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CALABI–YAU THREEFOLDS OF BORCEA–VOISIN, ANALYTIC TORSION, AND BORCHERDS PRODUCTS

by

Ken-ichi Yoshikawa

Dedicated to Professor Jean-Michel Bismut on his sixtieth birthday

Abstract. — For a class of Borcea–Voisin threefolds, we give an explicit formula for the BCOV invariant [**3**], [**14**] as a function on the moduli space. For those Calabi–Yau threefolds, the BCOV invariant is expressed as the Petersson norm of the tensor product of a certain Borcherds lift on the Kähler moduli of a Del Pezzo surface and the Dedekind η-function. As a by-product, we construct an automorphic form on the orthogonal modular variety associated to the odd unimodular lattice of signature $(2, m)$, $m \leq 10$, which vanishes exactly on the Heegner divisor of norm (-1)-vectors.

***Résumé* (Variétés de Calabi-Yau de dimension trois de type Borcea–Voisin, torsion analytique, et produits de Borcherds)**

Pour une classe de variétés de Borcea–Voisin, nous donnons une formule explicite de l'invariant de BCOV [**3**], [**14**] comme une fonction sur l'espace de modules. Pour ces variétés de Calabi–Yau de dimension trois, l'invariant de BCOV s'exprime comme la norme du produit tensoriel d'un relèvement de Borcherds à l'espace des modules kählériens d'une surface de Del Pezzo et de la fonction η de Dedekind. Nous construisons une forme automorphe sur la variété modulaire orthogonale associée au réseau unimodulaire impair de signature $(2, m)$, $m \leq 10$, qui s'annule exactement sur le diviseur de Heegner des vecteurs de norme -1.

1. Introduction

In [**33**], Ray–Singer introduced the notion of analytic torsion for compact Kähler manifolds. Their definition was extended to arbitrary holomorphic Hermitian vector bundles over a compact Kähler manifold by Quillen [**32**] and Bismut–Gillet–Soulé [**7**]. Let $\xi \to X$ be a holomorphic Hermitian vector bundle over a compact Kähler

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manifold and let $\zeta_q(s)$ be the spectral zeta function of the Hodge–Kodaira Laplacian acting on the space of $(0,q)$-forms on X with values in ξ. Then the real number

$$\tau(X,\xi) = \exp[-\sum_{q\geq 0}(-1)^q q\,\zeta_q'(0)]$$

is called the analytic torsion of ξ. The most fundamental results in the theory of analytic torsion such as the first variational formula, the second variational formula and the comparison formula for complex immersions were obtained by Bismut–Gillet–Soulé and Bismut–Lebeau as the corresponding results in the theory of Quillen metrics, i.e., the anomaly formula, the curvature formula and the immersion formula for Quillen metrics [**7**], [**8**],...

In [**3**], Bershadsky–Cecotti–Ooguri–Vafa introduced the following combination of analytic torsions for a compact Kähler manifold X

$$\prod_{p\geq 0} \tau(X,\Omega_X^p)^{(-1)^p p},$$

which we call the BCOV torsion of X. They studied the BCOV torsion as a function on the moduli space of Calabi–Yau threefolds and used it to extend the mirror symmetry conjecture to higher-genus Gromov–Witten invariants [**2**], [**3**].

In [**14**], the notion of BCOV invariant was introduced for Calabi–Yau threefolds by Fang–Lu–Yoshikawa, which they obtained using the BCOV torsion and a certain Bott–Chern secondary class. (See Sect. 5.1 for the definition.) The BCOV invariant of a Calabi–Yau threefold X is denoted by $\tau_{\mathrm{BCOV}}(X)$. Then $\tau_{\mathrm{BCOV}}(X)$ depends only on the isomorphism class of X, while the BCOV torsion does depend on the choice of a Kähler metric on X. Because of this invariance property, the BCOV invariant τ_{BCOV} gives rise to a function on the moduli space of Calabi–Yau threefolds and is identified with the partition function F_1 in [**3**]. In this paper, we give an explicit formula for the BCOV invariant for a class of Calabi–Yau threefolds studied by Borcea [**9**] and Voisin [**36**]. (See [**14**] for some other examples including the quintic mirror threefolds and the FHSV models.) Let us explain our results.

Let S be a $K3$ surface and let $\theta\colon S \to S$ be an anti-symplectic holomorphic involution. Let T be an elliptic curve and let $-1_T\colon T \to T$ be the involution defined as $-1_T(x) = -x$. Let $X_{(S,\theta,T)}$ be the blow-up of the orbifold $(S\times T)/\theta\times(-1)_T$ along the singular locus. Then $X_{(S,\theta,T)}$ is a smooth Calabi–Yau threefold equipped with the following two fibrations. Let $\pi_1\colon X_{(S,\theta,T)} \to S/\theta$ be the elliptic fibration with constant fiber T induced from the projection $(S\times T)/\theta\times(-1)_T \to S/\theta$ and let $\pi_2\colon X_{(S,\theta,T)} \to T/(-1_T)$ be the $K3$-fibration with constant fiber S induced from the projection $(S\times T)/\theta\times(-1)_T \to T/(-1_T)$. The triplet $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is called the Borcea–Voisin threefold associated with (S,θ,T). The moduli space of the triplet $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is determined by the lattice $H^2_-(S,\mathbf{Z})$, the anti-invariant part of the

θ-action on $H^2(S,\mathbf{Z})$. By [**28**], $H^2_-(S,\mathbf{Z})$ is isometric to a primitive 2-elementary sublattice of the $K3$-lattice $\mathbb{L}_{K3}$. Let $\Lambda \subset \mathbb{L}_{K3}$ be a sublattice of rank $r(\Lambda)$. A Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ is of type Λ if $H^2_-(S,\mathbf{Z})$ is isometric to Λ. Since θ is anti-symplectic, there exist Borcea–Voisin threefolds of type Λ if and only if $\Lambda \subset \mathbb{L}_{K3}$ is a primitive 2-elementary sublattice of signature $(2, r(\Lambda)-2)$.

Some Borcea–Voisin threefolds are related to Del Pezzo surfaces. Let V be a Del Pezzo surface and set $\deg V = c_1(V)^2 \in \mathbf{Z}_{>0}$. Let $H(V,\mathbf{Z})$ be the total cohomology group of V, which is equipped with the cup-product $\langle\cdot,\cdot\rangle_V$. Then the sublattice $H^2(V,\mathbf{Z}) \subset H(V,\mathbf{Z})$ is Lorentzian. Let $H(V,\mathbf{Z})(2)$ be the lattice $(H(V,\mathbf{Z}), 2\langle\cdot,\cdot\rangle_V)$. By the classification of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ [**29**], there exist Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$. Let us explain their moduli space briefly.

Let $\mathcal{K}_V \subset H^2(V,\mathbf{R})$ be the Kähler cone of V, let $\mathcal{C}_V^+ \subset H^2(V,\mathbf{R})$ be the component of the positive cone of $H^2(V,\mathbf{R})$ with $\mathcal{K}_V \subset \mathcal{C}_V^+$ and let $\mathrm{Eff}(V) \subset H^2(V,\mathbf{Z})$ be the set of effective classes on V. The tube domain $H^2(V,\mathbf{R}) + i\,\mathcal{C}_V^+$ is isomorphic to a bounded symmetric domain of type IV and its subdomain $H^2(V,\mathbf{R}) + i\,\mathcal{K}_V$ is called the complexified Kähler cone of V. Let $\mathfrak{H}$ be the complex upper half-plane. By assigning $(X_{(S,\theta,T)},\pi_1,\pi_2)$ the periods of (S,θ) and T, the coarse moduli space of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ is isomorphic to the quotient of the tube domain $(H^2(V,\mathbf{R})+i\,\mathcal{C}_V^+)\times\mathfrak{H}$ by the group $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$ with some divisor removed (cf. Theorem 3.7), where $O^+(H(V,\mathbf{Z}))$ is the group of isometries of $H(V,\mathbf{Z})$ preserving $H^2(V,\mathbf{R}) + i\,\mathcal{C}_V^+$. Hence τ_{BCOV} is regarded as an $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$-invariant function on a certain Zariski open subset of $(H^2(V,\mathbf{R})+i\,\mathcal{C}_V^+)\times\mathfrak{H}$. The goal of this paper is to give an explicit formula for τ_{BCOV} as a function on $(H^2(V,\mathbf{R}) + i\,\mathcal{C}_V^+)\times\mathfrak{H}$ for Borcea–Voisin threefolds of type $H^2(V,\mathbf{Z})(2)$. Let us explain the infinite product appearing in the formula.

After Borcherds [**12**] and Gritsenko–Nikulin [**16**], we introduce the following infinite product $\Phi_V(z)$ on the complexified Kähler cone $H^2(V,\mathbf{R}) + i\,\mathcal{K}_V$:

$$\Phi_V(z) = e^{\pi i\langle c_1(V),z\rangle_V} \prod_{\alpha\in\mathrm{Eff}(V)} \left(1-e^{2\pi i\langle\alpha,z\rangle_V}\right)^{c^{(0)}_{\deg V}(\alpha^2)}$$
$$\times \prod_{\beta\in\mathrm{Eff}(V),\ \beta/2\equiv c_1(V)/2 \mod H^2(V,\mathbf{Z})} \left(1-e^{\pi i\langle\beta,z\rangle_V}\right)^{c^{(1)}_{\deg V}(\beta^2/4)},$$

where $c^{(0)}_k(m)$ and $c^{(1)}_k(m)$ are the m-th Fourier coefficients of the modular forms

$$f^{(0)}_k(\tau) = \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}\,\theta_{\mathbb{A}_1^+}(\tau)^k, \quad f^{(1)}_k(\tau) = -8\,\eta(4\tau)^8\eta(2\tau)^{-16}\,\theta_{\mathbb{A}_1^++\frac{1}{2}}(\tau)^k,$$

respectively. Here $\eta(\tau)$ is the Dedekind η-function and $\theta_{\mathbb{A}_1^+}(\tau)$, $\theta_{\mathbb{A}_1^++1/2}(\tau)$ are the theta series of the A_1-lattice. Let $A_{H(V,\mathbf{Z})(2)}$ be the discriminant group of the lattice $H(V,\mathbf{Z})(2)$ and let $\{\mathbf{e}_\gamma\}_{\gamma\in A_{H(V,\mathbf{Z})(2)}}$ be the standard basis of $\mathbf{C}[A_{H(V,\mathbf{Z})(2)}]$, the group ring of $A_{H(V,\mathbf{Z})(2)}$. In Sects. 4.3, 4.4 and 6.2, we shall prove that $\Phi_V(2z)^2$ is the Borcherds lift [**12**] of the $\mathbf{C}[A_{H(V,\mathbf{Z})(2)}]$-valued elliptic modular form

$$f^{(0)}_{\deg V}(\tau)\,\mathbf{e}_0+\sum_{\gamma\in A_{H(V,\mathbf{Z})(2)}}\ \sum_{m\equiv 2\gamma^2 \mod 4} c^{(0)}_{\deg V}(m)\,q^{m/4}\,\mathbf{e}_\gamma+f^{(1)}_{\deg V}(\tau)\,\mathbf{e}_{\mathbf{1}_{H(V,\mathbf{Z})(2)}}$$

with respect to the lattice $H(V,\mathbf{Z})(2)$. Here $\mathbf{1}_{H(V,\mathbf{Z})(2)}\in A_{H(V,\mathbf{Z})(2)}$ is the characteristic element and $q=\exp(2\pi i\tau)$. As a result, $\Phi_V(z)$ converges when $(\operatorname{Im} z)^2\gg 0$ and extends to an automorphic form on $H^2(V,\mathbf{R})+i\,\mathscr{C}^+_V$ for $O^+(H(V,\mathbf{Z}))$ of weight $\deg V+4$ vanishing exactly on the Heegner divisor of norm (-1)-vectors of $H(V,\mathbf{Z})$. If $\operatorname{Exc}(V)\subset H^2(V,\mathbf{Z})$ denotes the exceptional classes on V, the following functional equations hold by the automorphic property of $\Phi_V(z)$ (cf. Sect. 6.3):

(a) $\Phi_V(z+l)=\Phi_V(z)$ for all $l\in H^2(V,\mathbf{Z})$ with $\langle l,c_1(V)\rangle_V\equiv 0 \mod 2$.
(b) $\Phi_V(g(z))=\pm\Phi_V(z)$ for all $g\in O^+(H^2(V,\mathbf{Z}))$.
(c) $\Phi_V(-\frac{z}{\langle z,z\rangle_V}+\delta)=-(-\langle z,z\rangle_V)^{\deg V+4}\,\Phi_V(z+\delta)$ for all $\delta\in\operatorname{Exc}(V)$.
(d) $\Phi_V(-\frac{2z}{\langle z,z\rangle_V})=(-\frac{\langle z,z\rangle_V}{2})^{\deg V+4}\,\Phi_V(z)$.

Since $c_1(V)/2$ is a Weyl vector of $H^2(V,\mathbf{Z})(2)$, the Fourier expansion of $\Phi_V(2z)$ is of Lie type in the sense of [**18**] by (a), (b). Hence there exists a Borcherds superalgebra whose denominator function is $\Phi_V(2z)$. This Borcherds superalgebra is obtained as an automorphic correction [**17**] of the Kac-Moody algebra defined by the generalized Cartan matrix $(2\langle c_1(E),c_1(E')\rangle_V)_{E,E'\in\operatorname{Exc}(V)}$. (See Question 4.4.)

Let $\|\Phi_V\|$ and $\|\eta\|$ be the Petersson norms of $\Phi_V(z)$ and $\eta(\tau)$, respectively. Then $\|\Phi_V\|^2\cdot\|\eta^{24}\|^2$ is a function on $(H^2(V,\mathbf{R})+i\,\mathscr{C}^+_V)\times\mathfrak{H}$ invariant under the action of $O^+(H(V,\mathbf{Z}))\times SL_2(\mathbf{Z})$. The following (cf. Theorems 5.7 and 6.4) is the main result of this paper.

Theorem 1.1. — *If V is a Del Pezzo surface with $1\le\deg V\le 6$, then there exists a constant $C_{\deg V}$ depending only on $\deg V$ such that the following equation of functions on the moduli space of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ holds:*

$$\tau_{\mathrm{BCOV}}=C_{\deg V}\,\|\Phi_V\|^2\cdot\|\eta^{24}\|^2.$$

Under the identification of τ_{BCOV} with F_1 in B-model [**2**], [**3**], it follows from Theorem 1.1 that the conjecture of Harvey–Moore [**19**, Sect. 7] holds for Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ when $1\le\deg V\le 6$, since Φ_V is the denominator function of a Borcherds superalgebra.

After Theorem 1.1, the conjecture of Bershadsky–Cecotti–Ooguri–Vafa [**2**], [**3**] seems to predict that the elliptic Gromov–Witten invariants of the mirror of Borcea–Voisin threefolds of type $H(V,\mathbf{Z})(2)$ are expressed as certain linear combinations of the Fourier coefficients $c^{(0)}_{\deg V}(m)$, $c^{(1)}_{\deg V}(m)$. If this is the case, the invariant of $K3$

surfaces with involution constructed in [**37**] would be the Borcherds lift of an elliptic modular form whose Fourier coefficients are elliptic Gromov–Witten invariants of some Calabi–Yau threefolds by the structure theorem [**38**, Th. 0.1]. However, since the Borcea–Voisin construction of mirrors [**9**], [**36**] does not apply to Borcea–Voisin threefolds of type $H(V, \mathbf{Z})(2)$, we do not know the existence of mirrors for those Borcea–Voisin threefolds as well as their elliptic Gromov–Witten invariants.

This paper is organized as follows. In Sect. 2, we recall some definitions and results about lattices. In Sect. 3, we recall Borcea–Voisin threefolds and study their moduli space. In Sect. 4, we introduce the automorphic form Φ_m, which will be identified with $\varPhi_V$ in Sect. 6. In Sect. 5, we recall the BCOV invariant of a Calabi–Yau threefold and we prove the main theorem. In Sect. 6, we rewrite the automorphic form Φ_m as an automorphic form on the complexified Kähler cone of a Del Pezzo surface to give an identification between Φ_m and $\varPhi_V$.

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2. Lattices and orthogonal modular varieties

A free $\mathbf{Z}$-module of finite rank endowed with a non-degenerate, integral, symmetric bilinear form is called a lattice. We often identify a non-degenerate, integral, symmetric matrix with the corresponding lattice. The rank of a lattice L is denoted by $r(L)$. The signature of L is denoted by $\operatorname{sign}(L) = (b^+(L), b^-(L))$. A lattice L is *Lorentzian* if $\operatorname{sign}(L) = (1, r(L) - 1)$. For a lattice L with bilinear form $\langle \cdot, \cdot \rangle$, we denote by $L(k)$ the lattice with bilinear form $k\langle \cdot, \cdot \rangle$. The set of roots of L is defined by $\Delta_L := \{d \in L;\ \langle d, d \rangle = -2\}$. The isometry group of L is denoted by $O(L)$. For $r \in L \otimes \mathbf{R}$, the reflection $s_r \in O(L \otimes \mathbf{R})$ is defined by $s_r(x) = x - 2\frac{\langle x, r\rangle}{\langle r, r\rangle} r$ for $x \in L \otimes \mathbf{R}$. If $\delta \in L$ and $\delta^2 = -1$ or $\delta^2 = -2$, then $s_\delta \in O(L)$. The subgroup of $O(L)$ generated by the reflections $\{s_\delta\}_{\delta \in \Delta_L}$ is called the *Weyl group* of L and is denoted by $W(L)$. The dual lattice of L is defined by $L^\vee := \operatorname{Hom}_{\mathbf{Z}}(L, \mathbf{Z}) \subset L \otimes \mathbf{Q}$. We set $A_L := L^\vee / L$. A lattice L is *unimodular* if $A_L = 0$. A lattice L is *even* if $\langle x, x \rangle \in 2\mathbf{Z}$ for all $x \in L$. A lattice is *odd* if it is not even. A sublattice $M \subset L$ is *primitive* if L/M has no torsion elements.

