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Duality of the spaces of linear functionals on dual vector spaces


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DUALITY OF THE SPACES OF LINEAR FUNCTIONALS
ON DUAL VECTOR SPACES;

By H. S. ALLEN.

1. Dual linear vector spaces have been studied by Dieudonné [1] and Jacobson [2] and self-dual spaces by Rickart [3], [4]. In this paper it is established that the spaces of linear functionals on dual spaces are dual spaces and that the space of linear functionals on a self-dual space is self-dual and of the same type (symmetric or unitary), assuming that the characteristic of the field of scalars is not two.

2. Let \( E \) and \( F \) be left and right linear vector spaces over a field \( K \). Suppose there is a bilinear functional \( (x, y) \) defined on \( E \times F \) to \( K \) which is non-degenerate, i.e., \( (x, y) = 0 \) for all \( x \) (resp. all \( y \)) implies \( y = 0 \) (resp. \( x = 0 \)) then \( E \) and \( F \) are said to be dual spaces relative to \( (x, y) \). Let \( F^* \) and \( E^* \) be the left and right \( K \)-spaces whose elements are the linear functionals on \( F \) and \( E \), the algebraic operations of addition and scalar multiplication being defined as in Bourbaki [5]. If \( x \in E \) and \( f \in E^* \) we write \( f(x) = \langle x, f \rangle \): the spaces \( E \) and \( E^* \) are dual spaces relative to \( (x, f) \). If \( y \in F \) and \( g \in F^* \) we write \( g(y) = \langle g, y \rangle \): the spaces \( F^* \) and \( F \) are dual spaces relative to \( (g, y) \). We prove the following theorem.

Theorem 1. — If the characteristic of \( K \neq 2 \), then \( F^* \) and \( E^* \) are dual spaces.

If \( y_1 \in F \), the functional \( f_1 \) defined on \( E \) by \( \langle x, f_1 \rangle = (x, y_1) \) belongs to \( E^* \) and the mapping \( y_1 \to f_1 \) is an isomorphic mapping of \( F \) on a subspace \( M \) of \( E^* \). We shall denote this correspondence by writing \( f_1^2 = y_1 \). Similarly \( E \) is isomorphic to a subspace \( N \) of \( F^* \) under a mapping \( x_1 \to g_1 \) where \( \langle g_1, y \rangle = (x_1, y) \) and we write \( g_1^3 = x_1 \).

There is a subspace \( Q \) of \( E^* \) which is the algebraic complement of \( M \) (i.e., \( E^* \) is the direct sum \( M + Q \)) and a subspace \( P \) of \( F^* \) which is the algebraic complement of \( N \). Suppose \( g \in F^* \) and \( f \in E^* \). Let \( g = g_1 + g_2 \) where \( g_1 \in N \), \( g_2 \in P \) and \( f = f_1 + f_2 \) where \( f_1 \in M \), \( f_2 \in Q \). If \( g_1^3 = x_1 \) and \( f_1^3 = y_1 \) we define

\[
\{g, f\} = \frac{1}{2} [\langle x_1, f_1 \rangle + \langle g_1, y_1 \rangle].
\]
It is easily proved that the functional \( \langle g, f \rangle \) is bilinear. Suppose \( f \) is fixed and \( \langle g, f \rangle = 0 \) for every \( g \). Taking \( g_2 = 0 \) we obtain
\[
\langle g, y_1 \rangle = \langle g_1, y_1 \rangle = \langle x_1, y_1 \rangle = \langle x_1, f_1 \rangle \quad \text{and} \quad 0 = \langle g_1, f \rangle = \frac{1}{2} \langle x_1, f + f_1 \rangle.
\]
This holds for every \( x_1 \in E \) and it follows that
\[
f + f_1 = 2f_1 + f = 0.
\]
Hence \( 2f_1 = -f_2 \in M \cap Q \) and therefore \( f_1 = 0, f_2 = 0 \) and \( f = 0 \). Similarly \( \langle g, f \rangle = 0 \) for every \( g \) implies \( g = 0 \). It follows that \( E^* \) and \( E^* \) are dual spaces relative to \( \langle g, f \rangle \).

