BULLETIN DE LA S. M. F.

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Bulletin de la S. M. F., tome 80 (1952), p. 233-235
<http://www.numdam.org/item?id=BSMF_1952__80__233_0>

DUALITY OF THE SPACES OF LINEAR FUNCTIONALS ON DUAL VECTOR SPACES;

By H. S. Allen.

1. Dual linear vector spaces have been studied by Dieudonné [1] and Jacobson [2] and self-dual spaces by Rickart [3], [4]. In this paper it is established that the spaces of linear functionals on dual spaces are dual spaces and that the space of linear functionals on a self-dual space is self-dual and of the same type (symmetric or unitary), assuming that the characteristic of the sfield of scalars is not two.

2. Let E and F be left and right linear vector spaces over a sfield K. Suppose there is a bilinear functional (x, y) defined on $E \times F$ to K which is non-degenerate, i. e. $(x, y) = 0$ for all x (resp. all y) implies $y = 0$ (resp. $x = 0$) : then E and F are said to be dual spaces relative to (x, y). Let F^* and E^* be the left and right K-spaces whose elements are the linear functionals on F and E, the algebraic operations of addition and scalar multiplication being defined as in Bourbaki [5]. If $x \in E$ and $f \in E^*$ we write $f(x) = \langle x, f \rangle$: the spaces E and E^* are dual spaces relative to $\langle x, f \rangle$. If $y \in F$ and $g \in F^*$ we write $g(y) = \langle g, y \rangle$: the spaces F^* and F are dual spaces relative to $\langle g, y \rangle$. We prove the following theorem.

Theorem 1. — *If the characteristic of* K $\neq 2$, *then* F^* *and* E^* *are dual spaces.*

If $y_1 \in F$, the functional f_1 defined on E by $\langle x, f_1 \rangle = (x, y_1)$ belongs to E^* and the mapping $y_1 \to f_1$ is an isomorphic mapping of F on a subspace M of E^*. We shall denote this correspondence by writing $f_1^\alpha = y_1$. Similarly E is isomorphic to a subspace N of F^* under a mapping $x_1 \to g_1$ where $\langle g_1, y \rangle = (x_1, y)$ and we write $g_1^\beta = x_1$.

There is a subspace Q of E^* which is the algebraic complement of M (i. e. E^* is the direct sum $M + Q$) and a subspace P of F^* which is the algebraic complement of N. Suppose $g \in F^*$ and $f \in E^*$. Let $g = g_1 + g_2$ where $g_1 \in N$, $g_2 \in P$ and $f = f_1 + f_2$ where $f_1 \in M$, $f_2 \in Q$. If $g_1^\beta = x_1$ and $f_1^\alpha = y_1$ we define

$$\{g, f\} = \frac{1}{2}[\langle x_1, f \rangle + \langle g, y_1 \rangle].$$

It is easily proved that the functional $\{g, f\}$ is bilinear. Suppose f is fixed and $\{g, f\} = o$ for every g. Taking $g_2 = o$ we obtain

$$\langle g, y_1 \rangle = \langle g_1, y_1 \rangle = (x_1, y_1) = \langle x_1, f_1 \rangle \quad \text{and} \quad o = \{g_1, f\} = \frac{1}{2}\langle x_1, f + f_1 \rangle.$$

This holds for every $x_1 \in E$ and it follows that

$$f + f_1 = 2f_1 + f_2 = o.$$

Hence $2f_1 = -f_2 \in M \cap Q$ and therefore $f_1 = o$, $f_2 = o$ and $f = o$. Similarly $\{g, f\} = o$ for every f implies $g = o$. It follows that F^* and E^* are dual spaces relative to $\{g, f\}$.

3. A left vector space E over a sfield K is said to be *self-dual* if there is an involution $a \to a^J$ in K and a scalar product (x, y) defined on $E \times E$ to K with the properties (i)(x, y) is linear in x for every y, (ii)$(x, y) = o$ for all y implies $x = o$, (iii)$(y, x) = e(x, y)^J$ where $e = \pm 1$ is a constant independent of x and y. A self-dual space is said to be *symplectic* if every vector is isotropic, i. e. $(x, x) = o$. If there exist non-isotropic vectors in the space, the space is said to be *unitary*, (Rickart [3], [4]). As before E^* will denote the space of linear functionals on E. We prove the following result.

THEOREM 2. — *If the left vector space* E *over a sfield* K *of characteristic* $\neq 2$ *is self-dual, then the right* K*-space* E^* *is self-dual. The space* E^* *is symplectic or unitary according as* E *is symplectic or unitary.*

Let E_r be the right K-space whose elements are the elements of E with addition defined as on E and scalar multiplication defined by $xa = a^J x (x \in E, a \in K)$. Then E and E_r are dual spaces relative to (x, y). The space E_r is isomorphic to a subspace M of E^* : we have $x_1 \to X_1$ where $\langle x, X_1 \rangle = (x, x_1)$ and we write $x_1 = X_1^\alpha$. There is a subspace Q of E^* which is the algebraic complement of M. Suppose X and Y in E^*. Let $X = X_1 + X_2$, where $X_1 \in M$, $X_2 \in Q$ and $Y = Y_1 + Y_2$ where $Y_1 \in M$, $Y_2 \in Q$. Let $X_1^\alpha = x_1$ and $Y_1^\alpha = y_1$. We define the functional

$$[X, Y] = \frac{1}{2}[e\langle x_1, Y \rangle + \langle y_1, X \rangle^J]$$

on $E^* \times E^*$ to K. It is easily verified that $[X, Y]$ is linear in Y for every fixed X. Suppose $[X, Y] = o$ for every X. Take $X_2 = o$ and we obtain

$$\begin{aligned} o = [X_1, Y] &= \frac{1}{2}[e\langle x_1, Y \rangle + \langle y_1, X_1 \rangle^J] \\ &= \frac{1}{2}[e\langle x_1, Y \rangle + (y_1, x_1)^J] = \frac{1}{2}e[\langle x_1, Y \rangle + (x_1, y_1)] \\ &= \frac{1}{2}e[\langle x_1, Y \rangle + \langle x_1, Y_1 \rangle] = \frac{1}{2}e\langle x_1, Y + Y_1 \rangle. \end{aligned}$$

This holds for every $x_1 \in E$ and it follows as in theorem 1 that $Y = o$. Evidenthy $[X, Y] = e[Y, X]^J$ and hence E^* is self-dual with respect to $[X, Y]$.

If E is unitary we may suppose that (x, y) is hermitian, i. e. $e = 1$ as indicated by Rickart [3], [4]. There is an element $x_1 \in E$ such that $(x_1, x_1) \neq 0$. If $X_1^{\alpha} = x_1$ we have $[X_1, X_1] = (x_1, x_1)$ and E^* is unitary.

If E is symplectic the form (x, y) is skew-hermitian, i. e. $e = -1$ and K is a field (Rickart). In this case $a^J = a$ for every $a \in K$ and E^* is symplectic.

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