2.1. 2-elementary lattices.

— Set $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$. An even lattice L is 2-*elementary* if there is an integer $l \geq 0$ with $A_L \cong \mathbf{Z}_2^l$. For a 2-elementary lattice L, we set $l(L) := \dim_{\mathbf{Z}_2} A_L$.

Let $\mathbb{U} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and let $\mathbb{A}_1$, $\mathbb{E}_8$ be the *negative-definite* Cartan matrix of type A_1, E_8 respectively, which are identified with the corresponding even lattices. Then $\mathbb{U}$ and

$\mathbb{E}_8$ are unimodular, and $\mathbb{A}_1$ is 2-elementary. The lattice

$$\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$$

is called the $K3$ *lattice*. For a sublattice $\Lambda \subset \mathbb{L}_{K3}$, set $\Lambda^{\perp} := \{l \in \mathbb{L}_{K3};\ \langle l, \Lambda \rangle = 0\}$.

For a primitive 2-elementary Lorentzian sublattice $M \subset \mathbb{L}_{K3}$, let I_M be the involution on $M \oplus M^{\perp}$ defined as $I_M(x,y) = (x,-y)$ for $(x,y) \in M \oplus M^{\perp}$. Then I_M extends uniquely to an involution on $\mathbb{L}_{K3}$ by [**28**, Cor. 1.5.2].

Let L be an even 2-elementary lattice. Since A_L is a vector space over $\mathbf{Z}_2$, the mapping $A_L \ni \gamma \to \gamma^2 \in \frac{1}{2}\mathbf{Z}/\mathbf{Z} \cong \mathbf{Z}_2$ is $\mathbf{Z}_2$-linear. Since the discriminant bilinear form on A_L is non-degenerate, there is a unique element $\mathbf{1}_L \in A_L$ such that $\langle \gamma, \mathbf{1}_L \rangle \equiv \gamma^2 \mod \mathbf{Z}$ for all $\gamma \in A_L$. If $L = L' \oplus L''$, then $\mathbf{1}_L = \mathbf{1}_{L'} \oplus \mathbf{1}_{L''}$.

2.2. Lorentzian lattices. — Let L be a Lorentzian lattice. The set $\mathscr{C}_L := \{v \in L \otimes \mathbf{R};\ v^2 > 0\}$ is called the positive cone of L, which consists of two connected components. Let $\mathscr{C}_L^+$ be one of the connected components of $\mathscr{C}_L$. For $\lambda \in L \otimes \mathbf{R}$, we set $h_\lambda := \{v \in \mathscr{C}_L^+;\ \langle v, \lambda \rangle = 0\}$. Define $(\mathscr{C}_L^+)^o := \mathscr{C}_L^+ \setminus \bigcup_{\delta \in \Delta_L} h_\delta$. The Weyl group $W(L)$ acts simply transitively on the set of connected components of $(\mathscr{C}_L^+)^o$. Each connected component of $(\mathscr{C}_L^+)^o$ is called a *Weyl chamber* of L. Let $\mathscr{W}$ be a Weyl chamber of L. A hyperplane $h_d \subset L \otimes \mathbf{R}$, $d \in \Delta_L^+$ is called a *wall* of $\mathscr{W}$ if $\dim(h_d \cap \overline{\mathscr{W}}) = r(L) - 1$, where $\overline{\mathscr{W}}$ is the closure of $\mathscr{W}$ in $L \otimes \mathbf{R}$. We set $\Pi(L, \mathscr{W}) := \{d \in \Delta_L;\ d \cdot \mathscr{W} > 0,\ h_d \text{ is a wall of } \mathscr{W}\}$, which is the minimal set of roots defining $\mathscr{W}$, i.e.,

$$\mathscr{W} = \{v \in \mathscr{C}_L^+;\ \langle v, d \rangle > 0,\ \forall\, d \in \Pi(L, \mathscr{W})\}. \tag{2.1}$$

In (2.1), each inequality $\langle v, d \rangle > 0$, $d \in \Pi(L, \mathscr{W})$ is essential. A vector $\varrho \in L \otimes \mathbf{Q}$ is called a *Weyl vector* of $(L, \mathscr{W})$ if $\langle \varrho, d \rangle = 1$ for all $d \in \Pi(L, \mathscr{W})$.

2.3. Lattices of signature $(2,n)$**.** — Let Λ be a lattice with $\mathrm{sign}(\Lambda) = (2, r(\Lambda) - 2)$. Define

$$\Omega_\Lambda := \{[\eta] \in \mathbf{P}(\Lambda \otimes \mathbf{C});\ \langle \eta, \eta \rangle = 0,\ \langle \eta, \bar{\eta} \rangle > 0\}.$$

Then Ω_Λ consists of two connected components $\Omega_\Lambda^{\pm}$, each of which is isomorphic to a bounded symmetric domain of type IV of dimension $r(\Lambda) - 2$. The group $O(\Lambda)$ acts on Ω_Λ projectively. We set $O^+(\Lambda) := \{g \in O(\Lambda);\ g(\Omega_\Lambda^{\pm}) = \Omega_\Lambda^{\pm}\}$. Then $O^+(\Lambda)$ acts on Ω_Λ^+ properly discontinuously, and the quotient

$$\mathscr{M}_\Lambda := \Omega_\Lambda / O(\Lambda) = \Omega_\Lambda^+ / O^+(\Lambda)$$

is an analytic space. The Baily–Borel–Satake compactification of $\mathscr{M}_\Lambda$ is denoted by $\mathscr{M}_\Lambda^*$. Then $\mathscr{M}_\Lambda^*$ is an irreducible normal projective variety with $\dim(\mathscr{M}_\Lambda^* \setminus \mathscr{M}_\Lambda) \leq 1$. For $\lambda \in \Lambda \otimes \mathbf{R}$, set

$$H_\lambda := \{[\eta] \in \Omega_\Lambda;\ \langle \eta, \lambda \rangle = 0\}.$$

Then $H_\lambda \neq \varnothing$ if and only if $\langle \lambda, \lambda \rangle < 0$. We define

$$\mathscr{D}_\Lambda := \bigcup_{d \in \Delta_\Lambda} H_d, \qquad \Omega_\Lambda^o := \Omega_\Lambda \setminus \mathscr{D}_\Lambda.$$

The reduced divisor $\mathscr{D}_\Lambda$ is called the *discriminant locus* of Ω_Λ. We define the subsets $H_d^o \subset H_d$ ($d \in \Delta_\Lambda$) and $\mathscr{D}_\Lambda^o \subset \mathscr{D}_\Lambda$ by

$$H_d^o := \{[\eta] \in \Omega_\Lambda^+;\ O^+(\Lambda)_{[\eta]} = \{\pm 1, \pm s_d\}\}, \qquad \mathscr{D}_\Lambda^o := \sum_{d \in \Delta_\Lambda / \pm 1} H_d^o.$$

Since $O(\Lambda)$ preserves $\mathscr{D}_\Lambda$ and $\mathscr{D}_\Lambda^o$, we define

$$\overline{\mathscr{D}}_\Lambda := \mathscr{D}_\Lambda / O(\Lambda), \qquad \overline{\mathscr{D}}_\Lambda^o := \mathscr{D}_\Lambda^o / O(\Lambda) \subset \overline{\mathscr{D}}_\Lambda.$$

Then $\overline{\mathscr{D}}_\Lambda^o \cap \operatorname{Sing} \mathscr{M}_\Lambda = \varnothing$ by [**38**, Prop. 1.9 (5)] and $\Omega_\Lambda^o \cup \mathscr{D}_\Lambda^o$ is a Zariski open subset of Ω_Λ such that $\Omega_\Lambda \setminus (\Omega_\Lambda^o \cup \mathscr{D}_\Lambda^o)$ has codimension at least 2 by [**37**, Prop. 1.9 (2)].

When $\Lambda = \mathbb{U}(N) \oplus L$, a vector of $\Lambda \otimes \mathbf{C}$ is denoted by (m, n, v), where $m, n \in \mathbf{C}$ and $v \in L \otimes \mathbf{C}$. The tube domain $L \otimes \mathbf{R} + i\,\mathscr{C}_L$ is identified with Ω_Λ via the map

$$L \otimes \mathbf{R} + i\,\mathscr{C}_L \ni z \to [(-z^2/2, 1/N, z)] \in \Omega_\Lambda \subset \mathbf{P}(\Lambda \otimes \mathbf{C}), \qquad z \in L \otimes \mathbf{C}. \tag{2.2}$$

The component of Ω_Λ corresponding to $L \otimes \mathbf{R} + i\,\mathscr{C}_L^+$ via (2.2) is written as Ω_Λ^+.

3. Calabi–Yau threefolds of Borcea–Voisin

An irreducible, smooth, compact Kähler n-fold X with canonical line bundle K_X is *Calabi–Yau* if

$$(1) \quad K_X \cong \mathscr{O}_X, \qquad (2) \quad H^q(X, \mathscr{O}_X) = 0 \quad (0 < q < n).$$

A two-dimensional Calabi–Yau manifold is called a $K3$ surface. In this section, we recall a class of Calabi–Yau threefolds studied by Borcea [**9**] and Voisin [**36**].

3.1. $K3$ surfaces with involution and their moduli space. — Let S be a $K3$ surface. Then $H^2(S, \mathbf{Z})$ endowed with the cup-product pairing is isometric to the $K3$ lattice $\mathbb{L}_{K3}$. An isometry of lattices $\alpha\colon H^2(S, \mathbf{Z}) \cong \mathbb{L}_{K3}$ is called a *marking* of S, and the pair (S, α) is called a *marked $K3$ surface.* The period of a marked $K3$ surface (S, α) is defined by

$$\pi(S, \alpha) := [\alpha(\eta)] \in \mathbf{P}(\mathbb{L}_{K3} \otimes \mathbf{C}), \qquad \eta \in H^0(S, K_S) \setminus \{0\}.$$

Let $M \subset \mathbb{L}_{K3}$ be a sublattice. A $K3$ surface equipped with a holomorphic involution $\theta\colon S \to S$ is called a *2-elementary $K3$ surface of type M* if

$$\theta^* = \alpha^{-1} \circ I_M \circ \alpha, \qquad \theta^*|_{H^0(S, K_S)} = -1.$$

By the global Torelli theorem [**31**], [**13**] and by [**28**, Cor. 1.5.2], there exists a 2-elementary $K3$ surface of type M if and only if $M \subset \mathbb{L}_{K3}$ is a primitive 2-elementary Lorentzian sublattice.

Let (S,θ) be a 2-elementary $K3$ surface of type M and let α be a marking with $\theta^* = \alpha^{-1} \circ I_M \circ \alpha$. Let $\eta \in H^0(S, K_S) \setminus \{0\}$. Then $\pi(S,\alpha) \in \Omega^o_{M^\perp}$. By [**37**, Th. 1.8] and [**38**, Prop. 11.2], the $O(M^\perp)$-orbit of $\pi(S,\alpha)$ is independent of the choice of a marking α with $\theta^* = \alpha^{-1} \circ I_M \circ \alpha$. The period of (S,θ) is defined as the $O(M^\perp)$-orbit

$$\varpi_M(S,\theta) := O(M^\perp) \cdot \pi(S,\alpha) \in \Omega_{M^\perp}/O(M^\perp) = \mathcal{M}_{M^\perp}.$$

By [**37**, Th. 1.8], the period map induces an isomorphism from the coarse moduli space of 2-elementary $K3$ surfaces of type M to the analytic space

$$\mathcal{M}^o_{M^\perp} := \Omega^o_{M^\perp}/O(M^\perp) = (\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp})/O^+(M^\perp).$$

Theorem 3.1. — *Let $x \in \overline{\mathcal{D}}^o_{M^\perp}$ and let $C \subset \mathcal{M}^*_{M^\perp}$ be an irreducible projective curve passing through x. Assume that $x \in C \setminus \operatorname{Sing} C$ and that C intersects $\overline{\mathcal{D}}^o_{M^\perp}$ transversally at x. Then there exist a pointed smooth projective curve (B,y), a neighborhood U of y, a holomorphic map $f\colon (B,y) \to (C,x)$, a smooth projective threefold $\mathcal{W}$ with an involution $\theta\colon \mathcal{W} \to \mathcal{W}$, and a surjective holomorphic map $p\colon \mathcal{W} \to B$ satisfying the following properties:*

(1) *$f(B) = C$ and the map $f|_U\colon (U,y) \to (f(U),x)$ is an isomorphism.*
(2) *The projection $p\colon \mathcal{W} \to B$ is $\mathbf{Z}_2$-equivariant with respect to the $\mathbf{Z}_2$-action on $\mathcal{W}$ induced by θ and with respect to the trivial $\mathbf{Z}_2$-action on B.*
(3) *For every $b \in U \setminus \{y\}$, $(\mathcal{W},\theta)|_{p^{-1}(b)}$ is a 2-elementary $K3$ surface of type M such that $\varpi_M((\mathcal{W},\theta)|_{p^{-1}(b)}) = f(b)$.*

Proof. — See [**37**, Th. 2.8]. □

For a 2-elementary $K3$ surface (S,θ), we define $S^\theta := \{x \in S;\ \theta(x) = x\}$.

Proposition 3.2. — *Let (S,θ) be a 2-elementary $K3$ surface of type M and set*

$$g(M) := (22 - r(M) - l(M))/2, \qquad k(M) := (r(M) - l(M))/2.$$

If $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, $\mathbb{U} \oplus \mathbb{E}_8(2)$, then there exist a smooth irreducible curve C of genus $g(M)$ and (-2)-curves $E_1, \ldots, E_{k(M)}$ such that $S^\theta = C \amalg E_1 \amalg \cdots \amalg E_{k(M)}$.

Proof. — See [**29**, Th. 4.2.2]. □

3.2. Elliptic curves and elliptic fibrations. — Let $\mathfrak{H} = \{\tau \in \mathbf{C};\ \operatorname{Im}\tau > 0\}$ be the complex upper half-plane and let $\mathfrak{M}$ be the modular curve

$$\mathfrak{M} := SL_2(\mathbf{Z})\backslash\mathfrak{H}.$$

For an elliptic curve T, let $\Omega(T) \in \mathfrak{M}$ denote the period of T. Let $-1_T\colon T \to T$ be the holomorphic involution that assigns $x \in T$ the inverse $-x \in T$. Let $j(T) \in \mathbf{C}$ denote the value of the j-invariant of T. If T is isomorphic to the cubic curve of $\mathbf{P}^2$ defined by the inhomogeneous equation $y^2 = 4x^3 - g_2\,x - g_3$, then

$$j(T) = \frac{g_2^3}{g_2^3 - 27\,g_3^2}.$$

The j-invariant induces an identification between $\mathfrak{M}$ with the complex plane $\mathbf{C}$.

Let S be a compact complex surface, let B be a compact Riemann surface, and let $f\colon S \to B$ be a surjective holomorphic map. We set $S_b := f^{-1}(b)$ for $b \in B$. Let $\Delta_{S/B} \subset B$ be the set of critical values of f. Then $f\colon S \to B$ is an *elliptic fibration* if S_b is an elliptic curve for every $b \in B \setminus \Delta_{S/B}$. The analytic invariant of an elliptic fibration $f\colon S \to B$ is the meromorphic function on B defined as $j_{S/B}(b) := j(S_b)$ for $b \in B \setminus \Delta_{S/B}$. For an elliptic fibration $f\colon S \to B$, we set $B^o := B \setminus \Delta_{S/B}$, $S^o := f^{-1}(B^o)$ and $f^o := f|_{S^o}$.

Let $f\colon S \to B$ be an elliptic fibration with a holomorphic section $\sigma\colon B \to S$. By [**1**, Chap. V Prop. 9.1], the elliptic fibration $f^o\colon S^o \to B^o$ is canonically isomorphic to the Jacobian fibration $(R^1 f_* \mathcal{O}_S / R^1 f_* \mathbf{Z})|_{B^o} \to B^o$ such that $\sigma(b)$ is identified with the identity element of the Jacobian $H^1(S_b, \mathcal{O}_{S_b})/H^1(S_b, \mathbf{Z})$. Hence there exists a holomorphic involution -1_{S^o} on S^o such that $-1_{S^o}|_{S_b} = -1_{S_b}$ for all $b \in B^o$. When -1_{S^o} extends to a holomorphic involution on S, we call the elliptic fibration $f\colon S \to B$ with a holomorphic section *admissible*.