3. A left vector space \( E \) over a field \( K \) is said to be self-dual if there is an involution \( a \mapsto a^t \) in \( K \) and a scalar product \( \langle x, y \rangle \) defined on \( E \times E \) to \( K \) with the properties (i) \( \langle x, y \rangle \) is linear in \( x \) for every \( y \), (ii) \( \langle x, y \rangle = 0 \) for all \( y \) implies \( x = 0 \), (iii) \( \langle y, x \rangle = e \langle x, y \rangle \) where \( e = \pm 1 \) is a constant independent of \( x \) and \( y \). If every vector is isotropic, i.e. \( \langle x, x \rangle = 0 \). If there exist non-isotropic vectors in the space, the space is said to be unitary, (Rickart [3], [4]). As before \( E^* \) will denote the space of linear functionals on \( E \). We prove the following result.

**Theorem 2.** — If the left vector space \( E \) over a field \( K \) of characteristic \( \neq 2 \) is self-dual, then the right \( K \)-space \( E^* \) is self-dual. The space \( E^* \) is symplectic or unitary according as \( E \) is symplectic or unitary.

Let \( E_r \) be the right \( K \)-space whose elements are the elements of \( E \) with addition defined as on \( E \) and scalar multiplication defined by \( xa = a^t x (x \in E, a \in K) \). Then \( E \) and \( E_r \) are dual spaces relative to \( \langle x, y \rangle \). The space \( E_r \) is isomorphic to a subspace \( M \) of \( E^* \) : we have \( x_1 \mapsto X_1 \) where \( \langle x, X_1 \rangle = \langle x, x_1 \rangle \) and we write \( x_1 = X_1^t \). There is a subspace \( Q \) of \( E^* \) which is the algebraic complement of \( M \). Suppose \( X \) and \( Y \) in \( E^* \). Let \( X = X_1 + X_2 \), where \( X_1 \in M, X_2 \in Q \) and \( Y = Y_1 + Y_2 \), where \( Y_1 \in M, Y_2 \in Q \). Let \( X_r = x_1 \) and \( Y_r = y_1 \). We define the functional
\[
\langle [X, Y] \rangle = \frac{1}{2} \left[ e \langle x_1, Y \rangle + \langle y_1, X \rangle \right]
\]
on \( E^* \times E^* \) to \( K \). It is easily verified that \( [X, Y] \) is linear in \( Y \) for every fixed \( X \). Suppose \( [X, Y] = 0 \) for every \( X \). Take \( X_2 = 0 \) and we obtain
\[
o = \langle [X, Y] \rangle = \frac{1}{2} \left[ e \langle x_1, Y \rangle + \langle y_1, X \rangle \right]
\]
\[
= \frac{1}{2} \left[ e \langle x_1, Y \rangle + \langle y_1, x_1 \rangle \right] = \frac{1}{2} e \left[ \langle x_1, Y \rangle + \langle x_1, y_1 \rangle \right]
\]
\[
= \frac{1}{2} \left[ e \langle x_1, Y \rangle + \langle x_1, Y \rangle \right] = \frac{1}{2} e \langle x_1, Y + Y_1 \rangle.
\]
This holds for every \( x_1 \in E \) and it follows as in theorem 1 that \( Y = 0 \). Evidently \( [X, Y] = e [Y, X]^t \) and hence \( E^* \) is self-dual with respect to \( [X, Y] \). If \( E \) is unitary we may suppose that \( \langle x, y \rangle \) is hermitian, i.e. \( e = 1 \) as indi-
cated by Rickart [3], [4]. There is an element $x \in E$ such that $(x, x) \neq 0$.
If $X_1 = x_1$ we have $[X_1, X_1] = (x_1, x_1)$ and $E^*$ is unitary.
If $E$ is symplectic the form $(x, y)$ is skew-hermitian, i. e. $c = -1$ and $K$ is a
field (Rickart). In this case $\alpha^t = \alpha$ for every $\alpha \in K$ and $E^*$ is symplectic.

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