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Limit in terms of continuous transformation

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LIMIT IN TERMS OF CONTINUOUS TRANSFORMATION;

By Norbert Wiener.

1. The *calcul fonctionnel*, or the study of the limit properties of an abstract assemblage, has been investigated in the course of the last fifteen years from a number of distinct standpoints. In addition to the notion of sequential limit (1) which furnished the starting-point of Fréchet's well known thesis, and the more restrictive concept of *écart* (or *distance*, as Fréchet, now calls it), there is Riesz' non-sequential limit (2), which Fréchet, in turn, has discussed as a special case of an extremely general notion of neighborhood (3). If, however, we consider the *calcul fonctionnel* with reference to the obviously intimate bearing which it has on analysis situs — which has been defined essentially as the study of the invariants of the group of all bicontinuous biunivocal transformations (4) — there is another avenue of approach to which our attention is immediately directed. This paper will be devoted to the discussion of the derivation of limit-properties from those of continuous transformations.

This work has been carried out in France with the aid of much advice and many important suggestions from Professor Fréchet, to whom I wish to express my sincerest thanks.

2. Let us start with a class Σ of biunivocal transformations of all the elements of a class C. On the hypothesis that these are to be considered as bicontinuous, how should we naturally proceed

(1) *Sur quelques points du Calcul fonctionnel* (*Rend. Cir. Mat. Palermo*, vol. XXII, p. 4). This article will in the future be referred to as *Thesis*.

(2) In a Communication read before the International Congress of Mathematicians of Rome.

(3) *Sur la notion de voisinage dans les ensembles abstraits* (*Bulletin des Sciences mathématiques*, 2d séries, vol. XLII). Hereinafter to be referred to as *V*.

(4) A definition to this effect is to be found in the article on *Analysis situs* by Dehn and Heegard, in the *Encyklopädie der Mathematischen Wissenschaften*.

to define the limit-elements of E, a sub-class of C? It may be shown that in an n-space if the operations of Σ are bicontinuous in any ordinary sense, if A is a limit-element of E, every transformation of Σ which leaves invariant every element of E except A will leave A invariant also. It may furthermore be shown that if A is not a limit-element of E, there is some little region containing A but no point of E which we may permute by some transformation of Σ in such a way as to change A but leave each point of E invariant. We shall follow out the obvious analogy and make the following definition :

An element A *of* C *will be said to be a limit-element of a sub-set* E *of* C, *when and only when every transformation of* Σ *that leaves invariant all the elements of* E, *except possibly* A, *also leaves* A *invariant.* That this definition is natural over a wide set of cases will appear in what follows.

3. A system in which limit-element is défined in this manner will be called a system (s). It becomes a matter of interest to discover when limit-properties, as defined in a system (s), are really invariant under all the transformations of Σ, for only then will the transformations of Σ be in any true sense continuous. Now, let us transform C by the transformation T of Σ, and let us represent the transform of A by T(A), the set of transforms of elements of E (which does not contain A) by T((E)). If limit-properties are to be left invariant, if S is any transformation of Σ, then if, whenever B belongs to E, $S(T(B)) = T(B)$, we shall have $S(T(A)) = T(A)$. In other words, if we write the transformation which changes A into R(S(A)), $R|S$, and the transformation which changes R(A) into A, $\breve{R}$, it will follow that *every transformation of the form* $\breve{T}|S|T$ *which leaves each term of* E *invariant will also have to leave* A *invariant*, if S and T belong to Σ. If the operations of Σ are to be bicontinuous, the same statement applies to transformations of the form $T|S|\breve{T}$. These two conditions together are necessary and *sufficient* that Σ consist of bicontinuous operations. We shall speak of any system (Σ) satisfying these conditions as a system (J).

4. It will be observed that if Σ is a group, an (s) is a (J). It will

also be observed that in a (J) there is always a group of bicontinuous, biunivocal operations generated by the members of Σ. It does not follow, however, that these cases are identical as to their limit-properties, for though a bicontinous, biunivocal transformation which keeps invariant every element of E will also transform every limit-element of E into a limit-element of E, it does not necessarily result that it will keep every limit-element of E invariant.