3.3. Borcea–Voisin threefolds and their moduli space.

— Let (S, θ) be a 2-elementary $K3$ surface. Let T be an elliptic curve. Let $T[2]$ denote the 2-torsion points of T, which is the set of fixed points of -1_T.

Define a holomorphic involution on $S \times T$ by $\iota := \theta \times (-1_T)$, which acts trivially on $H^0(S \times T, K_{S\times T})$. By identifying the generator of $\mathbf{Z}_2$ with the involutions θ, -1_T and ι, the group $\mathbf{Z}_2$ acts holomorphically on S, T, $S \times T$, respectively. The set of fixed points of ι, $(S \times T)^\iota = S^\theta \times T[2]$, is the disjoint union of four copies of the curve S^θ. After Borcea [**9**] and Voisin [**36**], we make the following

Definition 3.3. — For a 2-elementary $K3$ surface (S, θ) and an elliptic curve T, let $X_{(S,\theta,T)}$ be the resolution of $S \times T/\mathbf{Z}_2$ defined as the blow-up of $S \times T/\mathbf{Z}_2$ along $\mathrm{Sing}\,(S\times T/\mathbf{Z}_2) \cong (S\times T)^\iota$. Let $\pi_1\colon X_{(S,\theta,T)} \to S/\mathbf{Z}_2$ and $\pi_2\colon X_{(S,\theta,T)} \to T/\mathbf{Z}_2$ be the projections induced from the projections $\mathrm{pr}_1\colon S\times T \to S$ and $\mathrm{pr}_2\colon S\times T \to T$, respectively. The triplet $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ is called the *Borcea–Voisin threefold associated with* (S, θ, T). Two Borcea–Voisin threefolds $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ and $(X_{(S',\theta',T')}, \pi_1', \pi_2')$ are isomorphic if there exist isomorphisms of complex manifolds

$$f\colon X_{(S,\theta,T)} \to X_{(S',\theta',T')}, \qquad g\colon S/\mathbf{Z}_2 \to S'/\mathbf{Z}_2, \qquad h\colon T/\mathbf{Z}_2 \to T'/\mathbf{Z}_2$$

such that $\pi_1' \circ f = g \circ \pi_1$ and $\pi_2' \circ f = h \circ \pi_2$.

By Borcea [**9**] and Voisin [**36**], $X_{(S,\theta,T)}$ is a Calabi–Yau threefold, which is equipped with the elliptic fibration $\pi_1 \colon X_{(S,\theta,T)} \to S/\mathbf{Z}_2$ with constant fiber T and with the $K3$-fibration $\pi_2 \colon X_{(S,\theta,T)} \to T/\mathbf{Z}_2$ with constant fiber S.

We recall another construction of $X_{(S,\theta,T)}$. Let $q \colon \widetilde{S \times T} \to S \times T$ be the blow-up of $S \times T$ along the curve $\Sigma_{(S,\theta,T)} := S^\theta \times T[2] = (S \times T)^{\theta \times (-1_T)}$. Let $\widetilde{\theta \times (-1_T)}$ be the involution on $\widetilde{S \times T}$ induced from $\theta \times (-1_T)$. We consider the $\mathbf{Z}_2$-action on $\widetilde{S \times T}$ induced from $\widetilde{\theta \times (-1_T)}$, so that $q \colon \widetilde{S \times T} \to S \times T$ is $\mathbf{Z}_2$-equivariant. Since $\theta \times (-1_T)$ acts as -1 on the normal bundle $N_{\Sigma_{(S,\theta,T)}/(S \times T)}$, $\widetilde{\theta \times (-1_T)}$ acts trivially on the exceptional divisor $q^{-1}(\Sigma_{(S,\theta,T)})$. Hence

$$(\widetilde{S \times T})^{\widetilde{\theta \times (-1_T)}} = q^{-1}(\Sigma_{(S,\theta,T)}).$$

Since $\widetilde{\theta \times (-1_T)}$ acts as the reflection with respect to the hypersurface $q^{-1}(\Sigma_{(S,\theta,T)})$, we have $K_{\widetilde{S \times T}} \cong \mathcal{O}_{\widetilde{S \times T}}(q^{-1}(\Sigma_{(S,\theta,T)}))$ and $K_{\widetilde{S \times T}/\mathbf{Z}_2} \cong \mathcal{O}_{\widetilde{S \times T}/\mathbf{Z}_2}$. Hence $\widetilde{S \times T}/\mathbf{Z}_2$ is a Calabi–Yau threefold. The natural projection $(\widetilde{S \times T})/\mathbf{Z}_2 \to (S \times T)/\mathbf{Z}_2$ induces an isomorphism

$$X_{(S,\theta,T)} \cong (\widetilde{S \times T})/\mathbf{Z}_2 = (\widetilde{S \times T})/\widetilde{\theta \times (-1_T)}. \tag{3.1}$$

By (3.1), the projections $\pi_1 \colon X_{(S,\theta,T)} \to S/\mathbf{Z}_2$ and $\pi_2 \colon X_{(S,\theta,T)} \to T/\mathbf{Z}_2$ are induced from the projections $\mathrm{pr}_1 \colon \widetilde{S \times T} \to S$ and $\mathrm{pr}_2 \colon \widetilde{S \times T} \to T$.

Definition 3.4. — Let $\Lambda \subset \mathbb{L}_{K3}$ be a primitive 2-elementary sublattice with signature $(2, r(\Lambda) - 2)$. A Borcea–Voisin threefold $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ is of *type* Λ if there exists an isometry of lattices $H^2_-(S, \mathbf{Z}) := \{l \in H^2(S, \mathbf{Z});\ \theta^* l = -l\} \cong \Lambda$.

Notice that when $X_{(S,\theta,T)}$ is a Borcea–Voisin threefold of type Λ, (S, θ) is a 2-elementary $K3$ surface of type $\Lambda^\perp$.

Lemma 3.5. — *Let (S,θ) and (S',θ') be 2-elementary $K3$ surfaces of type $\Lambda^\perp$, and let T and T' be elliptic curves. Then the Borcea–Voisin threefolds $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ and $(X_{(S',\theta',T')}, \pi_1', \pi_2')$ are isomorphic if and only if $(S,\theta) \cong (S',\theta')$ and $T \cong T'$.*

Proof. — Let $f \colon X_{(S,\theta,T)} \to X_{(S',\theta',T')}$, $g \colon S/\mathbf{Z}_2 \to S'/\mathbf{Z}_2$ and $h \colon T/\mathbf{Z}_2 \to T'/\mathbf{Z}_2$ be isomorphisms as in Definition 3.3. Let $\bar{t} = \{\pm t\} \in T/\mathbf{Z}_2$ be a regular value of π_2 and set $\overline{t'} := h(\bar{t}) \in T'/\mathbf{Z}_2$. Since $t \neq -t$, we have $\pi_2^{-1}(\bar{t}) = (S \times \{t\} \amalg S \times \{-t\})/\mathbf{Z}_2 \cong S$. Similarly, we have $(\pi_2')^{-1}(\overline{t'}) \cong S'$. We obtain the involutions $\theta \colon S \to S$ and $\theta' \colon S' \to S'$ as the non-trivial covering transformations of the projections $\pi_1 \colon S = \pi_2^{-1}(\bar{t}) \to S/\mathbf{Z}_2$ and $\pi_1' \colon S' = (\pi_2')^{-1}(\overline{t'}) \to S'/\mathbf{Z}_2$, respectively. The isomorphism of fibers $f|_{\pi_2^{-1}(\bar{t})}\colon \pi_2^{-1}(\bar{t}) \to (\pi_2')^{-1}(\overline{t'})$ is an isomorphism from S to S' such that $\theta = (f|_{\pi_2^{-1}(\bar{t})})^{-1} \circ \theta' \circ f|_{\pi_2^{-1}(\bar{t})}$. This proves that $(S,\theta) \cong (S',\theta')$.

Let $x \in (S \setminus S^{\theta})/\mathbf{Z}_2$ be a regular value of π_1 and set $x' := g(x) \in S'/\mathbf{Z}_2$. Since $T = \pi_1^{-1}(x)$ and $T' = \pi_1^{-1}(x')$, the map $f|_{\pi_1^{-1}(x)}$ is an isomorphism from T to T'.

Conversely, if $(S,\theta) \cong (S',\theta')$ and $T \cong T'$, then it is obvious by construction that $(X_{(S,\theta,T)}, \pi_1, \pi_2) \cong (X_{(S',\theta',T')}, \pi_1', \pi_2')$. This proves the lemma. □

By Lemma 3.5, the following definition makes sense.

Definition 3.6. — Let $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ be a Borcea–Voisin threefold of type Λ. The point $\widetilde{\varpi}_{\Lambda}(X_{(S,\theta,T)}, \pi_1, \pi_2)$ is defined as the pair of the periods of (S,θ) and T, i.e.,

$$\widetilde{\varpi}_{\Lambda}(X_{(S,\theta,T)}, \pi_1, \pi_2) := (\varpi_{\Lambda^{\perp}}(S,\theta), \Omega(T)) \in \mathcal{M}^o_{\Lambda} \times \mathfrak{M}.$$

Let $p\colon \mathcal{X} \to B$ be a proper, surjective holomorphic submersion between smooth complex spaces. Let $p_1\colon (\mathcal{S}, \vartheta) \to B$ be a family of 2-elementary $K3$ surfaces of type Λ and let $p_2\colon \mathcal{T} \to B$ be a family of elliptic curves with a holomorphic section. Then $\mathcal{T}$ is equipped with an involution $-1_{\mathcal{T}}$ which induces $-1_{p_2^{-1}(b)}$ for every $b \in B$. With respect to the trivial $\mathbf{Z}_2$-action on B, $p_2\colon \mathcal{T} \to B$ is $\mathbf{Z}_2$-equivariant. Let $\pi_1\colon \mathcal{X} \to \mathcal{S}/\mathbf{Z}_2$ and $\pi_2\colon \mathcal{X} \to \mathcal{T}/\mathbf{Z}_2$ be surjective holomorphic maps such that $p = p_1 \circ \pi_1 = p_2 \circ \pi_2$. Then the quintet $(p\colon \mathcal{X} \to B, p_1\colon (\mathcal{S},\vartheta) \to B, p_2\colon \mathcal{T} \to B, \pi_1, \pi_2)$ is called a *family of Borcea–Voisin threefold of type* Λ if $(p^{-1}(b), \pi_1|_{p^{-1}(b)}, \pi_2|_{p^{-1}(b)})$ is a Borcea–Voisin threefold of type Λ for all $b \in B$.

Theorem 3.7. — *The coarse moduli space of Borcea–Voisin threefolds of type Λ is isomorphic to $\mathcal{M}^o_{\Lambda} \times \mathfrak{M}$ via the map $\widetilde{\varpi}_{\Lambda}$.*

Proof. — By Lemma 3.5, the set of isomorphism classes of Borcea–Voisin threefold of type Λ is identified with $\mathcal{M}^o_{\Lambda} \times \mathfrak{M}$ via the map $\widetilde{\varpi}_{\Lambda}$. Since the period map $\varpi_{\Lambda^{\perp}}$ (resp. Ω) is holomorphic for every family of 2-elementary $K3$ surfaces of type $\Lambda^{\perp}$ (resp. elliptic curves), $\widetilde{\varpi}_{\Lambda}$ is also holomorphic for every family of Borcea–Voisin threefold of type Λ by Definition 3.6. □

By Theorem 3.7 and [**38**, Cor. 8.3], the coarse moduli space of Borcea–Voisin threefolds of type Λ is quasi-affine if $r(\Lambda) \leq 12$.

3.4. Degenerations of Borcea–Voisin threefolds

Theorem 3.8. — *Let $(\mathfrak{p}, \mathfrak{q}) \in \overline{\mathcal{D}}^o_{\Lambda} \times \mathfrak{M}$ and let $C \subset \mathcal{M}^*_{\Lambda}$ be an irreducible projective curve passing through $\mathfrak{p}$. Assume that $\mathfrak{p} \in C \setminus \operatorname{Sing} C$ and that C intersects $\overline{\mathcal{D}}^o_{\Lambda}$ transversally at $\mathfrak{p}$. Then there exist an irreducible projective fourfold $\mathcal{X}$, a pointed compact Riemann surface $(B, \mathfrak{b})$, a neighborhood U of $\mathfrak{b}$, a surjective flat holomorphic map $\pi\colon \mathcal{X} \to B$, and a holomorphic map $f\colon (B,\mathfrak{b}) \to (C,\mathfrak{p})$ satisfying*

(1) *$f(B) = C$ and the map $f|_U\colon (U, \mathfrak{b}) \to (f(U), \mathfrak{p})$ is an isomorphism;*
(2) *for all $b \in U \setminus \{\mathfrak{b}\}$, $\pi^{-1}(b)$ is the Calabi–Yau threefold underlying a Borcea–Voisin threefold $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ of type Λ such that*

$$\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (f(b), \mathfrak{q}).$$

Proof. — By Theorem 3.1, there exist a pointed smooth projective curve $(B, \mathfrak{b})$, a neighborhood U of $\mathfrak{b}$, a holomorphic map $f\colon (B, \mathfrak{b}) \to (C, \mathfrak{p})$, a smooth projective threefold $\mathcal{W}$ with an involution $\theta\colon \mathcal{W} \to \mathcal{W}$, and a surjective holomorphic map $p\colon \mathcal{W} \to B$ satisfying Theorem 3.1 (1), (2), (3).

Let T be an elliptic curve with $\Omega(T) = \mathfrak{q} \in \mathfrak{M}$. Let Σ be the union of all 2-dimensional components of $(\mathcal{W} \times T)^{\theta \times (-1_T)} = \mathcal{W}^\theta \times T[2]$. Let $q\colon \widetilde{\mathcal{W} \times T} \to \mathcal{W} \times T$ be the blow-up of $\mathcal{W} \times T$ along Σ. Since $\theta \times (-1_T)$ acts as -1 on the normal bundle $N_{\Sigma/(\mathcal{W} \times T)}$ and since $q^{-1}(\Sigma) = \mathbf{P}(N_{\Sigma/(\mathcal{W} \times T)})$, $\theta \times (-1_T)$ lifts to an involution $\mathcal{I}$ on $\widetilde{\mathcal{W} \times T}$, which acts trivially on the exceptional divisor $q^{-1}(\Sigma)$.

We consider the $\mathbf{Z}_2$-action on $\widetilde{\mathcal{W} \times T}$ induced from $\mathcal{I}$, so that $q\colon \widetilde{\mathcal{W} \times T} \to \mathcal{W} \times T$ is $\mathbf{Z}_2$-equivariant. Set $\mathcal{X} := (\widetilde{\mathcal{W} \times T})/\mathbf{Z}_2 = (\widetilde{\mathcal{W} \times T})/\mathcal{I}$. Then $\mathcal{X}$ is an irreducible projective fourfold. Since the projections $p\colon \mathcal{W} \to B$, $\mathrm{pr}_1\colon \mathcal{W} \times T \to \mathcal{W}$, and $q\colon \widetilde{\mathcal{W} \times T} \to \mathcal{W} \times T$ are $\mathbf{Z}_2$-equivariant, the composite $p \circ \mathrm{pr}_1 \circ q\colon \widetilde{\mathcal{W} \times T} \to B$ is $\mathbf{Z}_2$-equivariant and induces a surjective holomorphic map $\pi\colon \mathcal{X} \to B$. Since $\mathcal{X}$ is irreducible and $\dim B = 1$, $\pi\colon \mathcal{X} \to B$ is a flat holomorphic map.

