H.S. Allen

Duality of the spaces of linear functionals on dual vector spaces


<http://www.numdam.org/item?id=BSMF_1952__80__233_0>
DUALITY OF THE SPACES OF LINEAR FUNCTIONALS
ON DUAL VECTOR SPACES;

By H. S. ALLEN.

1. Dual linear vector spaces have been studied by Dieudonné [1] and Jacobson [2] and self-dual spaces by Rickart [3], [4]. In this paper it is established that the spaces of linear functionals on dual spaces are dual spaces and that the space of linear functionals on a self-dual space is self-dual and of the same type (symmetric or unitary), assuming that the characteristic of the field of scalars is not two.

2. Let $E$ and $F$ be left and right linear vector spaces over a field $K$. Suppose there is a bilinear functional $(x, y)$ defined on $E \times F$ to $K$ which is non-degenerate, i.e. $(x, y) = 0$ for all $x$ (resp. all $y$) implies $y = 0$ (resp. $x = 0$): then $E$ and $F$ are said to be dual spaces relative to $(x, y)$. Let $F^*$ and $E^*$ be the left and right $K$-spaces whose elements are the linear functionals on $F$ and $E$, the algebraic operations of addition and scalar multiplication being defined as in Bourbaki [5]. If $x \in E$ and $f \in E^*$ we write $f(x) = \langle x, f \rangle$: the spaces $E$ and $E^*$ are dual spaces relative to $\langle x, f \rangle$. If $y \in F$ and $g \in F^*$ we write $g(y) = \langle g, y \rangle$: the spaces $F^*$ and $F$ are dual spaces relative to $\langle g, y \rangle$. We prove the following theorem.

Theorem 1. — If the characteristic of $K \neq 2$, then $F^*$ and $E^*$ are dual spaces.

If $y_1 \in F$, the functional $f_1$ defined on $E$ by $\langle x, f_1 \rangle = (x, y_1)$ belongs to $E^*$ and the mapping $y_1 \to f_1$ is an isomorphic mapping of $F$ on a subspace $M$ of $E^*$. We shall denote this correspondence by writing $f_1 = y_1$. Similarly $E$ is isomorphic to a subspace $N$ of $F^*$ under a mapping $x_1 \to g_1$ where $\langle g_1, y \rangle = (x_1, y)$ and we write $g_1^* = x_1$.

There is a subspace $Q$ of $E^*$ which is the algebraic complement of $M$ (i.e. $E^*$ is the direct sum $M + Q$) and a subspace $P$ of $F^*$ which is the algebraic complement of $N$. Suppose $g \in F^*$ and $f \in E^*$. Let $g = g_1 + g_2$ where $g_1 \in N, g_2 \in P$ and $f = f_1 + f_2$ where $f_1 \in M, f_2 \in Q$. If $g_1^* = x_1$ and $f_2^* = y_1$ we define

$$\{ g, f \} = \frac{1}{2} [\langle x_1, f \rangle + \langle g, y_1 \rangle].$$
It is easily proved that the functional \([ g, f ]\) is bilinear. Suppose \(f\) is fixed and \([ g, f ] = 0\) for every \(g\). Taking \(g_2 = 0\) we obtain
\[
\langle g_1, y_1 \rangle = \langle g_1, y_1 \rangle = \langle x_1, y_1 \rangle = \langle x_1, f_1 \rangle \quad \text{and} \quad 0 = [ g_1, f ] = \frac{1}{2} \langle x_1, f + f_1 \rangle.
\]
This holds for every \(x_1 \in E\) and it follows that
\[
f + f_1 = 2f_1 = 0.
\]
Hence \(2f_1 = -f_2 \in M \cap Q\) and therefore \(f_1 = 0, f_2 = 0\) and \(f = 0\). Similarly \([ g, f ] = 0\) for every \(f\) implies \(g = 0\). It follows that \(E^*\) and \(E^*\) are dual spaces relative to \([ g, f ]\).