The case when Σ consists precisely of the group of *all* bicontinuous, biunivocal transformations is especially interesting; we shall denominate it (J_1). If we call a set E *closed* if it contains all its limit-elements — if, that is, it contains all those elements that are left invariant by every transformation which belongs to Σ and leaves every element of E invariant — then it is easy to see that a bicontinuous transformation is precisely one which leaves all closed sets closed. We thus get as the necessary and sufficient condition that Σ should *contain* all bicontinuous biunivocal transformations.

A. *If* R *is a biunivocal transformation of* C *such that* R *and* $\breve{R}$ *both leave all closed sets closed, it belongs to* Σ.

If Σ *consists* of all bicontinuous, biunivocal transformations, it must also satisfy the group-conditions.

B. *If* R *belongs to* Σ, *so does* $\breve{R}$, and.

C. *If* R *and* S *belong to* Σ, *so does* $R \mid S$.

A, B, and C are moreover *sufficient* to show that Σ consists in all bicontinuous, biunivocal transformations, for it results from B and C that if S and T belong to Σ, so do $S|T|\breve{S}$ and $\breve{S}|T|S$, so that S and $\breve{S}$ simply permute the operations of Σ and change no limit — properties. Needless to say, every (J_1) is a (J).

4. Not every system in which limit-properties are defined, nor even every system in which sequential limit is defined, nor finally every system in which distance is defined is a (J). Consider the set of points on a number-line consisting of $x = 0$ and $x = \frac{1}{n}$ for all integral values of n. Manifestly, every transformation leaving limit-properties invariant will leave $x = 0$ inva-

riant. Hence any biunivocal, bicontinuous transformation leaving all members of any set E invariant leaves $x = 0$ invariant, and $x = 0$ is a limit-element of any set. However, this does not agree with our original notion of limit.

The next problem is therefore to determine under what circumstances a system in which limit-element is defined belongs to one of the classes (J) or (J_1). To begin with we shall search for the necessary condition that a (V) [1] or system in which neighborhood is defined, be a (J). We may make use of the fact that our notion of limit necessarily satisfies the two conditions :

1° Every limit-element of a set E is also a limit-element of every set containing E;

2° The fact that an element A is or *is* not a limit-element of E is not affected by adjoining A to E.

Fréchet has shown that under these conditions, if we define a neighborhood of A as a set V_A of elements such that the set of all elements not in V_A does not have A as a limit-element, then the necessary and sufficient condition for a set E to have A as a limit-element is for E to have elements in every V_A. In terms of Σ, a neighborhood V_A of A will be a set of elements such that there is at least one transformation of Σ leaving invariant every element of C not in V_A but changing A. Since in a (J) every member of Σ is bicontinuous, a *necessary* condition that our system be a (J) is that given any element A and any neighborhood of A, there is at least one biunivocal transformation R of the whole of C such that :

(1) If B does not belong to V_A, $R(B) = B$;
(2) $R(A) \neq A$;
(3) If D is a limit-element of E, R(D) is a limit-element of R((E));
(4) If R(D) is a limit-element of R((E)), D is a limit-element of E.

If to these conditions be adjoined the condition that, if R be a biunivocal transformation of the whole of C satisfying the conditions (3) and (4) just mentioned, then if E is any set of elements each of which remains unchanged under R [2] and A is a limit-

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[1] *V.*, p. 4.
[2] *Thesis*, Also. *V*, p. 1.

element of E, $R(A) = A$, we get a *necessary* condition that our system be a (J_1). This is, moreover, *sufficient*, for clearly the new system Σ' consisting of all biunivocal transformations of C satisfying (3) and (4) will be a (J_1). It only remains to prove that it leads to our original notion of limit. Obviously any set having A as a limit in our new sense will have terms in each V_A, and will consequently have A as a limit in our original sense. Moreover, every set E having A as a limit in our original sense will have at least one term in each V_A, for otherwise there would be at least one biunivocal transformation satisfying (3) and (4) leaving each term of E invariant but changing A, which would be contrary to our new assumption.