For $b \in U \setminus \{\mathfrak{b}\}$, set $W_b := p^{-1}(b)$, $\theta_b := \theta|_{W_b}$ and $\Sigma_b := \Sigma \cap (W_b \times T)$. Then (W_b, θ_b) is a 2-elementary $K3$ surface of type $\Lambda^\perp$ and $\Sigma_b = W_b^{\theta_b} \times T[2]$ by Theorem 3.1 (3). Let $q_b\colon \widetilde{W_b \times T} \to W_b \times T$ be the blow-up along Σ_b. Since $W_b \times T$ intersects Σ transversely, we get $q^{-1}(W_b \times T) = \widetilde{W_b \times T}$ and $q_b = q|_{q^{-1}(W_b \times T)}$. Thus

$$(p \circ \mathrm{pr}_1 \circ q)^{-1}(b) = q^{-1} \circ (\mathrm{pr}_1)^{-1} \circ p^{-1}(b) = q^{-1}(W_b \times T) = \widetilde{W_b \times T}. \tag{3.2}$$

Since $p \circ \mathrm{pr}_1 \circ q$ is $\mathbf{Z}_2$-equivariant, $\mathcal{I}$ preserves the fibers of $p \circ \mathrm{pr}_1 \circ q$. Set $\mathcal{I}_b := \mathcal{I}|_{\widetilde{W_b \times T}}$. Since $q \circ \mathcal{I} \circ q^{-1}|_{(\mathcal{W} \times T) \setminus \Sigma} = \theta \times (-1_T)|_{(\mathcal{W} \times T) \setminus \Sigma}$ by the definition of $\mathcal{I}$, we get

$$q \circ \mathcal{I}_b \circ q^{-1}|_{(W_b \times T) \setminus \Sigma_b} = \theta_b \times (-1_T)|_{(W_b \times T) \setminus \Sigma_b}.$$

Since $q_b|_{(\widetilde{W_b \times T}) \setminus q_b^{-1}(\Sigma_b)}\colon (\widetilde{W_b \times T}) \setminus q_b^{-1}(\Sigma_b) \to (W_b \times T) \setminus \Sigma_b$ is an isomorphism,

$$\mathcal{I}_b|_{(\widetilde{W_b \times T}) \setminus q^{-1}(\Sigma_b)} = q_b^{-1} \circ (\theta_b \times (-1_T)) \circ q_b|_{(\widetilde{W_b \times T}) \setminus q^{-1}(\Sigma_b)} = \widetilde{\theta_b \times (-1_T)}|_{(\widetilde{W_b \times T}) \setminus q^{-1}(\Sigma_b)}$$

for all $b \in U \setminus \{\mathfrak{b}\}$. Since both of $\mathcal{I}_b$ and $\widetilde{\theta_b \times (-1_T)}$ are defined on $\widetilde{W_b \times T}$, this implies that

$$\mathcal{I}_b = \widetilde{\theta_b \times (-1_T)}. \tag{3.3}$$

By (3.1), (3.2), (3.3), we get

$$(3.4)\qquad \pi^{-1}(b) = (p \circ \mathrm{pr}_1 \circ q)^{-1}(b)/\mathbf{Z}_2 = \widetilde{(W_b \times T)}/\mathscr{I}_b = X_{(W_b,\theta_b,T)}.$$

Consider the projections $\pi_{1,b}\colon X_{(W_b,\theta_b,T)} \to W_b/\mathbf{Z}_2$ and $\pi_{2,b}\colon X_{(W_b,\theta_b,T)} \to T/\mathbf{Z}_2$. Then the triplet $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ is a Borcea–Voisin threefold of type Λ. Since $\varpi_{\Lambda^\perp}(W_b, \theta_b) = f(b)$ by Theorem 3.1 (3) and since $\Omega(T) = \mathfrak{q}$, we get $\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (f(b), \mathfrak{q})$ by (3.4). This completes the proof. □

Theorem 3.9. — *Let $\mathfrak{p} \in \mathscr{M}_\Lambda^o$. Let $p\colon \mathscr{E} \to B$ be an admissible elliptic fibration over a compact Riemann surface with a holomorphic section such that $\mathscr{E}$ is projective. Then there exist an irreducible projective fourfold $\mathscr{X}$ and a surjective flat holomorphic map $\pi\colon \mathscr{X} \to B$ such that $\pi^{-1}(b)$ is the Calabi–Yau threefold underlying a Borcea–Voisin threefold $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ of type Λ such that*

$$\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (\mathfrak{p}, \Omega(p^{-1}(b))), \qquad b \in B^o.$$

Proof. — Set $E_b := p^{-1}(b)$ for $b \in B^o$. Let $-1_{\mathscr{E}}$ be the holomorphic involution on $\mathscr{E}$ preserving the fibers of p such that $-1_{\mathscr{E}}|_{E_b} = -1_{E_b}$ for all $b \in B^o$. Let (S, θ) be a 2-elementary $K3$ surface of type $\Lambda^\perp$ with $\mathfrak{p} = \varpi_{\Lambda^\perp}(S, \theta)$. Then $S \times \mathscr{E}$ is equipped with the $\mathbf{Z}_2$-action induced from the involution $\theta \times (-1_{\mathscr{E}})$. Let $\mathscr{E}[2]$ denote the set of fixed points of $-1_{\mathscr{E}}$. The fixed point set of $\theta \times (-1_{\mathscr{E}})$ is given by $S^\theta \times \mathscr{E}[2]$. Since $\dim \mathscr{E}[2] = 1$, we get $\dim(S^\theta \times \mathscr{E}[2]) = 2$, where $S^\theta \times \mathscr{E}[2]$ may not be pure dimensional. Let Σ be the union of all 2-dimensional components of $S^\theta \times \mathscr{E}[2]$. Then Σ is the disjoint union of smooth complex surfaces. Let $q\colon \widetilde{S \times \mathscr{E}} \to S \times \mathscr{E}$ be the blow-up along Σ. As in the proof of Theorem 3.8, $\theta \times (-1_{\mathscr{E}})$ lifts to an involution $\mathscr{I}$ on $\widetilde{S \times \mathscr{E}}$, which induces a $\mathbf{Z}_2$-action on $\widetilde{S \times \mathscr{E}}$. Then $q\colon \widetilde{S \times \mathscr{E}} \to S \times \mathscr{E}$ is $\mathbf{Z}_2$-equivariant.

Set $\mathscr{X} := (\widetilde{S \times \mathscr{E}})/\mathbf{Z}_2$, which is an irreducible projective fourfold. Since the projections $q\colon \widetilde{S \times \mathscr{E}} \to S \times \mathscr{E}$, $\mathrm{pr}_2\colon S \times \mathscr{E} \to \mathscr{E}$, and $p\colon \mathscr{E} \to B$ are $\mathbf{Z}_2$-equivariant, the composite map $p \circ \mathrm{pr}_2 \circ q\colon \widetilde{S \times \mathscr{E}} \to B$ is $\mathbf{Z}_2$-equivariant and induces a holomorphic surjection $\pi\colon \mathscr{X} \to B$.

Let $b \in B^o$. Let $\widetilde{S \times E_b}$ be the blow-up of $S \times E_b$ along $S^\theta \times E_b[2] = (S \times E_b) \cap \Sigma$. Since $S \times E_b$ intersects Σ transversally, we get $q^{-1}(S \times E_b) = \widetilde{S \times E_b}$. Thus

$$(3.5)\qquad (p \circ \mathrm{pr}_2 \circ q)^{-1}(b) = q^{-1} \circ (\mathrm{pr}_2)^{-1} \circ p^{-1}(b) = q^{-1}(S \times E_b) = \widetilde{S \times E_b}.$$

Since $p \circ \mathrm{pr}_2 \circ q$ is $\mathbf{Z}_2$-equivariant, $\mathscr{I}$ preserves the fibers of $p \circ \mathrm{pr}_2 \circ q$. Set $\mathscr{I}_b := \mathscr{I}|_{\widetilde{S \times E_b}}$. Since $-1_{\mathscr{E}}|_{E_b} = -1_{E_b}$, we get

$$(3.6)\qquad I_b = \widetilde{\theta \times (-1_{E_b})}$$

as before in the proof of Theorem 3.8. By (3.5), (3.6), we get

$$(3.7)\qquad \pi^{-1}(b) = (p \circ \mathrm{pr}_2 \circ q)^{-1}(b)/\mathbf{Z}_2 = \widetilde{(S \times E_b)}/\mathscr{I}_b = X_{(S,\theta,E_b)}.$$

Consider the projections $\pi_{1,b}\colon X_{(S,\theta,E_b)} \to S/\mathbf{Z}_2$ and $\pi_{2,b}\colon X_{(S,\theta,E_b)} \to E_b/\mathbf{Z}_2$. Then $(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b})$ is a Borcea–Voisin threefold of type Λ. Since $\varpi_{\Lambda^\perp}(S,\theta) = \mathfrak{p}$, we get $\widetilde{\varpi}_\Lambda(\pi^{-1}(b), \pi_{1,b}, \pi_{2,b}) = (\mathfrak{p}, \Omega(E_b))$. □

Example 3.10. — We consider the pencil of plane cubics

$$S := \{((x:y:z),(t_0:t_1)) \in \mathbf{P}^2 \times \mathbf{P}^1;\ t_0 y^2 z = 4t_0 x^3 - 3t_1 x z^2 - t_0 z^3\},$$

$B := \mathbf{P}^1$, $p := \mathrm{pr}_2\colon S \to \mathbf{P}^1$. Then $p\colon S \to \mathbf{P}^1$ is an elliptic fibration equipped with a section $\sigma\colon \mathbf{P}^1 \ni t = (t_0 : t_1) \to ((0:1:0), t) \in S$. When t is a regular value of p, $\sigma(t)$ is the identity element of $p^{-1}(t)$. The involution

$$-1_S\colon S \ni ((x:y:z),(t_0:t_1)) \to ((x:-y:z),(t_0:t_1)) \in S$$

induces the map $-1_{p^{-1}(t)}$ when t is a regular value of p. Let $(\mathscr{E}, -1_{\mathscr{E}}) \to (S, -1_S)$ be an equivariant resolution of the singularity of S and set $\widetilde{p} := q \circ p$. Then $\widetilde{p}\colon \mathscr{E} \to \mathbf{P}^1$ is an admissible elliptic fibration with section. Since $j_{\mathscr{E}/\mathbf{P}^1}(t) = \frac{27t^3}{27(t^3-1)}$, $1/j_{\mathscr{E}/\mathbf{P}^1}(t)$ is a local coordinate of $\mathbf{P}^1$ near the set $\{(t_0:t_1) \in \mathbf{P}^1;\ t_0^3 = t_1^3\} \subset \Delta_{\mathscr{E}/\mathbf{P}^1}$.

3.5. Borcea–Voisin threefolds of exceptional type.

— Let 1_k denote the $k \times k$-identity matrix. For $\ell, m \in \mathbf{Z}$, we set

$$\mathbb{I}_{\ell,m} := \begin{pmatrix} 1_\ell & 0 \\ 0 & -1_m \end{pmatrix}, \qquad \mathbb{I}_{\ell,m}(2) = 2\begin{pmatrix} 1_\ell & 0 \\ 0 & -1_m \end{pmatrix},$$

which are identified with the corresponding lattices. Then $\mathbb{I}_{1,m}$ is an odd unimodular lattice and $\mathbb{I}_{1,m}(2)$ is a 2-elementary lattice. For $m \geq 0$, we define

$$\Lambda_m := \mathbb{U}(2) \oplus \mathbb{I}_{1,m-1}(2) \quad (m \geq 1), \qquad \Lambda_0 := \mathbb{I}_{2,0}(2).$$

By the classification of primitive 2-elementary Lorentzian sublattices of $\mathbb{L}_{K3}$ [**29**, p. 1434 Table 1], there exists a Borcea–Voisin threefold of type Λ_m if $0 \leq m \leq 9$.

Remark 3.11. — Let X be the Calabi–Yau threefold underlying a Borcea–Voisin threefold of type Λ and let $\pi\colon (\mathfrak{X}, X) \to (\mathrm{Def}(X), [X])$ be the Kuranishi family of X. We define the Borcea–Voisin locus $\mathrm{Def}(X)_{\mathrm{BV}} \subset \mathrm{Def}(X)$ as follows: $u \in \mathrm{Def}(X)_{\mathrm{BV}}$ if there exist a 2-elementary $K3$ surface (S_u, θ_u) of type $\Lambda^\perp$ and an elliptic curve T_u such that $\pi^{-1}(u) = X_{(S_u,\theta_u,T_u)}$. Comparing $\dim \mathrm{Def}(X)$ (cf. [**9**], [**36**]) and $\dim(\mathscr{M}^o_\Lambda \times \mathfrak{M})$, we have $\mathrm{Def}(X) = \mathrm{Def}(X)_{\mathrm{BV}}$ if and only if Λ is isometric to one of Λ_m $(0 \leq m \leq 9)$, $\mathbb{U}(2) \oplus \mathbb{U}(2)$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. When Λ is isometric to one of these lattices, then the Weil–Petersson metric on $\mathrm{Def}(X)$ coincides with the Bergman metric on $\Omega_\Lambda \times \mathfrak{H}$ (cf. Proof of Lemma 5.8). Notice that even if the moduli space is covered by a bounded symmetric domain, the Weil–Petersson metric does not necessarily coincide with the Bergman metric. For example, the moduli space of quintic mirror threefolds is covered by $\mathfrak{H}$, but the curvature of the Weil–Petersson metric is positive on some domain of the moduli space.

Lemma 3.12. — *Let $(X_{(S,\theta,T)}, \pi_1, \pi_2)$ be a Borcea–Voisin threefold of type Λ. If Λ is isometric to one of Λ_m $(0 \leq m \leq 9)$, $\mathbb{U}(2) \oplus \mathbb{U}(2)$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, then*

$$h^{1,2}(X_{(S,\theta,T)}) + \frac{\chi(X_{(S,\theta,T)})}{12} + 3 = 14. \tag{3.8}$$

Proof. — Set $N := \dim H^0(S^\theta, \mathbf{C})$ and $N' := \frac{1}{2}\dim H^1(S^\theta, \mathbf{C})$. By [**9**], [**36**], we get

$$h^{1,1}(X_{(S,\theta,T)}) = 11 - 5N - N', \qquad h^{1,2}(X_{(S,\theta,T)}) = 11 + 5N' - N. \tag{3.9}$$

Assume $\Lambda \cong \Lambda_m$ $(0 \leq m \leq 9)$ or $\Lambda \cong \mathbb{U}(2) \oplus \mathbb{U}(2)$. Set $r := r(\Lambda)$, $r^\perp := r(\Lambda^\perp)$ and $l^\perp := l(\Lambda^\perp) = l(\Lambda)$. Then $r^\perp = 22 - r$ and $l^\perp = r$. By Proposition 3.2,

$$N = 1 + \frac{r^\perp - l^\perp}{2}, \qquad N' = 11 - \frac{r^\perp + l^\perp}{2}. \tag{3.10}$$

By (3.9) and (3.10), we get

$$h^{1,1}(X_{(S,\theta,T)}) = 5r^\perp - 39, \qquad h^{1,2}(X_{(S,\theta,T)}) = 21 - r^\perp. \tag{3.11}$$

Since $\chi(X_{(S,\theta,T)}) = 2(h^{1,1}(X_{(S,\theta,T)}) - h^{1,2}(X_{(S,\theta,T)}))$, we get

$$\chi(X_{(S,\theta,T)}) = 12(r^\perp - 10). \tag{3.12}$$

The result follows from (3.11) and (3.12) in this case.

Assume $\Lambda \cong \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$. Then $\Lambda^\perp \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$ and a 2-elementary $K3$ surface of type $\Lambda^\perp$ is the universal covering of an Enriques surface. Hence $N = N' = 0$ in this case. Since $h^{1,1}(X_{(S,\theta,T)}) = h^{1,2}(X_{(S,\theta,T)}) = 11$ and $\chi(X_{(S,\theta,T)}) = 0$ in this case, we get the result. □

4. Odd unimodular lattices and Borcherds products

In this section, we assume that Λ is a lattice of signature $(2, r(\Lambda) - 2)$.

4.1. Automorphic forms. — We fix a vector $l_\Lambda \in \Lambda \otimes \mathbf{R}$ with $\langle l_\Lambda, l_\Lambda \rangle \geq 0$. Hence $H_{l_\Lambda} = \varnothing$. We define

$$j_\Lambda(\gamma, [z]) := \frac{\langle \gamma(z), l_\Lambda \rangle}{\langle z, l_\Lambda \rangle}, \qquad [z] \in \Omega^+_\Lambda, \quad \gamma \in O^+(\Lambda).$$

Then $j_\Lambda(\gamma, \cdot)$ is a nowhere vanishing holomorphic function on Ω^+_Λ. A holomorphic function $f \in \mathcal{O}(\Omega^+_\Lambda)$ is called an *automorphic form on Ω^+_Λ for $O^+(\Lambda)$ of weight p* if

$$f(\gamma \cdot [z]) = \chi(\gamma)\, j_\Lambda(\gamma, [z])^p\, f([z]), \qquad [z] \in \Omega^+_\Lambda, \quad \gamma \in O^+(\Lambda),$$

where $\chi \in \mathrm{Hom}(O^+(\Lambda), \mathbf{C}^*)$ is a character. For an automorphic form f on Ω^+_Λ for $O^+(\Lambda)$ of weight p, the Petersson norm $\|f\|$ is the C^∞ function on Ω^+_Λ defined as

$$\|f([z])\|^2 := K_\Lambda([z])^p\,|f([z])|^2, \qquad K_\Lambda([z]) := \frac{\langle z, \bar{z}\rangle}{|\langle z, l_\Lambda\rangle|^2}.$$

Since $O^+(\Lambda)/[O^+(\Lambda), O^+(\Lambda)]$ is finite when $r(\Lambda) \geq 5$, $\|f\|^2$ is $O^+(\Lambda)$-invariant.

Let ω_Λ be the Kähler form of the Bergman metric on Ω^+_Λ:

$$\omega_\Lambda := -dd^c \log K_\Lambda = \frac{1}{2\pi i}\partial\bar{\partial}\log K_\Lambda.$$

For a divisor D on Ω^+_Λ, δ_D denotes the Dirac δ-current on Ω^+_Λ with support D.

