basis e_i, the cube I corresponds to the unit cube $0 \leqq x_i \leqq 1$. In this same basis the first cell in F, is given by the equations : $x_i \geqq 0$; and $\sum_1^l d_i x_i \leqq 1$ where d_i are the coefficients of the dominant root. Hence the volume of Δ in the euclidean volume of the x_i is $\left\{ l! \prod_1^l d_i \right\}^{-1}$. The volume of I is clearly 1. The number, $Q(1)$, of cells in I is therefore given by

$$Q(1) = l! \prod_1^l d_i.$$

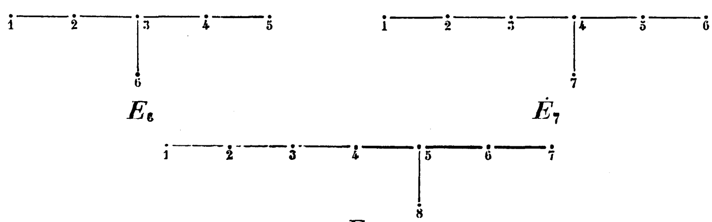
This proves the formula.

As an application of this new relation, we will compute the Betti numbers of E_6, E_7 and E_8. These numbers are of course known already. They were first announced by Chih-Ta Yen in his *Comptes rendus* Note : *Sur les polynomes de Poincaré des groupes de Lie exceptionels* (*C. R. Acad. Sc.*, vol. **228**, 1949, p. 628-630). They were explicitly derived by Borel and Chevalley in their paper : *The Betti numbers of the exceptional groups* (*Memoirs Math. Soc.*, N° **14**). Our relation gives a considerably shorter way to the result for E_7 and especially for E_8. However this happens purely by chance — the $(m_i - 1)$ all turn out to be primes !

The coefficients, d_i, of the dominant roots are well known. They are listed, for instance, in the paper by J. de Siebenthal : *Sur les sous-groupes fermés...* (*Com. Math. Helv.*, vol. **25**, 1951, p. 210-256). We will compute the $a_i's$ for the three groups E_6, E_7, E_8. This is done most efficiently from the Cartan integers, as I learned from A. Shapiro.

In terms of the bounding forms, φ_i, of F, the Cartan integers can be defined in the following manner : Let R_i be the reflection of $\mathbf{t}$ in the plane $\varphi_i = 0$. Then R_i induces a transformation R_i^* on the linear forms of $\mathbf{t}$.

Further $R_i^* \varphi_j = \varphi_j - \alpha_{ij} \varphi_i (i, j = 1, \ldots, l)$. The numbers α_{ij} are precisely the Cartan integers. These integers are determined in turn by the Schlaefly figure of the group in question. For E_6, E_7 and E_8 these fiigures are reproduced below



E_8

and in these cases they determine the Cartan integers according to the following simple convention

$$\alpha_{ii} = 2,$$
$$\alpha_{ij} = \begin{cases} -1 & \text{if vertex } i \text{ is joined to vertex } j \text{ by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Consider now the sum of the positive root forms, $a = \sum_1^{2m} \theta^k$. Under every $R_i^\star$ $(i = 1, \ldots, l)$, φ_i is changed into $-\varphi_i$ while the other terms of a are permuted. Hence

$$(13.8) \qquad R_i^\star a = a - 2\varphi_i \qquad (i = 1, \ldots, l).$$

If $a = \sum a_j \varphi_j$, then $R_i^\star a$ can also be computed directly

$$R_i^\star a = \sum_{j}{}_1^l a_j (\varphi_j - \alpha_{ji}\varphi_i).$$

Equating this expression with the right hand side of (13.8) we obtain

$$(13.9) \qquad \sum a_j \alpha_{jk} = 2.$$

These linear conditions determine a_j uniquely. In terms of the diagram, (13.8) is equivalent to the following recursion relation

$2a_j = 2 +$ sum of the a_i's at adjacent vertices on the diagram.