5. The assumption that every biunivocal bicontinuous transformation of the whole of C, if it leaves every element of a set invariant, leaves every element of the derivative invariant, will clearly be satisfied if whenever A is a limit-element of E and B is any element, A is a limit-element of a sub-set of E which does not have B for a limit-element. Let it be observed that this is a sufficient condition for a (J), to be a (J_1). This property of limit will always occur when our system is what Fréchet calls an (L), in which the derivative of a set may be defined in terms of sequential limit. It is an interesting matter to find a whole wide set of cases which all fulfill both this and the other part of the necessary and sufficient condition for a (J_1).

We shall say that a set σ of entities is a *vector family* if there are associated with it operations $\oplus$, $\odot$, and $\| \ \|$ satisfying the following conditions :

(1) If ξ and η belong to σ, $\xi \oplus \eta$ belongs to σ,

(2) If ξ belongs to σ and n is a real number $\geqq 0$, $n \odot \xi$ belongs to σ,

(3) If ξ belongs to σ, $\|\xi\|$ is a real number $\geqq 0$,

(4) $n \odot (\xi \oplus \eta) = (n \odot \xi) \oplus (n \odot \eta)$,

(5) $(m \odot \xi) \oplus (n \odot \xi) = (m + n) \odot \xi$,

(6) $\|m \odot \xi\| = m . \|\xi\|$,

(7) $\|\xi \oplus \eta\| \leqq \|\xi\| + \|\eta\|$,

(8) $m \odot (n \odot \xi) = mn \odot \xi$.

We shall say that a set E of elements is a system (Ve) if there is a vector-family σ such that.

I. If A and B belong to, E, there is associated with them a single member AB of σ;

II. $\|AB\| = \|BA\|$;

III. Given an element A of E and an element ξ of σ, there is an element B of E such hat $AB = \xi$;

IV. $AC = AB \oplus BC$;

V. $\|AB\| = 0$ when and only when $A = B$;

VI. If $AB = CD$, $BA = DC$.

It will be seen that $\|AB\|$ is an écart in Fréchet's original sense [1], and that we can say that a sub-class F of E has the limit-element A when, given any positive number ε, there is always a member B of F other than A, such that $\|AB\| < \varepsilon$.

Let us suppose given some set F consisting of all the elements B such that $\|AB\| < \varepsilon$, where ε is some positive quantity. Clearly every neighborhood of A will contain the whole of some set of the sort. Let us consider the following transformation : if $\|AB\| \geqq \varepsilon$, let B be unchanged, but if $\|AB\| < \varepsilon$, let B be changed into the element C such that $AC = \frac{\|AB\|}{\varepsilon} \odot AB$. The existence and biunivocality of this transformation will be guaranteed by our assumptions. It will also be bicontinuous, as the following argument will prove.

Let B and C be any two elements in F, and let their transforms by our transformation be B' and C'. Let D be the element such that $AD = \frac{\|AB'\|}{\|AB\|} \odot AC$. I wish to find an upper bound for $\|B'C'\|$ in terms of $\|BC\|$. We have

$$
\begin{aligned}
\|B'C'\| &\leqq \|B'D\| + \|DC'\| \\
&\leqq \|B'A \oplus AD\| + \|DC'\| \\
&\leqq \left\| \frac{\|AB'\|}{\|AB\|} \odot (BA \oplus AC) \right\| + \left\| \left\{ \frac{\|AB'\|}{\|AB\|} \odot CA \right\} \oplus \left\{ \frac{\|AC'\|}{\|AC\|} \odot AC \right\} \right\| \\
&\leqq \frac{\|AB\|}{\varepsilon} \|BC\| + \left| \frac{\|AB\| - \|AC\|}{\varepsilon} \right| \|AC\| \\
&\leqq \|BC\| \left\{ \frac{\|AB\|}{\varepsilon} + \frac{\|AC\|}{\varepsilon} \right\} \\
&\leqq 2\|BC\|
\end{aligned}
$$

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(1) *Thesis*, p. 30.