4.2. Borcherds product associated with 2-elementary lattices. — For $\tau \in \mathfrak{H}$, set $q = e^{2\pi i\tau}$. The Dedekind η-function is defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n).$$

The theta series of the positive-definite A_1-lattice $\mathbb{A}_1^+ = \langle 2\rangle$ are defined by

$$\theta_{\mathbb{A}_1^+}(\tau) := \sum_{n\in\mathbf{Z}} q^{n^2}, \qquad \theta_{\mathbb{A}_1^+ + 1/2}(\tau) := \sum_{n\in\mathbf{Z}} q^{(n+\frac{1}{2})^2}.$$

Define $f_k^{(0)}(\tau), f_k^{(1)}(\tau) \in \mathcal{O}(\mathfrak{H})$ and the series $\{c_k^{(0)}(\ell)\}_{\ell\in\mathbf{Z}}$, $\{c_k^{(1)}(\ell)\}_{\ell\in\mathbf{Z}+k/4}$ by

$$\begin{aligned} f_k^{(0)}(\tau) &= \textstyle\sum_{l\in\mathbf{Z}} c_k^{(0)}(\ell)\, q^\ell &&:= \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}\,\theta_{\mathbb{A}_1^+}(\tau)^k, \\ f_k^{(1)}(\tau) &= \textstyle\sum_{l\in k/4+\mathbf{Z}} 2c_k^{(1)}(\ell)\, q^\ell &&:= -16\,\eta(4\tau)^8\eta(2\tau)^{-16}\,\theta_{\mathbb{A}_1^+ + 1/2}(\tau)^k. \end{aligned}$$

We define holomorphic functions $g_k^{(i)}(\tau) \in \mathcal{O}(\mathfrak{H})$, $i \in \mathbf{Z}/4\mathbf{Z}$ by

$$g_k^{(i)}(\tau) := \sum_{\ell\equiv i \bmod 4} c_k^{(0)}(\ell)\, q^{\ell/4}.$$

Let $\mathbf{C}[A_\Lambda]$ be the group ring of the discriminant group A_Λ and let $\{\mathbf{e}_\gamma\}_{\gamma\in A_\Lambda}$ be its standard basis. Recall that the element $\mathbf{1}_\Lambda \in A_\Lambda$ was defined in Sect. 2.1. If Λ is 2-elementary and $r(\Lambda) \leq 12$, the $\mathbf{C}[A_\Lambda]$-valued holomorphic function on $\mathfrak{H}$

$$F_\Lambda(\tau) := f^{(0)}_{12-r(\Lambda)}(\tau)\,\mathbf{e}_0 + 2^{\frac{r(\Lambda)-l(\Lambda)}{2}} \sum_{\gamma\in A_\Lambda} g^{(2\gamma^2)}_{12-r(\Lambda)}(\tau)\,\mathbf{e}_\gamma + f^{(1)}_{12-r(\Lambda)}(\tau)\,\mathbf{e}_{\mathbf{1}_\Lambda}$$

is a modular form for $Mp_2(\mathbf{Z})$ of type ρ_Λ in the sense of [**12**, Sect. 2] by [**38**, Th. 7.7].

Let $N \in \{1, 2\}$ and let L be a 2-elementary Lorentzian lattice. Let $\mathcal{W}$ be a Weyl chamber of L. We set $\Lambda := \mathbb{U}(N) \oplus L$ and $l_\Lambda = (1, 0, 0)$ in Sect. 4.1. By [**12**, Th. 13.3], the following infinite product on $L \otimes \mathbf{R} + i\,\mathcal{W}$ converges absolutely when $(\operatorname{Im} z)^2 \gg 0$ and it extends to an automorphic form on Ω_Λ^+ for $O^+(\Lambda)$:

$$
\begin{aligned}
\Psi_\Lambda(z, F_\Lambda) := e^{2\pi i \langle \varrho(L, F_L, \mathcal{W}), z\rangle} & \prod_{\lambda \in L,\, \lambda\cdot\mathcal{W}>0,\, \lambda^2 \geq -2} \left(1 - e^{2\pi i \langle \lambda, z\rangle}\right)^{c^{(0)}_{12-r(\Lambda)}(\lambda^2/2)} \\
\times & \prod_{\lambda \in 2L^\vee,\, \lambda\cdot\mathcal{W}>0,\, \lambda^2 \geq -2} \left(1 - e^{\pi i N \langle \lambda, z\rangle}\right)^{2^{\frac{r(\Lambda)-l(\Lambda)}{2}} c^{(0)}_{12-r(\Lambda)}(\lambda^2/2)} \\
\times & \prod_{\lambda \in (\mathbf{1}_L + L),\, \lambda\cdot\mathcal{W}>0,\, \lambda^2 \geq 0} \left(1 - e^{2\pi i \langle \lambda, z\rangle}\right)^{2c^{(1)}_{12-r(\Lambda)}(\lambda^2/2)},
\end{aligned}
\tag{4.1}
$$

where $\varrho(L, F_L, \mathcal{W}) \in L \otimes \mathbf{Q}$ is the Weyl vector of $(L, F_L, \mathcal{W})$. See [**12**, Th. 10.4] for an explicit formula for $\varrho(L, F_L, \mathcal{W})$. We refer to [**38**] for more about $\Psi_\Lambda(\cdot, F_\Lambda)$.

4.3. A Borcherds product associated with Λ_m. — Let $m \geq 1$. We fix a basis $\{h, d_1, \cdots, d_{m-1}\}$ of $\mathbb{I}_{1,m-1}(2)$ over $\mathbf{Z}$ such that

$$\langle h, h\rangle = 2, \qquad \langle h, d_i\rangle = 0, \qquad \langle d_i, d_j\rangle = -2\delta_{ij} \quad (1 \leq i, j \leq m-1).$$

We define

$$\varrho_m := \frac{1}{2}(3h - d_1 - \cdots - d_{m-1}) \in \mathbb{I}_{1,m-1}(2)^\vee = \mathbb{I}_{1,m-1}(1/2)$$

and

$$\Pi_m := \{d \in \Delta_{\mathbb{I}_{1,m-1}(2)};\ \langle \varrho_m, d\rangle = 1\}.$$

When $m \leq 9$, $\varrho_m^2 > 0$ and Π_m is finite. See [**27**, Th. 26.2] for an explicit formula for Π_m. Let $\mathcal{W}_m$ be the Weyl chamber of $\mathcal{C}_{\mathbb{I}_{1,m-1}(2)}$ containing ϱ_m. Set

$$\operatorname{Aut}(\mathcal{W}_m) := \{g \in O(\mathbb{I}_{1,m-1}(2));\ g(\mathcal{W}_m) = \mathcal{W}_m\}.$$

Proposition 4.1. — *If $1 \leq m \leq 9$, then the following hold:*

(1) *ϱ_m is a Weyl vector of $(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m)$.*
(2) *Π_m is the set of simple roots of $(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m)$.*
(3) $\mathcal{W}_m = \{v \in \mathbb{I}_{1,m-1}(2) \otimes \mathbf{R};\ v^2 > 0,\ \langle v, d\rangle > 0\ \forall\, d \in \Pi_m\}$.
(4) $\{v \in \mathbb{I}_{1,m-1}(2) \otimes \mathbf{R};\ \langle v, d\rangle \geq 0,\ \forall\, d \in \Pi_m\} \subset \overline{\mathcal{C}^+_{\mathbb{I}_{1,m-1}(2)}} \subset \sum_{d \in \Pi_m} \mathbf{R}_{\geq 0} d$.

Proof. — Since $\varrho(\mathbb{I}_{1,m-1}(2), F_{\mathbb{I}_{1,m-1}(2)}, \mathcal{W}_m) = 2\varrho_m$ by [**12**, Th. 10.4], we get (1) by [**38**, Th. 7.11 (2)]. We get the inclusion $\Pi(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m) \subset \Pi_m$ by the definition of a Weyl vector of $(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m)$. We prove the converse inclusion. Let $\delta \in \Pi(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m)$. Since $\operatorname{Aut}(\mathcal{W}_m)$ acts transitively on Π_m by [**27**, Cor. 26.7 (ii)],

$$\Pi_m = \operatorname{Aut}(\mathcal{W}_m) \cdot \delta \subset \operatorname{Aut}(\mathcal{W}_m) \cdot \Pi(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m).$$

Since $\operatorname{Aut}(\mathcal{W}_m)$ preserves $\Pi(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m)$, we get $\Pi_m \subset \Pi(\mathbb{I}_{1,m-1}(2), \mathcal{W}_m)$. This proves (2). We get (3) by (2.1) and (2).

Since $O(\mathbb{I}_{1,m-1}(2))/W(\mathbb{I}_{1,m-1}(2))$ is finite by [**29**, Cor. 4.2.3], the first inclusion of (4) follows from [**30**, Th. 1.4.3 and (1.4.5)]. Since $\overline{\mathscr{C}^{+}_{\mathbb{I}_{1,m-1}(2)}}$ is a self-dual cone and since $\sum_{d\in\Pi_m}\mathbf{R}_{\geq 0}d$ is the dual cone of $\{v\in\mathbb{I}_{1,m-1}(2)\otimes\mathbf{R};\ \langle v,d\rangle\geq 0,\ \forall\, d\in\Pi_m\}$, the second inclusion of (4) is a consequence of the first inclusion of (4). □

Theorem 4.2. — *If $1\leq m\leq 10$, then the following hold:*

(1) *There exists an automorphic form Φ_m on $\Omega^{+}_{\Lambda_m}$ for $O^{+}(\Lambda_m)$ of weight $14-m$ with zero divisor $\mathcal{D}_{\Lambda_m}$ such that*

$$\Phi_m(z)^2=\Psi_{\Lambda_m}(z,F_{\Lambda_m}).$$

(2) *The following identity holds for $z\in\mathbb{I}_{1,m-1}(2)\otimes\mathbf{R}+i\,\mathcal{W}_m$ with $(\operatorname{Im} z)^2\gg 0$:*

$$\Phi_m(z)=e^{2\pi i\langle\varrho_m,z\rangle}\prod_{\delta\in\{0,1\}}\prod_{\lambda\in\Pi_m^{+(\delta)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{c^{(\delta)}_{10-m}(\lambda^2/2)},$$

where $\Pi_m^{+(\delta)}:=\{\lambda\in\delta\varrho_m+\mathbb{I}_{1,m-1}(2);\ \lambda\cdot\mathcal{W}_m>0,\ \lambda^2\geq 2(\delta-1)\}$.

Proof. — Since $r(\Lambda_m)=l(\Lambda_m)$, we deduce from [**38**, Th. 8.1] that the weight of $\Psi_{\Lambda_m}(z,F_{\Lambda_m})$ is $2(14-m)$ and that $\operatorname{div}(\Psi_{\Lambda_m}(z,F_{\Lambda_m}))=2\,\mathcal{D}_{\Lambda_m}$. We set $\varphi=\|\Psi_{\Lambda_m}(z,F_{\Lambda_m})\|$ in [**37**, Th. 3.17]. Since we may choose $\nu(\Lambda_m^{\perp})=1$ in [**37**, Th. 3.17], we get the existence of an automorphic form Φ_m on $\Omega^{+}_{\Lambda_m}$ for $O^{+}(\Lambda_m)$ of weight $14-m$ with zero divisor $\mathcal{D}_{\Lambda_m}$. Comparing the weights and zeros, we get $\Phi_m^2=\Psi_{\Lambda_m}(\cdot,F_{\Lambda_m})$. This proves (1).

By [**12**, Th. 10.4], we get $\varrho(L,F_L,\mathcal{W})=2\varrho_m$ when $L=\mathbb{I}_{1,m-1}(2)$ and $\mathcal{W}=\mathcal{W}_m$. Since $\mathbb{I}_{1,m-1}(2)=\mathbb{A}_1^{+}\oplus\mathbb{A}_1\oplus\cdots\oplus\mathbb{A}_1$ and since $\mathbf{1}_{\mathbf{Z}h}=h/2$, $\mathbf{1}_{\mathbf{Z}d_i}=d_i/2$, we get $\mathbf{1}_{\mathbb{I}_{1,m-1}(2)}=(h+d_1+\cdots+d_{m-1})/2\equiv\varrho_m \mod \mathbb{I}_{1,m-1}(2)$. Since $L=\mathbb{I}_{1,m-1}(2)=2L^{\vee}$, $N=2$ and $r(\mathbb{I}_{2,m}(2))=l(\mathbb{I}_{2,m}(2))$ in (4.1), we get

$$\begin{aligned}\Phi_m(z)^2&=\Psi_{\Lambda_m}(z,F_{\Lambda_m})\\&=e^{2\pi i\langle 2\varrho_m,z\rangle}\prod_{\lambda\in\Pi_m^{+(0)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{2c^{(0)}_{10-m}(\frac{\lambda^2}{2})}\prod_{\lambda\in\Pi_m^{+(1)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{2c^{(1)}_{10-m}(\frac{\lambda^2}{2})}\\&=\left[e^{2\pi i\langle\varrho_m,z\rangle}\prod_{\delta\in\{0,1\}}\prod_{\lambda\in\Pi_m^{+(\delta)}}\left(1-e^{2\pi i\langle\lambda,z\rangle}\right)^{c^{(\delta)}_{10-m}(\lambda^2/2)}\right]^2.\end{aligned}$$

This proves (2). □

We study the invariance property of Φ_m. Recall that $W(\mathbb{I}_{1,m-1}(2))$ is the Weyl group of $\mathbb{I}_{1,m-1}(2)$. By Proposition 4.1 (3) and the definition of Π_m, we have

$$\operatorname{Aut}(\mathcal{W}_m)=\{g\in O^{+}(\mathbb{I}_{1,m-1}(2));\ g(\varrho_m)=\varrho_m\}.$$

By [**27**, Th. 23.9], $\mathrm{Aut}(\mathcal{W}_m)$ $(4 \leq m \leq 9)$ is isomorphic to the Weyl group of the root system of type $A_1 \times A_2$, A_4, D_5, E_6, E_7, E_8, respectively. Since the Weyl group $W(\mathbb{I}_{1,m-1}(2))$ acts transitively on the set of Weyl chambers of $\mathbb{I}_{1,m-1}(2)$, $O^+(\mathbb{I}_{1,m-1}(2))$ is generated by the reflection groups $W(\mathbb{I}_{1,m-1}(2))$ and $\mathrm{Aut}(\mathcal{W}_m)$.

Proposition 4.3. — *If $1 \leq m \leq 9$, then the following hold:*

(1) *For all $r \in \mathbb{I}_{1,m-1}(2)^\vee$ with $\langle r, \varrho_m\rangle \equiv 0 \mod 2$,*

$$\Phi_m(z + r) = \Phi_m(z).$$

(2) *For all $w \in W(\mathbb{I}_{1,m-1}(2))$,*

$$\Phi_m(w(z)) = \det(w)\, \Phi_m(z).$$

(3) *For all $g \in \mathrm{Aut}(\mathcal{W}_m)$,*

$$\Phi_m(g(z)) = \Phi_m(z).$$

Proof. — We get (1) by the infinite product expansion of Φ_m in Theorem 4.2 (2). Since $\mathrm{Aut}(\mathcal{W}_m)$ preserves ϱ_m and $\mathcal{W}_m$, $\Pi_m^{+(0)}$ and $\Pi_m^{+(1)}$ are $\mathrm{Aut}(\mathcal{W}_m)$-invariant. We get (3) by the infinite product expansion of Φ_m in Theorem 4.2.(2).

Since $O^+(\mathbb{I}_{1,m-1}(2)) \subset O^+(\mathbb{U}(2) \oplus \mathbb{I}_{1,m-1}(2))$ and since Φ_m is an automorphic form for $O^+(\mathbb{U}(2) \oplus \mathbb{I}_{1,m-1}(2))$, there is a character $\epsilon \in \mathrm{Hom}(O^+(\mathbb{I}_{1,m-1}(2)), \mathbf{C}^*)$ such that $\Phi_m(g(z)) = \epsilon(g)\, \Phi_m(z)$ for all $g \in O^+(\mathbb{I}_{1,m-1}(2))$. Since $W(\mathbb{I}_{1,m-1}(2))$ is generated by the reflections $\{s_\delta;\ \delta \in \Delta_{\mathbb{I}_{1,m-1}(2)}\}$, it suffices to prove $\epsilon(s_\delta) = -1$ for all $\delta \in \Delta_{\mathbb{I}_{1,m-1}(2)}$. Since $s_\delta^2 = 1$, we get $\epsilon(s_\delta) \in \{\pm 1\}$. If $\epsilon(s_\delta) = 1$, the vanishing order of Φ_m along the divisor H_δ would be an even integer, which contradicts Theorem 4.2 (1), i.e., $\mathrm{div}(\Phi_m) = \mathcal{D}_{\mathbb{I}_{1,m-1}(2)}$. Hence we get $\epsilon(s_\delta) = -1$. □

Question 4.4. — By Proposition 4.3 (1) and the infinite product expansion in Theorem 4.2, $\Phi_m(z)$ has a Fourier expansion with integral Fourier coefficients. By the same argument as in [**17**, Proof of Th. 2.3 (a)] (cf. [**21**]), we see that $\Phi_m(z)$ has a Fourier expansion of Lie type in the sense of [**18**, Def. 2.5.1]. Namely, the Fourier expansion of $\Phi_m(z)$ with respect to the cusp defined by a primitive isotropic vector of $\mathbb{U}(2)$ is of the form:

$$\sum_{w \in W(\mathbb{I}_{1,m-1}(2))} \det(w)\, \{e^{2\pi i\langle w(\varrho_m), z\rangle} - \sum_{r \in (\mathbb{I}_{1,m-1}(2) + \mathbf{Z}\varrho_m) \cap \overline{\mathcal{W}_m} \setminus \{0\}} m(r)\, e^{2\pi i \langle w(\varrho_m + r), z\rangle}\},$$

where $m(r) \in \mathbf{Z}$ for all $r \in (\mathbb{I}_{1,m-1}(2) + \mathbf{Z}\varrho_m) \cap \overline{\mathcal{W}_m} \setminus \{0\}$. To get this Fourier expansion, we used Propositions 4.1 and 4.3 (1), (2) instead of [**17**, Prop. 2.2, Eqs (2.2), (2.3)]. Since $\Phi_m(z)$ has a Fourier expansion of Lie type, there exists by [**17**, Sect. 3 and p. 222 Statement 6.8'], [**18**, Sect. 2.5] a Borcherds superalgebra $\mathfrak{g}_m$ such that $\mathfrak{g}_m$ is an automorphic correction of the Kac–Moody algebra defined by the generalized Cartan matrix $\langle d, \delta\rangle_{d,\delta \in \Pi_m}$ and such that $\Phi_m(z)$ is the denominator function of $\mathfrak{g}_m$.