3. A left vector space \(E\) over a sfield \(K\) is said to be self-dual if there is an involution \(a \mapsto a^t\) in \(K\) and a scalar product \((x, y)\) defined on \(E \times E\) to \(K\) with the properties (i) \((x, y)\) is linear in \(x\) for every \(y\), (ii) \((x, y) = 0\) for all \(y\) implies \(x = 0\), (iii) \((y, x) = e(x, y)^t\) where \(e = \pm 1\) is a constant independent of \(x\) and \(y\). A self-dual space is said to be symplectic if every vector is isotropic, i.e. \((x, x) = 0\). If there exist non-isotropic vectors in the space, the space is said to be unitary, (Rickart [3], [4]). As before \(E^*\) will denote the space of linear functionals on \(E\). We prove the following result.

**Theorem 2.** — If the left vector space \(E\) over a sfield \(K\) of characteristic \(\neq 2\) is self-dual, then the right \(K\)-space \(E^*\) is self-dual. The space \(E^*\) is symplectic or unitary according as \(E\) is symplectic or unitary.

Let \(E_r\) be the right \(K\)-space whose elements are the elements of \(E\) with addition defined as on \(E\) and scalar multiplication defined by \(xa = a^t x (x \in E, a \in K)\). Then \(E\) and \(E_r\) are dual spaces relative to \((x, y)\). The space \(E_r\) is isomorphic to a subspace \(M\) of \(E^*\): we have \(x_1 \mapsto x_1\) where \(\langle x, x_1 \rangle = (x, x_1)\) and we write \(x_1 = X^r_1\). There is a subspace \(Q\) of \(E^*\) which is the algebraic complement of \(M\). Suppose \(X\) and \(Y\) in \(E^*\). Let \(X = X_1 + X_2\), where \(X_1 \in M, X_2 \in Q\) and \(Y = Y_1 + Y_2\) where \(Y_1 \in M, Y_2 \in Q\). Let \(X^r_1 = x_1\) and \(Y^r_1 = y_1\). We define the functional
\[
[X, Y] = \frac{1}{2} [e \langle x_1, Y \rangle + \langle y_1, X \rangle^t]
\]
on \(E^* \times E^*\) to \(K\). It is easily verified that \([X, Y]\) is linear in \(Y\) for every fixed \(X\). Suppose \([X, Y] = 0\) for every \(X\). Take \(X_2 = 0\) and we obtain
\[
o = [X_1, Y] = \frac{1}{2} [e \langle x_1, Y \rangle + \langle y_1, X_1 \rangle^t]
\]
\[
= \frac{1}{2} [e \langle x_1, Y \rangle + (y_1, x_1)^t] = \frac{1}{2} e[\langle x_1, Y \rangle + (x_1, y_1)]
\]
\[
= \frac{1}{2} e[\langle x_1, Y \rangle + \langle y_1, X_1 \rangle] = \frac{1}{2} e \langle x_1, Y \rangle + \langle y_1, X_1 \rangle.
\]
This holds for every \(x_1 \in E\) and it follows as in theorem 1 that \(Y = 0\). Evidently \([X, Y] = e[Y, X]^t\) and hence \(E^*\) is self-dual with respect to \([X, Y]\).

If \(E\) is unitary we may suppose that \((x, y)\) is hermitian, i.e. \(e = 1\) as indi-
There is an element $x_i \in E$ such that $(x_1, x_1) \neq 0$. If $X_i^2 = x_i$ we have $[X_1, X_1] = (x_1, x_1)$ and $E^\ast$ is unitary.

If $E$ is symplectic the form $(x, y)$ is skew-hermitian, i.e. $c = -i$ and $K$ is a field (Rickart). In this case $\alpha^2 = \alpha$ for every $\alpha \in K$ and $E^\ast$ is symplectic.

BIBLIOGRAPHIE.