Evidently, for a given B, if we have a set G such that, for every positive η, there is a member C of G such that $\|BC\| < \eta$, then the same statement will hold true of B' and the set G' made up of transforms of the elements of G.

Let us now proceed to find an upper bound for $\|BC\|$ in terms of $\|B'C'\|$. We have, proceeding as before, supposing that

$$AD' = \frac{\|AB\|}{\|AB'\|} \odot AC'$$

$$\begin{aligned}
\|BC\| &\leqq \|BA \oplus AD'\| + \|D'C\| \\
&\leqq \left\| \frac{\|AB\|}{\|AB'\|} \odot (B'A \oplus AC') \right\| + \left\| \left\{ \frac{\|AB\|}{\|AB'\|} \odot C'A \right\} \oplus \left\{ \frac{\|AC\|}{\|AC'\|} \odot AC' \right\} \right\| \\
&\leqq \sqrt{\frac{\varepsilon}{\|AB'\|}} \|B'C'\| + \left| \sqrt{\frac{\varepsilon}{\|AB'\|}} - \sqrt{\frac{\varepsilon}{\|AC'\|}} \right| \|AC'\| \\
&\leqq \|B'C'\| \left\{ \sqrt{\frac{\varepsilon}{\|AB'\|}} + \frac{\varepsilon}{\|AB'\| \left(\sqrt{\frac{\varepsilon}{\|AB'\|}} + \sqrt{\frac{\varepsilon}{\|AC'\|}} \right)} \right\} \\
&\leqq \|B'C'\| \left\{ \sqrt{\frac{\varepsilon}{\|AB'\|}} + \frac{\varepsilon}{2\|AB'\|} \right\}.
\end{aligned}$$

Unless B' is A, this shows that a set of transforms of members of G can have B' for a limit-element only when G has B for a limit-element. In the special case when B and A coincide, it is easily seen that $\|AC\|$ is small when and only when $\|AC'\|$ is small. Our transformation is thus bicontinuous and biunivocal when we consider elements B such that $\|AB\| < \varepsilon$. There is no difficulty in showing that the bicontinuity is valid over the whole system, for if $\|AB\| = \varepsilon$, then $AB = \frac{\|AB\|}{\varepsilon} \odot AB$, so that our transformations within and without the (sphere) $\|AB\| = \varepsilon$ make a precise join on the sphere. No element in the sphere except A if left invariant.

Now, let B be any element of a system (Ve). I say that given any neighborhood V_B of B, it will be possible to find a transformation of the sort just discussed which will change B but leave invariant every point not in V_B. Clearly, there will be some positive number η such that V_B will contain all the elements C such that $\|BC\| < \eta$. Let A be some element other than B — there always will be some such element — such that $\|AB\| < \frac{\eta}{2}$. Let ε be some number such that $\|AB\| < \varepsilon < \frac{\eta}{2}$. Then all the elements whose distance from A is less than ε will lie in V_B, while B will be one of these elements. Establish a transformation such as was discussed in the last paragraph. This will change B and leave invariant every element not in V_B, and will be biunivocal and bicontinuous. Hence the first part of the condition that our system be a (J_1) is satisfied. Since every (Ve) is an (L), the second part is also satisfied, and every (Ve) is a (J_1).