Since $\Phi_m(z)$ has the $\mathrm{Aut}(\mathcal{W}_m)$-invariance by Proposition 4.3 (3), it is very likely that there is an $\mathrm{Aut}(\mathcal{W}_m)$-action on $\mathfrak{g}_m$ inducing the $\mathrm{Aut}(\mathcal{W}_m)$-invariance of $\Phi_m(z)$.

In Theorem 6.4 below, we shall see that $\Phi_m(z)$ is regarded as an automorphic form on the Kähler moduli of a Del Pezzo surface of degree $10-m$. A more interesting question is the construction of $\Phi_m(z)$ from the geometry of Del Pezzo surface. Is $\Phi_m(z)$ (or equivalently $\Phi_V(z)$ in Sect. 6) related to the Borcherds superalgebra constructed in [**20**] for a Del Pezzo surface of degree $10-m$?

4.4. Borcherds products associated with the odd unimodular lattices. —

We identify $\mathbb{I}_{1,m-1}\otimes\mathbf{R}+i\,\mathcal{C}^+_{\mathbb{I}_{1,m-1}}$ with $\Omega^+_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ by the isomorphism (2.2).

Theorem 4.5. — *For $1\leq m\leq 10$, $\Phi_m(z/2)$ is an automorphic form on $\Omega^+_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ for $O^+(\mathbb{U}\oplus\mathbb{I}_{1,m-1})$ of weight $14-m$ with zero divisor $\sum_{d\in\mathbb{U}\oplus\mathbb{I}_{1,m-1},\,d^2=-1}H_d$.*

Proof. — Set $L=\mathbb{I}_{1,m-1}(2)$. Hence $L(\frac{1}{2})=\mathbb{I}_{1,m-1}$. By (2.2), $\Omega_{\mathbb{U}(2)\oplus L}=\Omega_{\Lambda_m}$ is isomorphic to $L\otimes\mathbf{R}+i\,\mathcal{C}_L$ via the map

$$\iota\colon L\otimes\mathbf{R}+i\,\mathcal{C}_L\ni z\to\left[\left(-\frac{1}{2}\langle z,z\rangle_L,\frac{1}{2},z\right)\right]\in\Omega_{\mathbb{U}(2)\oplus L}. \tag{4.2}$$

Identify $\mathbb{U}$ with $\mathbb{U}(2)$ via the identity map of the Abelian groups underlying them. The lattice $\mathbb{U}\oplus L(1/2)$ is an *odd* unimodular lattice. The map (2.2) gives the following identification between $L(1/2)\otimes\mathbf{R}+i\,\mathcal{C}_{L(1/2)}$ and $\Omega_{\mathbb{U}\oplus L(1/2)}$:

$$\iota'\colon L(1/2)\otimes\mathbf{R}+i\,\mathcal{C}_{L(1/2)}\ni z\to\left[\left(-\frac{1}{2}\langle z,z\rangle_{L(1/2)},1,z\right)\right]\in\Omega_{\mathbb{U}\oplus L(1/2)}. \tag{4.3}$$

The identity map of the free $\mathbf{Z}$-modules underlying $\Lambda_m=\mathbb{U}(2)\oplus L$ and $\mathbb{U}\oplus L(1/2)$ induces an isomorphism from $\Omega_{\mathbb{U}(2)\oplus L}$ to $\Omega_{\mathbb{U}\oplus L(1/2)}$. This isomorphism is denoted by $I\colon\Omega_{\mathbb{U}(2)\oplus L}\ni[z]\to[z]\in\Omega_{\mathbb{U}\oplus L(1/2)}$. By (4.2) and (4.3), we get

$$(\iota')^{-1}\circ I\circ\iota(z)=2z. \tag{4.4}$$

By (4.2), (4.3), (4.4), an automorphic form $\Psi(z)$ on $L(1/2)\otimes\mathbf{R}+i\,\mathcal{C}_{L(1/2)}$ for $O^+(\mathbb{U}\oplus L(1/2))$ is identified with the automorphic form $\Psi((\iota')^{-1}\circ I\circ\iota(z))=\Psi(2z)$ on $L\otimes\mathbf{R}+i\,\mathcal{C}_L$ for $O^+(\mathbb{U}(2)\oplus L)$ via the identity map $I\colon\Omega_{\mathbb{U}(2)\oplus L}\to\Omega_{\mathbb{U}\oplus L(1/2)}$. In particular, $\Phi_m(z/2)$ is an automorphic form on $\Omega^+_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ for $O^+(\mathbb{U}\oplus\mathbb{I}_{1,m-1})$ of weight $14-m$. Since the zero divisor of $\Phi_m(z/2)$ on $\Omega^+_{\mathbb{U}\oplus\mathbb{I}_{1,m-1}}$ coincides with the zero divisor of $\Phi_m(z)$ on $\Omega^+_{\mathbb{U}(2)\oplus\mathbb{I}_{1,m-1}(2)}$, we get

$$\mathrm{div}(\Phi_m(z/2))=\sum_{d\in\Delta_{\Lambda_m}}H_d=\sum_{d\in\mathbb{U}\oplus\mathbb{I}_{1,m-1},\,d^2=-1}H_d.$$

This proves the theorem. □

Remark 4.6. — Let $\mathfrak{e}$, $\mathfrak{e}'$ be primitive isotropic vectors of Λ_m. By [**28**, Prop. 1.17.1], there exists $g \in O(\Lambda_m)$ with $g(\mathfrak{e}) = \mathfrak{e}'$ if and only if $\mathfrak{e}^\perp/\mathfrak{e} \cong (\mathfrak{e}')^\perp/\mathfrak{e}'$. Since $\mathfrak{e}^\perp/\mathfrak{e}$ is a unimodular Lorentzian lattice of signature $(1, m-1)$, Λ_m has a unique $O(\Lambda_m)$-orbit of primitive isotropic vectors if $m \neq 2, 10$. If $m = 2, 10$, there exist two $O(\Lambda_m)$-orbits of primitive isotropic vectors: If we set $\mathbb{V} := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, then $\Lambda_2 = \mathbb{U} \oplus \mathbb{V}$ and $\Lambda_{10} = \mathbb{U} \oplus \mathbb{V} \oplus \mathbb{E}_8$. Let $\mathfrak{e}$ (resp. $\mathfrak{e}'$) be a primitive isotropic vector of $\mathbb{U}$ (resp. $\mathbb{V}$). Then $\mathfrak{e}^\perp/\mathfrak{e}$ is an odd unimodular lattice, while $(\mathfrak{e}')^\perp/\mathfrak{e}'$ is an even unimodular lattice. Hence $\mathfrak{e}$ and $\mathfrak{e}'$ do not lie on the same $O(\Lambda_m)$-orbit. Since the choice of an $O(\Lambda_m)$-orbit of an isotropic vector of Λ_m corresponds to the choice of a zero-dimensional cusp of $\mathcal{M}^*_{\Lambda_m}$, $\mathcal{M}^*_{\Lambda_m}$ has a unique zero-dimensional cusp if $3 \leq m \leq 9$.

4.5. The Borcherds Φ-function and Φ_{10}. — By [**12**, Th. 13.3], [**38**, Th. 8.1], $\Psi_{\mathbb{U}(2)\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2)}(\cdot, F_{\mathbb{U}(2)\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2)})$ is a meromorphic function on $\mathcal{M}_{\mathbb{U}(2)\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2)}$ without zeros and poles and hence is a constant function. By comparing the exponents of the infinite product (4.1), this implies that the Fourier coefficients of $f_0^{(0)}(\tau)$ and $f_0^{(1)}(\tau)$ satisfy the following relation:

$$c_0^{(0)}(2m) + c_0^{(1)}(2m) = 0, \qquad m \in \mathbf{Z}. \tag{4.5}$$

Since $\eta(2\tau)^{-16}\eta(4\tau)^8 = \prod_{n=1}^\infty (1-q^{2n})(1+q^{2n})^{-1}$, we get by the definition of $f_0^{(1)}(\tau)$

$$c_0^{(0)}(2m-1) = 0, \qquad m \in \mathbf{Z}. \tag{4.6}$$

Let $\Lambda = \mathbb{U}(2) \oplus \mathbb{U} \oplus \mathbb{E}_8(2)$. The weight of $\Psi_\Lambda(\cdot, F_\Lambda)$ is 4 by [**12**, Th. 13.3], [**38**, Th. 8.1]. The automorphic form $\Psi_\Lambda(\cdot, F_\Lambda)$ is the Borcherds Φ-function of dimension 10 (cf. [**11**]). We set $N = 2$, $L = \mathbb{U} \oplus \mathbb{E}_8(2)$ and $\rho = ((0,1), 0_{E_8(2)})$ in (4.1). Then $\varrho(L,, F_L \mathcal{W}) = \rho$ by [**12**, Th. 10.4]. Substituting this into (4.1) and using (4.5), (4.6), we get the expression in [**11**]:

$$\Psi_\Lambda(z, F_\Lambda) = e^{2\pi i\langle\rho,z\rangle} \prod_{\lambda \in \Delta_L^+ \cup (L \cap \overline{\mathcal{C}}_L^+)} (1 - e^{2\pi i\langle\lambda,z\rangle})^{\epsilon(\lambda) c_0^{(0)}(\lambda^2/2)},$$

which is the denominator function of the fake monster algebra [**10**, Sect. 14 Example 3]. Here $\epsilon(\lambda) = 1$ when $\lambda \in 2L^\vee$. When $\lambda \in L \setminus (2L^\vee)$, we set $\epsilon(\lambda) = 1$ if $\lambda^2/2 \notin 2\mathbf{Z}$ and $\epsilon(\lambda) = -1$ if $\lambda^2/2 \in 2\mathbf{Z}$. Then $\Psi_\Lambda(\cdot, F_\Lambda)$ is identified with Φ_{10} as follows.

Using the basis $\{h, d_1, \dots, d_9\}$ of $\mathbb{I}_{1,9}(2)$ with Gram matrix $\mathbb{I}_{1,9}(2)$, we define

$$K := \{k \in \mathbb{I}_{1,9}(2);\ \langle k, d_9\rangle = \langle k, 3h - \sum_{i=1}^8 d_i\rangle = 0\} \cong \mathbb{E}_8(2),$$

where the last isometry follows from e.g. [**27**, Th. 25.4]. We set

$$\mathfrak{f} := (3h - \sum_{i=1}^9 d_i)/2 = \varrho_9, \qquad \mathfrak{f}' := (3h - \sum_{i=1}^8 d_i + d_9)/2.$$

Then $\mathfrak{f}^2 = (\mathfrak{f}')^2 = 0$ and $\langle \mathfrak{f}, \mathfrak{f}' \rangle = 1$. We define $L := \mathbf{Z}\mathfrak{f} + \mathbf{Z}\mathfrak{f}' + \mathbf{Z}h + \sum_{i=1}^{9} \mathbf{Z}d_i$, which is equipped with the bilinear form induced from $\mathbb{I}_{1,9}(2)$. Since

$$(4.7)\quad \mathbf{Z}h \oplus \mathbf{Z}d_1 \oplus \cdots \oplus \mathbf{Z}d_9 = \mathbf{Z}(3h - \sum_{i=1}^{8} d_i) \oplus \mathbf{Z}d_9 \oplus K = \mathbf{Z}(\mathfrak{f}' + \mathfrak{f}) \oplus \mathbf{Z}(\mathfrak{f}' - \mathfrak{f}) \oplus K$$

and hence $L = \mathbf{Z}\mathfrak{f} \oplus \mathbf{Z}\mathfrak{f}' \oplus K$, we get $L \cong \mathbb{U} \oplus \mathbb{E}_8(2)$. Since $\mathbb{I}_{1,9}(2) \subset L$, we have the inclusion of lattices $\Lambda_{10} = \mathbb{U}(2) \oplus \mathbb{I}_{1,9}(2) \subset \mathbb{U}(2) \oplus L = \Lambda$, which yields the identification $\Omega_{\Lambda_{10}} = \Omega_\Lambda$. Since $O(\Lambda_{10}) = \{g \in O(\Lambda);\, g(\Lambda_{10}) = \Lambda_{10}\} \subset O(\Lambda)$, an automorphic form on $\Omega^+_{\Lambda_{10}}$ for $O^+(\Lambda_{10})$ is identified with an automorphic form on Ω^+_Λ for the cofinite subgroup $O^+(\Lambda_{10}) \subset O^+(\Lambda)$.

Theorem 4.7. — *Under the identification $\Omega^+_{\Lambda_{10}} = \Omega^+_\Lambda$ and the inclusion of groups $O^+(\Lambda_{10}) \subset O^+(\Lambda)$ induced from the inclusion of lattices $\Lambda_{10} \subset \Lambda$ as above,*

$$\Phi_{10} = \Psi_\Lambda(\cdot, F_\Lambda).$$

Proof. — We prove $\Delta_{\Lambda_{10}} = \Delta_\Lambda$. Since $\Lambda_{10} \subset \Lambda$ and hence $\Delta_{\Lambda_{10}} \subset \Delta_\Lambda$, it suffices to prove $\Delta_{\Lambda_{10}} \supset \Delta_\Lambda$. Let $d = (a, b, m, n, \lambda) \in \Delta_\Lambda$, where $(a, b) \in \mathbb{U}(2)$, $(m, n) \in \mathbb{U}$, and $\lambda \in \mathbb{E}_8(2)$. Since $d^2 = 4ab + 2mn + \lambda^2 = -2$ and $\lambda^2 \equiv 0 \mod 4$, we get $mn \equiv 1 \mod 2$ and hence $m \equiv n \equiv 1 \mod 2$. By (4.7), we get

$$m\,\mathfrak{f} + n\,\mathfrak{f}' + \lambda = \frac{m+n}{2}(\mathfrak{f} + \mathfrak{f}') + \frac{n-m}{2}(\mathfrak{f}' - \mathfrak{f}) + \lambda \in \mathbb{I}_{1,9}(2).$$

This proves $d \in \Lambda_{10} = \mathbb{U}(2) \oplus \mathbb{I}_{1,9}(2)$. Since $\Delta_{\Lambda_{10}} = \Delta_\Lambda$ via the inclusion $\Lambda_{10} \subset \Lambda$, both of Φ_{10} and $\Psi_\Lambda(\cdot, F_\Lambda)$ are automorphic forms on Ω^+_Λ for $O^+(\Lambda_{10})$ of weight 4 with zero divisor $\mathcal{D}_\Lambda$. Hence $\Phi_{10} = \mathrm{Const.}\,\Psi_\Lambda(\cdot, F_\Lambda)$ by the Koecher principle. Comparing $\lim_{z \to +i\infty} \Phi_{10}(z)$ and $\lim_{z \to +i\infty} \Psi_\Lambda(z, F_\Lambda)$, we get the result. □

5. The BCOV invariant of Borcea–Voisin threefolds

5.1. The BCOV invariant of Calabi–Yau threefolds. — Let X be a compact Kähler manifold with Kähler form γ. Let $D := \sqrt{2}(\bar\partial + \bar\partial^*)$ be the Dirac operator of (X, γ) and let $\square_{p,q} := D^2$ be the Laplacian of (X, γ) acting on (p, q)-forms on X. Let $\zeta_{p,q}(s)$ be the spectral zeta function of $\square_{p,q}$. After Ray-Singer [**33**], Bismut-Gillet-Soulé [**7**], and Bershadsky-Cecotti-Ooguri-Vafa [**3**], we make the following:

Definition 5.1. — The *BCOV torsion* of (X, γ) is the real number defined by

$$\mathcal{T}_{\mathrm{BCOV}}(X, \gamma) := \exp[-\sum_{p,q \geq 0} (-1)^{p+q} pq\, \zeta'_{p,q}(0)].$$

Assume that X is a Calabi–Yau n-fold. Let $\mathrm{Vol}(X, \gamma) = (2\pi)^{-n} \int_X \gamma^n / n!$ be the volume of (X, γ) and let $c_i(X, \gamma)$ denote the i-th Chern form of (TX, γ). Let η be a nowhere vanishing holomorphic n-form on X, whose L^2-norm is defined as $\|\eta\|_{L^2}^2 = (2\pi)^{-n}(\sqrt{-1})^{n^2}\int_X \eta\wedge\bar{\eta}$. Define

$$\mathcal{A}(X,\gamma) := \mathrm{Vol}(X,\gamma)^{\frac{\chi(X)}{12}}\exp\left[-\int_X \log\left(\frac{(\sqrt{-1})^{n^2}\eta\wedge\bar{\eta}}{\gamma^n/n!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|_{L^2}^2}\right)\frac{c_n(X,\gamma)}{12}\right].$$

Set $b_2(X) := \dim H^2(X,\mathbf{R})$. Let $\{\mathbf{e}_1,\dots,\mathbf{e}_{b_2(X)}\}$ be an integral basis of the free $\mathbf{Z}$-module $H^2(X,\mathbf{Z})_{\mathrm{fr}} := H^2(X,\mathbf{Z})/\mathrm{Torsion}$. Let κ be a Kähler class on X, and let $\mathrm{Vol}_{L^2}(H^2(X,\mathbf{Z}),\kappa)$ be the covolume of $H^2(X,\mathbf{Z})$ with respect to κ, i.e.,

$$\mathrm{Vol}_{L^2}(H^2(X,\mathbf{Z}),\kappa) := \det\left(\langle\mathbf{e}_i,\mathbf{e}_j\rangle_{L^2,\kappa}\right) = \mathrm{Vol}(H^2(X,\mathbf{R})/H^2(X,\mathbf{Z})_{\mathrm{fr}},\langle\cdot,\cdot\rangle_{L^2,\kappa}).$$

Definition 5.2. — When X is a Calabi–Yau threefold, define

$$\tau_{\mathrm{BCOV}}(X) := \frac{\mathcal{A}(X,\gamma)\,\mathcal{T}_{\mathrm{BCOV}}(X,\gamma)}{\mathrm{Vol}(X,\gamma)^3\,\mathrm{Vol}_{L^2}(H^2(X,\mathbf{Z}),[\gamma])}.$$

We call $\tau_{\mathrm{BCOV}}(X)$ the *BCOV invariant* of X.