Assuming that $a_1 = \lambda$, (13.8) allows one to compute all the a_i's in terms of λ, and finally yields a linear equation in λ. Proceeding in this fashion we obtain :

For E_6 : The a_i are given (in increasing subscript order), by : λ, $2(\lambda - 1)$, $3(\lambda - 2)$, $\frac{5}{2}(\lambda - 10)$, $2(\lambda - 8)$, $\frac{3}{2}(\lambda - 2)$. Finally $\lambda = 16$. Hence these numbers are : 2^4; $2.3.5$; $3.2.7$; $2.3.5$; 2^4; 11.2; and

$$\prod_1^6 a_i = 2^{12}.3^3.5^2.7.11.$$

The d_i's are given by 1, 2, 3, 2, 1, 2. Our relation therefore yields

$$(13.10) \qquad \prod_1^6 (m_i - 1) = 1.2^5.5.7.11.$$

For E_7 : The a_i's : λ, $2(\lambda-1)$, $3(\lambda-2)$, $4(\lambda-3)$, $3(\lambda-5)$, $2(\lambda-10)$, $(2\lambda-5)$, with $\lambda=27$. Hence $\prod_1^7 a_i = 2^9.3^6.5^2.7^2.11.13.17$. Here the d_i have the values 1, 2, 3, 4, 3, 2, 2. Therefore by (13.4)

$$\prod_1^7 (m_i-1) = 1.3^2.5.7.11.13.17. \tag{13.11}$$

For E_8 : The a_i's : λ, $2(\lambda-1)$, $3(\lambda-2)$, $4(\lambda-3)$, $5(\lambda-4)$, $\frac{7}{2}(\lambda-21)$, $2(\lambda-12)$, $\frac{5}{2}(\lambda-9)$, with $\lambda=58$. The d_i's : 2, 3, 4, 5, 6, 4, 2, 3. Therefore by (13.4)

$$\prod_1^8 (m_i-1) = 1.7.11.13.17.19.23.29. \tag{13.12}$$

The exponents, m_i, of a simple group G have the following elementary property

$$m_i \geq 2, \qquad m_i = 2 \quad \text{for precisely one } i, \tag{13.13}$$

$$\sum_1^l (2m_i-1) = \dim G. \tag{13.14}$$

The relation (13.4), *together with these two conditions uniquely determines the exponents m_i of E_6, E_7, and E_8.* [We set $\overline{m}_i = (m_i-1)$.]

E_6 : Let the $\overline{m}_i$ $(i=1, \ldots, 6)$ be ordered in increasing order. Then $\overline{m}_1 = 1$, dim $E_6 = 78$. Therefore

$$\overline{m}_2 + \overline{m}_3 + \ldots + \overline{m}_6 = 35.$$

From this sum we conclude that amongst the five numbers $\overline{m}_2, \ldots, \overline{m}_6$ there must be an odd number of odd numbers. From (13.10), we know that $\prod_2^6 \overline{m}_i = 2^5.5.7.11$. Therefore the only odd numbers that can occur are 5, 7, and 11. If only one of them occurred, say 11, then at least 2.5 and 2.7 must occur. But $11+10+14=35$. This is impossible as then all the others would have to be zero. The situation is even worse if only 7 or 5 occurred. Hence all three, 5, 7 and 11 occur among the $\overline{m}_i$ $(i=2, \ldots, 6)$. Say $\overline{m}_4 = 5$, $\overline{m}_5 = 7$, $\overline{m}_6 = 11$. Then $\overline{m}_2 + \overline{m}_3 = 12$, and again by (13.9), $\overline{m}_2 = 2^\alpha$, $\overline{m}_3 = 2^\beta$, $\alpha+\beta = 5$. This implies that $\overline{m}_2 = 4$, $\overline{m}_3 = 8$. Hence the $\overline{m}_i$ are given by

$$1, \quad 4, \quad 5, \quad 7, \quad 8, \quad 11,$$

and the Poincaré polynomial of E_6 is

$$(1+t^3)(1+t^9)(1+t^{11})(1+t^{15})(1+t^{17})(1+t^{23}).$$

E_7 : The dimension of E_7 is 133. Therefore if $\overline{m}_1 = 1$, then $\sum_2^7 \overline{m}_i = 62$.

By (13.12), the product of the $\overline{m}_i$ is the product of 3^2, 5, 7, 11, 13, 17. Further the sum of these numbers is precisely 62. They are therefore the only solutions, and the Poincaré polynomial of E_7 is given by

$$(1+t^3)(1+t^{11})(1+t^{15})(1+t^{19})(1+t^{23})(1+t^{27})(1+t^{35}).$$

E_8 : In this case $\prod_2^8 \overline{m}_i = 7.11.13.17.19.23.29$. As there are seven prime numbers there is no ambiguity, and the Poincaré polynomial of E_8 is

$$(1+t^3)(1+t^{15})(1+t^{23})(1+t^{27})(1+t^{35})(1+t^{39})(1+t^{47})(1+t^{59}).$$

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