6. Examples of systems (Ve) are the following.

(1) The system consists of all n-partite numbers, $(x_1, x_2, \ldots, x_n)$, σ likewise consists of all n-partite numbers. If $A = (x_1, x_2, \ldots, x_n)$, and $B = (y_1, y_2, \ldots, y_n)$, $AB = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$. If $\xi = (u_1, u_2, \ldots, u_n)$, $\|\xi\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$, and

$$k \odot \xi = (ku_1, ku_2, \ldots, ku_n).$$

If, moreover,

$$\eta = (v_1, v_2, \ldots, v_n),$$

$$\xi \oplus \eta = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

(2) The system consists of all ∞-partite numbers,

$$(x_1, x_2, \ldots, x_n, \ldots),$$

such that there is a finite X such that whatever n, $|x_n| \leqq X$. σ likewise consists of all such numbers. If $A = (x_1, x_2, \ldots, x_n, \ldots)$ and $B = (y_1, y_2, \ldots, y_n, \ldots)$, $AB = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n, \ldots)$. If $\xi = (u_1, u_2, \ldots, u_n, \ldots)$ and $\eta = (v_1, v_2, \ldots, v_n, \ldots)$,

$$\|\xi\| = \text{upper bound}_n |u_n|,$$

$k \odot \xi = (ku_1, ku_2, \ldots, ku_n, \ldots)$, and

$$\xi \oplus \eta = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots).$$

(3) The system consists of all ∞-partite numbers

$$(x_1, x_2, \ldots, x_n, \ldots),$$

such that the series $x_1^2 + x_2^2 + \ldots + x_n^2 + \ldots$ converges. σ likewise consists of all such numbers AB, $k \odot \xi$, and $\xi \oplus \eta$ are defined as in (2). If

$$\xi = (u_1, u_2, \ldots, u_n, \ldots), \qquad \|\xi\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2 + \ldots}.$$

(4) The system consists of all continuous functions of a real variable defined over a given closed interval. $\mathfrak{I}$ likewise consists of all such functions. The vector fg is the function $f(x) - g(x)$. If $\xi = f(x)$ and $\eta = g(x)$, $\|\xi\| = \max |f(x)|$, $k \odot \xi = kf(x)$, and $\xi \oplus \eta = f(x) + g(x)$.

7. In addition to these systems which are systems (V*e*) « im grossen », there are systems which may be said to be systems (V*e*) « im kleinen » or as we shall say, systems (V*e*'). We shall characterize them as follows : every point A has at least one neighborhood V_A which can be put into biunivocal bicontinuous correspondence with a set $V_{A'}$ consisting of all the elements B' in a (V*e*) such that $\|A' B'\| \leqq \varepsilon$ [1], in such a manner that A will correspond to A' and the set of all points in V_A that are limit-points of the set of all elements not in V_A shall correspond to the set of elements consisting of all elements B' such that $\|A'B'\| = \varepsilon$. It is clear that our argument by which we proved that every (V*e*) was a (J_1) will also prove that every (V*e*') is a (J_1). The points on a sphere or on a torus are examples of sets (V*e*').

8. Another example of a set (J_1) is Fréchet's E_ω [2]. This consists of all ∞-partite numbers $(x_1, x_2, \ldots, x_n, \ldots)$, with limit defined non-uniformly. As Fréchet has shown, in this space limit may be defined in terms of distance, the distance between two elements $(x_1, x_2, \ldots, x_n, \ldots)$ and $(x'_1, x'_2, \ldots, x'_n, \ldots)$ being defined to be $\sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - x'_n|}{1 + |x_n - x'_n|}$. Clearly, then, any neighborhood V_A of $(x_1, x_2, \ldots, x_n, \ldots)$ will contain for some $\varepsilon > 0$ all of the points $(x'_1, x'_2, \ldots, x'_n, \ldots)$ such that $\sum_{n=1}^{\infty} \frac{1}{n!} \frac{1 + |x_n - x'_n|}{|x_n - x'_n|} < \varepsilon$. Let us choose k in such a manner that $\sum_{n=k+1}^{\infty} \frac{1}{n!} > \frac{\varepsilon}{2}$. If, then, I choose a certain set of positive quantities $\eta_1, \eta_2, \ldots, \eta_k$ in such a manner

(1) Here $\varepsilon > 0$.
(2) *Thesis*, p. 39.