The following result is a consequence of the curvature formula for Quillen metrics [**7**, Th. 0.1].

Theorem 5.3. — *When X is a Calabi–Yau threefold, $\tau_{\mathrm{BCOV}}(X)$ is independent of the choice of a Kähler metric on X. In particular, $\tau_{\mathrm{BCOV}}(X)$ is an invariant of X.*

Proof. — See [**14**, Th. 4.16]. □

5.2. The singularity of the BCOV invariant. — The following result is an application of the immersion formula for Quillen metrics [**8**], [**5**] (cf. [**39**]).

Theorem 5.4. — *Let $\mathcal{X}$ be an irreducible projective algebraic fourfold and let S be a compact Riemann surface. Let $\pi\colon\mathcal{X}\to S$ be a surjective, flat holomorphic map. Let $\mathcal{D}\subset S$ be a reduced divisor and set $\mathcal{X}^o := \mathcal{X}\setminus\pi^{-1}(\mathcal{D})$, $S^o := S\setminus\mathcal{D}$, $\pi^o := \pi|_{\mathcal{X}^o}$. Let $0\in\mathcal{D}$, and let (U,t) be a coordinate neighborhood of S centered at 0 such that $U\setminus\{0\}$ is isomorphic to the unit punctured disc in $\mathbf{C}$. If $\pi^o\colon\mathcal{X}^o\to S^o$ is a smooth morphism whose fibers are Calabi–Yau threefolds, then there exists $\alpha\in\mathbf{R}$ such that*

$$\log\tau_{\mathrm{BCOV}}(X_t) = \alpha\log|t|^2 + O(\log(-\log|t|^2))\qquad(t\to 0).$$

Proof. — See [**14**, Th. 9.1]. □

For a Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ of type Λ, set

$$\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}(\varpi_{\Lambda^\perp}(S,\theta),\Omega(T)) := \tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}).$$

By Theorem 3.7, $\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}$ is a function on $\mathcal{M}^o_\Lambda\times\mathfrak{M}$.

Proposition 5.5. — *Let* $(\mathfrak{p},\mathfrak{q}) \in \overline{\mathcal{D}}^{o}_{\Lambda} \times \mathfrak{M}$ *and let* $C \subset \mathcal{M}^{*}_{\Lambda}$ *be an irreducible projective curve passing through* $\mathfrak{p}$. *Assume that* $\mathfrak{p} \in C \setminus \operatorname{Sing} C$ *and that* C *intersects* $\overline{\mathcal{D}}^{o}_{\Lambda}$ *transversally at* $\mathfrak{p}$. *Let* (V,s) *be a coordinate neighborhood of* $\mathfrak{p}$ *in* C *centered at* $\mathfrak{p}$ *satisfying* $(\mathcal{M}^{*}_{\Lambda} \setminus \mathcal{M}^{o}_{\Lambda}) \cap \overline{V} = \{\mathfrak{p}\}$ *and* $\sup_{z\in V}|s(z)| < 1$. *Then there exist constants* $\alpha \in \mathbf{R}$ *and* $K \in \mathbf{R}_{>0}$ *such that for all* $z \in V \setminus \{\mathfrak{p}\}$,

$$\left|\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}|_{V}(z,\mathfrak{q}) + \alpha \log|s(z)|^{2}\right| \leq K \log(-\log|s(z)|^{2}). \tag{5.1}$$

Proof. — Let $\pi\colon \mathcal{X} \to B$, $f\colon (B,\mathfrak{b}) \to (C,\mathfrak{p})$, and $(U,\mathfrak{b}) \subset (B,\mathfrak{b})$ be the same as in Theorem 3.8. Choosing U sufficiently small, $f^{*}s$ is a coordinate on U centered at $\mathfrak{b}$. It suffices to prove (5.1) when $V = f(U)$. By Theorem 5.4 applied to the family $\pi\colon \mathcal{X} \to B$, there exist constants $\alpha \in \mathbf{R}$ and $K \in \mathbf{R}_{>0}$ such that for all $b \in U \setminus \{\mathfrak{b}\}$,

$$\left|\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}|_{f(U)}(f(b),\mathfrak{q}) + \alpha \log|s(f(b))|^{2}\right| \leq K \log(-\log|s(f(b))|^{2}), \tag{5.2}$$

because $\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}|_{f(U)}(f(b),\mathfrak{q}) = \tau_{\mathrm{BCOV}}(X_{(W_b,\theta_b,T)})$ by Theorem 3.8 (2). By setting $z = f(b)$, Estimate (5.1) follows from (5.2). □

Proposition 5.6. — *Let* $\mathfrak{p} \in \mathcal{M}^{o}_{\Lambda}$. *Let* $p\colon \mathcal{E} \to B$ *be an admissible elliptic fibration over a compact Riemann surface with a holomorphic section such that* $\mathcal{E}$ *is projective. For* $\mathfrak{b} \in j^{-1}_{\mathcal{E}/B}(\{\infty\})$, *let* (V,s) *be a coordinate neighborhood of* $\mathfrak{b}$ *in* B *centered at* $\mathfrak{b}$ *satisfying* $\sup_{z\in V}|s(z)| < 1$ *and* $V \cap j^{-1}_{\mathcal{E}/B}(\{\infty\}) = \{\mathfrak{b}\}$. *Then there exist constants* $\beta \in \mathbf{R}$ *and* $K \in \mathbf{R}_{>0}$ *such that for all* $z \in V \setminus \{\mathfrak{b}\}$,

$$\left|\overline{\tau}^{\Lambda}_{\mathrm{BCOV}}(\mathfrak{p}, j(E_b)) + \beta \log|s(z)|^{2}\right| \leq K \log(-\log|s(z)|^{2}). \tag{5.3}$$

Proof. — Let $\pi\colon \mathcal{X} \to B$ be the same as in Theorem 3.9. The result follows from Theorem 5.4 applied to the family $\pi\colon \mathcal{X} \to B$. □

5.3. The BCOV invariant of Borcea–Voisin threefolds of type Λ_m. — Let $\Delta(\tau) := \eta(\tau)^{24} = q\prod_{n=1}^{\infty}(1-q^{n})^{24}$ be the Jacobi Δ-function. Then $\Delta(\tau)$ is a cusp form on $\mathfrak{H}$ for $SL_2(\mathbf{Z})$ of weight 12. Let $\|\Delta(\tau)\|^{2} := (\operatorname{Im}\tau)^{12}|\Delta(\tau)|^{2}$ be the Petersson norm of $\Delta(\tau)$, which is a $SL_2(\mathbf{Z})$-invariant C^{∞} function on $\mathfrak{H}$. We often regard $\|\Delta(\tau)\|^{2}$ as a function on $\mathfrak{M} = SL_2(\mathbf{Z})\backslash\mathfrak{H}$.

Theorem 5.7. — *Assume that* $m = 0$ *or* $4 \leq m \leq 9$ *and set* $\|\Phi_m\| := 1$ *when* $m = 0$. *Then there exists a constant* C_m *depending only on* m *such that for every Borcea–Voisin threefold* $(X_{(S,T)},\pi_1,\pi_2)$ *of type* Λ_m,

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}) = C_m\,\|\Phi_m(\varpi_{\Lambda^{\pm}_{m}}(S,\theta))\|^{2} \cdot \|\Delta(\Omega(T))\|^{2}. \tag{5.4}$$

Since Φ_m is the denominator function of a Borcherds superalgebra (cf. Question 4.4), Theorem 5.7 implies that the conjecture of Harvey–Moore [**19**, Sect. 7 Conjecture] holds for Borcea–Voisin threefolds of type Λ_m, $4 \leq m \leq 9$.

For the proof of Theorem 5.7, we need some intermediate results. Let

$$\Pi_m\colon \Omega_{\Lambda_m}\times\mathfrak{H}\to\mathcal{M}_{\Lambda_m}\times\mathfrak{M}$$

be the natural projection and set

$$\tau^{\Lambda_m}_{\mathrm{BCOV}}:=\Pi_m^*\,\overline{\tau}^{\Lambda_m}_{\mathrm{BCOV}}.$$

By Theorems 3.7 and 5.3, $\tau^{\Lambda_m}_{\mathrm{BCOV}}$ is an $O^+(\Lambda_m)\times SL_2(\mathbf{Z})$-invariant C^∞ function on $\Omega^o_{\Lambda_m}\times\mathfrak{H}$. Set

$$\overline{F}_m:=\log\left[\frac{\overline{\tau}^{\Lambda_m}_{\mathrm{BCOV}}}{\|\Phi_m\|^2\|\Delta\|^2}\right].$$

Then $\overline{F}_m$ is a function on $\mathcal{M}^o_{\Lambda_m}\times\mathfrak{M}$. Set

$$F_m:=\Pi_m^*\overline{F}_m,$$

which is an $O(\Lambda_m)^+\times SL_2(\mathbf{Z})$-invariant C^∞ function on $\Omega^o_{\Lambda_m}\times\mathfrak{H}$.

Lemma 5.8. — *If $0\le m\le 9$, then F_m is pluri-harmonic on $\Omega^o_{\Lambda_m}\times\mathfrak{H}$.*

Proof. — Let $X=X_{(S,\theta,T)}$ be the Calabi–Yau threefold underlying a Borcea–Voisin threefold of type Λ_m and let $\pi\colon(\mathfrak{X},X)\to(\mathrm{Def}(X),[X])$ be the Kuranishi family of X. Similarly, let $\pi'\colon((\mathfrak{S},\Theta),(S,\theta))\to(\mathrm{Def}(S,\theta),[(S,\theta)])$ and $\pi''\colon(\mathfrak{T},T)\to(\mathrm{Def}(T),[T])$ be the Kuranishi family of (S,θ) and T, respectively. Comparing the dimensions of the Kuranishi spaces (cf. Remark 3.11 and (3.11)), we have an isomorphism of germs $(\mathrm{Def}(S,\theta),[(S,\theta)])\times(\mathrm{Def}(T),[T])\cong(\mathrm{Def}(X),[X])$, which is induced by the map

$$(\mathrm{Def}(S,\theta),[(S,\theta)])\times(\mathrm{Def}(T),[T])\ni(s,t)\to[X_{(S_s,\theta_s,T_t)}]\in(\mathrm{Def}(X),[X]).$$

We regard $\mathrm{Def}(X)$ as a small open subset of $\Omega^o_{\Lambda_m}\times\mathfrak{H}$. Similarly, we regard $\mathrm{Def}(S,\theta)$ and $\mathrm{Def}(T)$ as small open subsets of $\Omega^o_{\Lambda_m}$ and $\mathfrak{H}$, respectively.

Let $\xi'\in H^0(\mathrm{Def}(S,\theta),\pi_*K_{\mathfrak{S}/\mathrm{Def}(S,\theta)})$, $\xi''\in H^0(\mathrm{Def}(T),\pi_*K_{\mathfrak{T}/\mathrm{Def}(T)})$ and $\xi\in H^0(\mathrm{Def}(X),\pi_*K_{\mathfrak{X}/\mathrm{Def}(X)})$ be nowhere vanishing relative canonical forms, respectively. Then $\xi|_{(s,t)}$ is a non-zero holomorphic 3-form on $X_{(S_s,\theta_s,T_t)}$. Let $\|\xi\|^2_{L^2}$ be the C^∞ function on $\mathrm{Def}(X)\subset\Omega^o_{\Lambda_m}\times\mathfrak{H}$ defined as

$$\|\xi\|^2_{L^2}(s,t)=\left|\int_{X_{(S_s,\theta_s,T_t)}}\xi|_{(s,t)}\wedge\overline{\xi|_{(s,t)}}\right|,\qquad(s,t)\in\mathrm{Def}(X).$$

We define the functions $\|\xi'\|^2_{L^2}\in C^\infty(\mathrm{Def}(S,\theta))$ and $\|\xi''\|^2_{L^2}\in C^\infty(\mathrm{Def}(T))$ in the same manner. Since the holomorphic 3-form $\xi'|_s\wedge\xi''|_t$ on $(S_s\times T_t)/\theta_s\times(-1)_{T_t}$ lifts to a holomorphic 3-form on $X_{(S_s,\theta_s,T_t)}$, there is a nowhere vanishing holomorphic function $\psi\in\mathcal{O}(\mathrm{Def}(X))$ such that

$$\|\xi\|^2_{L^2}=|\psi|^2\,\|\xi'\|^2_{L^2}\|\xi''\|^2_{L^2}.$$

Let ω_{WP} be the Weil-Petersson form on $\Omega^o_{\Lambda_m}\times\mathfrak{H}$. Then $\log\|\xi\|^2_{L^2}$ is a local potential function of ω_{WP} (cf. [**14**, Sect. 4.2]). Similarly, $\log\|\xi'\|^2_{L^2}$ is a local potential function of ω_{Λ_m} (cf. [**38**, Eq. (5.4)]). Let $\omega_{\mathfrak{H}}$ be the Kähler form of the Poincaré metric on $\mathfrak{H}$, i.e.,

$$\omega_{\mathfrak{H}} = -dd^c \log \operatorname{Im} \tau.$$

Then $\log \|\xi''\|_{L^2}^2$ is a local potential function of $\omega_{\mathfrak{H}}$.

Since ω_{WP}, ω_{Λ_m}, $\omega_{\mathfrak{H}}$ have potentials $\|\xi\|_{L^2}^2$, $\|\xi'\|_{L^2}^2$, $\|\xi''\|_{L^2}^2$ respectively, we have

$$\omega_{\mathrm{WP}}|_{\mathrm{Def}(X)} = -dd^c \log \|\xi\|_{L^2}^2 = -dd^c \log(\|\xi'\|_{L^2}^2 \|\xi''\|_{L^2}^2) = \omega_{\Lambda_m}|_{\mathrm{Def}(S,\theta)} + \omega_{\mathfrak{H}}|_{\mathrm{Def}(T)},$$

which implies the following equation of $(1,1)$-forms on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$:

$$\omega_{\mathrm{WP}} = \omega_{\Lambda_m} + \omega_{\mathfrak{H}}. \tag{5.5}$$

Let $\mathrm{Ric}(\omega_{\mathrm{WP}})$, $\mathrm{Ric}(\omega_{\Lambda_m})$, $\mathrm{Ric}(\omega_{\mathfrak{H}})$ be the Ricci-forms of ω_{WP}, ω_{Λ_m}, $\omega_{\mathfrak{H}}$, respectively. By (5.5), we get

$$\mathrm{Ric}(\omega_{\mathrm{WP}}) = \mathrm{Ric}(\omega_{\Lambda_m}) + \mathrm{Ric}(\omega_{\mathfrak{H}}) = -m\,\omega_{\Lambda_m} - 2\,\omega_{\mathfrak{H}}, \tag{5.6}$$

where we used [**22**, Th. 4.1] and the explicit formula for the Bergman kernel [**23**, p. 34] to get the second equality. Notice that $K_\Lambda([z])^{-(r(\Lambda)-2)}$ is the Bergman kernel of Ω_Λ up to a constant by [**23**, p. 34].