that $\sum_{n=1}^{k} \frac{1}{n!} \frac{\eta_n}{1+\eta_n} < \frac{\varepsilon}{2}$, a thing which is manifestly always possible, it will follow that V_A will contain every element $(x'_1, x'_2, \ldots, x'_n, \ldots)$, satisfying the finite set of conditions

$$x_1 - \eta_1 \leqq x'_1 \leqq x_1 + \eta_1, \qquad x_2 - \eta_2 \leqq x'_2 \leqq x_2 + \eta_2, \qquad x_k - \eta_k \leqq x'_k \leqq x_k + \eta_k.$$

Hence, if whenever V'_A is a neighborhood determined by a finite set of intervals in the above manner, there is a biunivocal, bicontinuous transformation changing $(x_1, x_2, \ldots, x_n, \ldots)$ but leaving invariant any element not in V'_A, our E_ω is a (J_1).

Now, we know there is a bicontinuous biunivocal transformation in the k-space made up of all elements $(x'_1, x'_2, \ldots, x'_k)$ which changes $(x_1, x_2, \ldots, x_k)$ but leaves invariant every element $(x'_1, x'_2, \ldots, x'_k)$ not satisfying simultaneously the conditions $x_1 - \eta_1 \leqq x'_1 \leqq x_1 + \eta_1, \ \ldots, \ x_k - \eta_k \leqq x'_k \leqq x_k + \eta_k$. Let us consider the transformation which affects the firts k coordinates of a point in E_ω in the above manner, but leaves all the other coordinates invariant. It is easy to see that this is bicontinuous and biunivocal, that it changes $(x_1, x_2, \ldots, x_n, \ldots)$, and that it leaves invariant every point not in V'_A. Hence E_ω is a (J_1).

9. Up to this point we have been considering the conditions that a (V) be a (J) or (J_1). Let us now reverse our point of view, and ask, given a (J) or (J_1), what are the conditions that it belong in one or another of the categories of Fréchet and Riesz. We have already seen (§ 4) that every (J) is a (V); the next most restricted classes so far discussed are those that also satisfy certain of the following conditions, numbered by Fréchet.

2. Given any two sets, E and F, $(E+F)'$ is contained in $E'+F'$.

3. A set consisting of a single element has no limit-element.

4. If A is a limit element of a set E, and if B is distinct from A, there is always at least one set F which has A for a limit-element without having B for a limit-element.

5. Given any set E, $(E)'$ is contained in E'.

In these condition, E' means the set of all limit-elements of E. Conditions 2, 3, and 4 make a set, what Fréchet calls after F. Riesz, who first investigated such sets, a set (R). Conditions 2, 3, and 5 have been found by Fréchet to form a combination even more important, for in every class (V) satisfying them the necessary, and ufficient condition for a set E to possess Borel's property is that every infinite sub-set of E should have at least one limit-element belonging to E (1).

Condition 2 may be written in the form : if A is a term which is neither a limit-element of E nor a limit-element of F, then it is not a limit-element of E + F. Stating this in terms of transformations, we get.

2′ *If there is a transformation from* Σ *changing the element* A, *but leaving all the elements of the class* E *invariant*, *and there is also a transformation from* Σ *changing* A *but leaving all the elements of the class* F *invariant*, *then there is a transformation from* Σ *changing* A, *but leaving all the elements of* E + F *invariant.*

In terms of transformation, 3 simply becomes.

3′ *Given any two elements*, A *and* B, *there is a transformation from* Σ *changing* A *but leaving* B *invariant.*

Condition 4 is rather awkward to translate; translating it literally, we get

4′ *If there is a set* E *not containing the element* A, *but such that every transformation from* Σ *that leaves all the elements of* E *invariant leaves* A *also invariant*, *then given any element* B, *distinct from* A, *there is a set* F *such that there is a transformation from* Σ *changing* B *but leaving each element of* F *invariant*, *while there is no transformation from* Σ *changing* A *but leaving each member of* F *invariant.*

10. It is interesting to remark that in sets (T) condition 5 is a consequence of 2′ and 3′. It results from 2′ and 3′ that the set of limit-elements of a class E is not affected by removing an element from E. Let A be any term in (E′)′. As an imn diate consequence of the definition of A, it is invariant under every transformation that leaver every element of E″ invariant. If A belongs to E′, our theorem needs no proof. If A does not belong to E′, on the other hand, E′, wich consits of all those elements B wich remain

(1) *V.*, p. 2, 3, 7.

invariant under all those transformations of Σ which leave each term of E — B invariant, whill consist of all those elements B which remain invariant under those transformations wich leave each term of E-A-B invariant. Hence every transformation that leaves each term of E — A invariant will leave each term of E′ invariant, and will hence leave A invariant. Hence again, A belongs to E′.

11. The next thing to investigate is the relation between systems (J) or (J_1) and systems in which sequence can be defined in terms of sequential limit-the systems (L) of Fréchet. We have already seen (§ 5) that a (J) which is an (L) may be considered as a (J_1) without change of limit-properties. There are, however, systems (J_1) that are not systems (L). Frechet [1] has given an example of a class (R) in wich no limit-point is the unique limit of any set. In this, the universe of discourse consists of all points on a line, while the set of limit-points of a classs is the set of its points of condensation. We shall show that is a (J_1).

We shall show (1°) that every biuniform transformation in this system, that retains limit-properties invariant, when it leaves every member of a class invariant, leaves every limit-element invariant, and (2°) that if an element A is not a limit-element of a class E, there is some transformation that is biuniform, that leaves all limit-properties invariant, that changes A, and that leaves invariant every element of E.

(1) Suppose E is a class of elements, and let A be a limit-element of E. Let A′ be any term distinct from A. Now, let F be some interval not containing A′ either in its interior or as an end-point but containing A in its interior and let G be the common part of E and F. Clearly A will be a limit-point of G, while A′ will not Therefore, by § 5, any transformation, which is bicontinuous and keeps every member of G invariant, cannot interchange A and A′. We are thus led into a contradiction unless we suppose that every bicontinuous biunivocal transformation wich leaves invariant every member of a class E also leaves invariant every limit-point.

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(1) *V.* p. 9.

(2) If an element A is not a limit-element of a class E, there is some interval F containing A in its interior and containing only a denumerable set of elements of E. Let us adjoin to this set all the rational points and end-points of F, always excluding A, however we thus get a dense denumerable series, forming a median class of F. It can hence be put into one-one correspondence with the set of rational numbers between, 0 and 1, inclusive, by a biunivocal transformation T which will determine a biunivocal, bicontinuous transformation of F into the whole interval from 0 to 1. This is a well-known theorem of Cantor. Now, let S be the following transformation of the interval $0 \geqq x \geqq 1$:

If $0 < x < \frac{1}{2}$ and x is irrational

$$x' = \frac{x}{2}.$$

If $\frac{1}{2} < x < 1$ and x is irrational

$$x' = \frac{(3x - 1)}{2}.$$

If x is rational

$$x' = x.$$

Let R be the transformation of our line which leaves invariant all the elements not in F, and in F is equivalent to $\breve{T}|S|T$. There is no difficulty in seeing that R is biunivocal and bicontinuous, that it leaves invariant every member of E except possibly A, and that it changes A. Hence, A is a limit of a set E when and only when every biunivocal, bicontinuous transformation which leaves invariant every element of E leaves A invariant. Our system is thus a (J_1).

12. An (R) that is a (J) may fail to be an (L) even though every limit-point of a set E is always a limit-point of a denumerable sub-set of E. For example, let C be made up of all points on a line L_1 and all but one of the points on a line L_2. Let this one point be Q. Let Σ consist of all bicontinuous biunivocal transformations of L_1, combined in all possible ways with all bicontinous, biunivocal transformations of L_2 leaving Q invariant, with the proviso that every transformation of L_2 wich leaves invariant points on every interval containing Q should be associated only with the identity transformation on L_1. This example may be shown to satisfy our conditions 2′, 3′, and 4′, and to be an (R). « Limit-point » will have the ordinary meaning, with the exception that when Q would ordinarily be a limit-point of a set on L_2, every point of L_1 will now be a limit-point of that set. It will be impossible to single out any point of L_1 as the unique limit of a sub-set of any such set. On the other hand, it will always be possible to single out a denumerable sub-set of any given set which approaches any given limit-point of the set.

A neighborhood of any point A on L_2 will be a set containing an interval containing A. A neighborhood of any point A on L_1 will consist of a set containing an interval on L_1 containing A and an interval on L_2 containing Q. It is interesting to observe that this set invalidates the condition given by Fréchet (1) as sufficient but not necessary to make an (R) set an (L) set — that every point A should determine a sequence $\{V_A^{(n)}\}$ of neighborhoods such that every neighborhood of A should contain at least one neighborhood belonging to this sequence — for we have here an (R) satisfying this sub-condition without being an (L).

This system is not a (J_1), for any transformation that is bicontinuous and biunivocal on L_1 but leaves L_2 untouched retains all limit-properties invariant, but does not always leave A invariant when it leaves every element of E invariant and A is a limit-element of E.

13. Though Fréchet's sufficient condition for an (R) to be an (L) breaks down, it becomes satisfactory if supplemented with a further condition. This condition is that given any two elements A and B, it is possible to find a neighborhood V_A of A and a neighborhood V_B of B that are mutually exclusive. For a class (R) subjected to no further condition, the two properties just mentioned will not be necessary to make it an (L). An example which is an (L) but does not satisfy the first condition has been given by Fréchet (2) — it consists of all continuous or

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(1) *V.* p. 11.

(2) *Relations entre les Notions de Limite et de Distance* (*Trans. Am. Math. Soc.*, vol. XIX, n° 1, p. 59).

discontinuous functions on a closed segment, with limit taken as not necessarily uniform. However, as this system does not have all derivative-sets closed, it is not a (J). It still remains, then, a possibility that these two conditions may be necessary to guarantee that an (R) which is also a (J) should be an (L).

That this possibility is not fulfilled, however the following example will show : let our space consist of all points on an ordinary three-space, and let Σ consist of all transformations that are biunivocal and bicontinuous on every plane containing the line $x = y = 0$, but which do not interchange points in any two such planes and which are not necessarily bicontinuous as between the several points in question. Here every point not on $x = y = 0$ will have as its neighborhood an ordinary planar neighborhood together with an arbitrary set of points from other planes while a point on the z-axis will have as a neighborhood an arbitrary selection of neighborhoods from the various planes, subject only to the condition that they all contain a segment of the z-axis containing in its interior the point in question. As there is otherwise no correlation between the neighborhoods of an axial point in the different planes, it is clear that no denumerable set of neighborhoods of such a point can be found such that every neighborhood of the point shall contain one of the set. For let $V_1, V_2, \ldots, V_n, \ldots$, be such a sequence of neighborhoods. Then we can select a set $P_1, P_2, \ldots, P_n, \ldots$ of axial planes, and determine (1) a neighborhood of our point in P_1 not containing all the points of V_1 in P_1, a neighborhood in P_2 not containing all the points of V_2 in P_2, and so on. We can, moreover, choose all these neighborhoods so that they will contain a given segment on the axis. Grouping them together and associating them with the whole of all planes not of the form P_n, we get a neighborhood V not containing any V_n in its entirety.

It is worthy of mention that this system is a (J_1).

14. What the precise necessary and sufficient conditions are that a set which is at once a (J) and an (R) be an (L) has not yet been determined. It appears difficult to obtain conditions

(1) The dependence of this on Zermelo's axiom is only specious.

which are more than reiterations of the fact that the set is an (L). It is reasonable to expect, however, that when the conditions are obtained, they will be two in number, and that one will guarantee that every set E having a limit-point A will have a denumerable sub-set having A as a limit, while the other will provide that it shall have a sub-set with A as its only limit.

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