Let $h^{1,2}$ and χ denote the Hodge number and the Euler characteristic of a Borcea–Voisin threefold of type Λ_m (cf. (3.11), (3.12)). By [**14**, Th. 4.14], Lemma 3.12, (5.5), (5.6), we get the following equation of C^∞ $(1,1)$-forms on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$:

$$dd^c \log \tau_{\mathrm{BCOV}}^{\Lambda_m} = -\left(h^{1,2} + \frac{\chi}{12} + 3\right) \omega_{\mathrm{WP}} - \mathrm{Ric}(\omega_{\mathrm{WP}}) = -(14-m)\,\omega_{\Lambda_m} - 12\,\omega_{\mathfrak{H}}. \tag{5.7}$$

Since Φ_m is an automorphic form on $\Omega^+_{\Lambda_m}$ for $O^+(\Lambda_m)$ of weight $14-m$ with zero divisor $\mathcal{D}_{\Lambda_m}$ by Theorem 4.2 and since $\Delta(\tau)$ is an elliptic modular form for $SL_2(\mathbf{Z})$ without zeros on $\mathfrak{H}$, we get the following equation on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$

$$-dd^c \log(\|\Phi_m\|^2 \|\Delta\|^2) = (14-m)\,\omega_{\Lambda_m} + 12\,\omega_{\mathfrak{H}},$$

which, together with (5.7), yields the desired equation $dd^c F_m = 0$ on $\Omega^o_{\Lambda_m} \times \mathfrak{H}$. This proves the lemma. □

Lemma 5.9. — *Let $\Delta \subset \mathbf{C}$ be the unit disc and set $\Delta^* := \Delta \setminus \{0\}$. Let f be a real-valued pluri-harmonic function on $\Delta^* \times \Delta^n$. Assume the existence of real-valued functions $\alpha(z)$ and $C(z)$ on Δ^n such that for all $|t| < \frac{1}{2}$ and $z \in \Delta^n$,*

$$|f(t,z) - \alpha(z) \log |t|^2| \le C(z) \log(-\log |t|).$$

Then $\alpha(z)$ is a constant function on Δ^n and there exists a real-valued pluri-harmonic function $\varphi(t,z)$ on Δ^{n+1} such that the following equation holds on $\Delta^* \times \Delta^n$:

$$f(t,z) = \alpha \log |t|^2 + \varphi(t,z), \qquad \alpha = \alpha(0).$$

In particular, the following identity of currents on Δ^{n+1} holds

$$dd^c f = \alpha\, \delta_{\{0\} \times \Delta^n}.$$

Proof. — Fix $z \in \Delta^n$. Since $dd^c f = 0$ on $\Delta^* \times \Delta^n$, we can put $P = f(\cdot, z)$, $\alpha = \alpha(z)$, $q = 0$ in [**37**, Prop. 3.11]. For each $z \in \Delta^n$, there exists by [**37**, Prop. 3.11] a harmonic function $\varphi(\cdot, z)$ on Δ satisfying the following the equation on $\Delta^* \times \{z\}$:

$$f(t, z) = \alpha(z) \log |t|^2 + \varphi(t, z).$$

By the same argument as in [**6**, pp. 54-75, Proof of Prop. 10.2 (ii)], $\alpha(z)$ is a constant function on Δ^n and $\varphi(t, z)$ is a pluri-harmonic function on Δ^{n+1}. □

Lemma 5.10. — *Let $0 \leq m \leq 9$. For every $d \in \Delta_{\Lambda_m}$, there exists $\alpha(d) \in \mathbf{R}$ such that the following equation of currents on $\Omega_{\Lambda_m} \times \mathfrak{H}$ holds:*

$$dd^c F_m = \sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_d \times \mathfrak{H}}. \tag{5.8}$$

Proof. — Since the result is obvious when $m = 0$, we assume $1 \leq m \leq 9$. By [**37**, Prop. 1.9 (2)], there is a Zariski closed subset $Z_m \subset \Omega_{\Lambda_m}$ of codimension ≥ 2 such that $\Omega^o_{\Lambda_m} \cup \mathscr{D}^o_{\Lambda_m} = \Omega_{\Lambda_m} \setminus Z_m$. Let $P \subset \Omega^o_{\Lambda_m} \cup \mathscr{D}^o_{\Lambda_m}$ be a small polydisc and set $H := P \cap \mathscr{D}^o_{\Lambda_m}$. Choosing P smaller if necessary, we may assume that H is a smooth hypersurfaces of P. By the same argument as in [**37**, Sect. 7 Step 1], there is a system of coordinates $(f_1, \ldots, f_m)$ on P such that $f_1, \ldots, f_m$ extend to meromorphic functions on $\mathscr{M}^*_{\Lambda_m}$ and such that H is defined by the equation $f_1 = 0$. By Proposition 5.5 and Lemma 5.8, there exist real-valued functions $\alpha(f_2, \ldots, f_m, \tau)$ and $C(f_2, \ldots, f_m, \tau)$ defined on $H \times \mathfrak{H}$ such that the following estimate holds on $(P \setminus H) \times \mathfrak{H}$:

$$|F_m(f_1, f_2, \ldots, f_m, \tau) - \alpha(f_2, \ldots, f_m, \tau) \log |f_1|^2| \leq C(f_2, \ldots, f_m, \tau) \log(-\log |f_1|^2).$$

By Lemma 5.9 applied to $F_m|_{(P \setminus H) \times \mathfrak{H}}$, α is a constant function on $H \times \mathfrak{H}$ and the equation of currents $dd^c F_m|_{P \times \mathfrak{H}} = \alpha\, \delta_{H \times \mathfrak{H}}$ holds on $P \times \mathfrak{H}$. This implies (5.8) on $\Omega^o_{\Lambda_m} \cup \mathscr{D}^o_{\Lambda_m} = \Omega_{\Lambda_m} \setminus Z_m$. By [**35**, p. 53 Th. 1], Eq. (5.8) holds on Ω_{Λ_m}. □

Lemma 5.11. — *Let $m = 0$ or $4 \leq m \leq 9$. Then F_m is pluri-harmonic on $\Omega_{\Lambda_m} \times \mathfrak{H}$. In particular, $\overline{F}_m$ extends to a pluri-harmonic function on $\mathscr{M}_{\Lambda_m} \times \mathfrak{M}$.*

Proof. — When $m = 0$, Ω_{Λ_m} is a point and $\Delta_{\Lambda_m} = \varnothing$. The result follows from (5.8) in this case. We assume $4 \leq m \leq 9$. By Lemma 5.9, it suffices to prove $\alpha(d) = 0$ for all $d \in \Delta_{\Lambda_m}/\{\pm 1\}$. Let $\gamma \in O^+(\Lambda_m)$. Since F_m is $O^+(\Lambda_m)$-invariant and hence $\gamma^* dd^c F_m = dd^c F_m$, we get by (5.8)

$$\sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_d \times \mathfrak{H}} = \gamma^*\Big(\sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_d \times \mathfrak{H}}\Big) = \sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \alpha(d)\, \delta_{H_{\gamma(d)} \times \mathfrak{H}}.$$

Hence $\alpha(\gamma(d)) = \alpha(d)$ for all $d \in \Delta_{\Lambda_m}/\{\pm 1\}$ and $\gamma \in O^+(\Lambda_m)$. Since $m \geq 4$, $\Delta_{\Lambda_m}/\{\pm 1\}$ consists of a unique $O^+(\Lambda_m)$-orbit by [**38**, Prop. 11.8]. There exists $\alpha \in \mathbf{R}$ such that $\alpha(d) = \alpha$ for all $d \in \Delta_{\Lambda_m}/\{\pm 1\}$. Replacing F_m by $-F_m$ if necessary, we may assume that $\alpha \geq 0$.

Let $\mathfrak{q} \in \mathfrak{H}$ be an arbitrary point. Set $f_m := F_m|_{\Omega_{\Lambda_m} \times \{\mathfrak{q}\}}$. Equation (5.8) restricted to $\Omega_{\Lambda_m} \times \{\mathfrak{q}\}$ yields that

$$dd^c f_m = dd^c(F_m|_{\Omega_{\Lambda_m} \times \{\mathfrak{q}\}}) = \alpha \sum_{d \in \Delta_{\Lambda_m}/\{\pm 1\}} \delta_{H_d \times \{\mathfrak{q}\}}. \tag{5.9}$$

Assume $\alpha \neq 0$. By (5.9) and the $O^+(\Lambda_m)$-invariance of $F_m|_{\Omega_{\Lambda_m} \times \{\mathfrak{q}\}}$, we may set $\varphi = f_m$, $p = q = 0$ in [**37**, Th. 3.17]. Then there would exist by [**37**, Th. 3.17] an integer $\nu \geq 1$ and an $O^+(\Lambda_m)$-invariant meromorphic function ψ on Ω_{Λ_m} such that

$$f_m = \alpha \log |\psi|^{2/\nu}, \qquad \operatorname{div}(\psi) = \nu \mathscr{D}_{\Lambda_m}.$$

Since $\dim(\mathscr{M}^*_{\Lambda_m} \setminus \mathscr{M}_{\Lambda_m}) \leq \dim \mathscr{M}^*_{\Lambda_m} - 2$ when $m \geq 3$, we deduce from the Levi extension theorem [**1**, Th. I.8.7] that ψ descends to a meromorphic function $\widetilde{\psi}$ on $\mathscr{M}^*_{\Lambda_m}$. Since $\operatorname{div}(\widetilde{\psi}) = \nu \overline{\mathscr{D}}_{\Lambda_m}$ by the relation $\operatorname{div}(\psi) = \nu \mathscr{D}_{\Lambda_m}$, we get a contradiction that the divisor of the meromorphic function $\widetilde{\psi}$ on the compact complex space $\mathscr{M}^*_{\Lambda_m}$ is non-zero and effective. Hence $\alpha(d) = \alpha = 0$ for all $d \in \Delta_{\Lambda_m}$. □

Lemma 5.12. — *Let* $\mathrm{pr}_2 \colon \mathscr{M}_{\Lambda_m} \times \mathfrak{M} \to \mathfrak{M}$ *be the projection. If* $m = 0$ *or* $4 \leq m \leq 9$, *then there exists a harmonic function* ϕ_m *on* $\mathfrak{M}$ *such that* $\overline{F}_m = (\mathrm{pr}_2)^* \phi_m$.

Proof. — Since $\mathscr{M}_{\Lambda_m}$ is a point when $m = 0$, the result is obvious in this case. We assume $4 \leq m \leq 9$. By Lemma 5.11, $\overline{F}_m$ extends to a pluri-harmonic function on $\mathscr{M}_{\Lambda_m} \times \mathfrak{M}$ when $4 \leq m \leq 9$. Since $\dim(\mathscr{M}^*_{\Lambda_m} \setminus \mathscr{M}_{\Lambda_m}) \leq \dim \mathscr{M}^*_{\Lambda_m} - 2$ when $m \geq 3$ and since $\mathscr{M}^*_{\Lambda_m}$ is normal, $\overline{F}_m$ extends to a pluri-harmonic function on $\mathscr{M}^*_{\Lambda_m} \times \mathfrak{M}$ by [**15**, Satz 4]. Since $\mathscr{M}^*_{\Lambda_m}$ is compact, $\overline{F}_m$ is constant on every slice $\mathscr{M}^*_{\Lambda_m} \times \{\mathfrak{q}\}$, $\mathfrak{q} \in \mathfrak{M}$, by the maximum principle. This proves the lemma. □

5.4. Proof of Theorem 5.7. — Let $\mathfrak{M}^*$ be the compactification of the modular curve $\mathfrak{M} = SL_2(\mathbf{Z}) \backslash \mathfrak{H}$ and set $\infty := \mathfrak{M}^* \setminus \mathfrak{M}$. The j-function induces an isomorphism $j \colon \mathfrak{M}^* \cong \mathbf{P}^1$ with $j(\infty) = \infty$ and $j(\mathfrak{M}) = \mathbf{C}$, such that $1/j$ is a local coordinate of $\mathfrak{M}^*$ centered at ∞. Since $j(\tau) = q^{-1} + O(1)$ and $\Delta(\tau) = q + O(q^2)$ near $\tau = +i\infty$, the following estimate holds near $j = \infty$:

$$\log \|\Delta\|^2 = \log |j|^2 + O(\log \log |j|). \tag{5.10}$$

Let (S, θ) be a 2-elementary $K3$ surface of type Λ_m with period $\mathfrak{p} \in \mathscr{M}^o_{\Lambda_m}$. Let $p \colon \mathscr{E} \to B$ be an admissible elliptic surface with a holomorphic section such that $\mathscr{E}$ is projective and such that there exists a singular fiber of type I_1, i.e., a nodal rational curve with a unique node. Such an elliptic fibration exists by Example 3.10. Set $E_b := p^{-1}(b)$ for $b \in B$. Let $\mathfrak{b} \in \Delta_{\mathscr{E}/B}$ be such that $E_{\mathfrak{b}}$ is a nodal rational curve with a unique node. Then $1/j_{\mathscr{E}/B}$ is a local coordinate of B near $\mathfrak{b}$. By (5.3), (5.10) and the definition of $\overline{F}_m$, there exists $\gamma \in \mathbf{R}$ such that as $b \to \mathfrak{b}$,

$$\begin{aligned}(5.11)\qquad \overline{F}_m(\mathfrak{p}, j_{\mathcal{E}/B}(b)) &= \log \overline{\tau}^{\Lambda_m}_{\mathrm{BCOV}}(\mathfrak{p}, j_{\mathcal{E}/B}(b)) - \log \|\Phi_m(\mathfrak{p})\|^2 - \log \|\Delta(\Omega(E_b))\|^2 \\ &= \gamma \log |j_{\mathcal{E}/B}(b)|^2 + O(\log\log |j_{\mathcal{E}/B}(b)|^2).\end{aligned}$$

Since $\overline{F}_m = (\mathrm{pr}_2)^*\phi_m$ and since ϕ_m is a harmonic function on $\mathfrak{M} = \mathbf{P}^1 \setminus \{\infty\}$, we deduce from (5.11) the following estimate near $j = \infty$:

$$(5.12)\qquad \phi_m(j) = \gamma \log |j|^2 + O(\log\log |j|).$$

Assume that $\gamma \neq 0$. Since ϕ_m is a harmonic function on $\mathfrak{M} = \mathbf{P}^1 \setminus \{\infty\}$, $\partial\phi_m$ must be a meromorphic 1-form on $\mathbf{P}^1$ with divisor $\mathrm{div}(\partial\phi_m) = -\{\infty\}$ by (5.12). Namely, $\partial\phi_m$ is a logarithmic 1-form on $\mathbf{P}^1$ with a unique pole at ∞. This contradicts the residue theorem. Hence $\gamma = 0$ and ϕ_m extends to a harmonic function on $\mathbf{P}^1$. By the maximum principle, ϕ_m is a constant. This proves that $\overline{F}_m = \mathrm{pr}_2^*\phi_m$ is also a constant. This completes the proof of Theorem 5.7. □

The proof contains technical difficulties when $1 \leq m \leq 3$; when $m = 3$, $\overline{\mathcal{D}}_{\Lambda_m}$ is not irreducible by [**38**, Prop. 11.8] and we can not get Lemma 5.11 by the same argument; when $m = 1, 2$, the boundary locus $\mathcal{M}^*_{\Lambda_m} \setminus \mathcal{M}_{\Lambda_m}$ is a divisor of $\mathcal{M}^*_{\Lambda_m}$ and the Hartogs extension theorem does not apply in Lemmas 5.11 and 5.12.

Conjecture 5.13. — Equation (5.4) holds when $1 \leq m \leq 3$.

5.5. Factorization of the BCOV invariant for Borcea–Voisin threefolds. — Let (X, γ) be a compact Kähler manifold. Let G be a compact Lie group acting holomorphically on X and preserving γ. Recall that $\square_{0,q}$ is the Laplacian acting on C^∞ $(0,q)$-forms on X. Let $\sigma(\square_{0,q})$ be the spectrum of $\square_{0,q}$. For $\lambda \in \sigma(\square_{0,q})$, let $E_{0,q}(\lambda)$ be the eigenspace of $\square_{0,q}$ with respect to the eigenvalue λ. Since G preserves γ, $E_{0,q}(\lambda)$ is a finite-dimensional unitary representation of G. For $g \in G$ and $s \in \mathbf{C}$, set

$$\zeta_{0,q}(g)(s) := \sum_{\lambda \in \sigma(\square_{0,q}) \setminus \{0\}} \mathrm{Tr}\,(g|_{E_{0,q}(\lambda)})\, \lambda^{-s}.$$

Then $\zeta_{0,q}(g)(s)$ converges absolutely when $\mathrm{Re}\, s > \dim X$, admits a meromorphic continuation to the complex plane $\mathbf{C}$, and is holomorphic at $s = 0$. The *equivariant analytic torsion* of (X, γ) is the class function on G defined by

$$\tau_G(X, \gamma)(g) := \exp[-\sum_{q \geq 0} (-1)^q q\, \zeta'_{0,q}(g)(0)].$$

When $g = 1$, $\tau_G(X, \gamma)(1)$ is denoted by $\tau(X, \gamma)$. We refer to [**4**], [**25**] for more about equivariant analytic torsion.

Let (S, θ) be a 2-elementary $K3$ surface of type M. Identify $\mathbf{Z}_2$ with the subgroup of $\mathrm{Aut}(S)$ generated by θ. Let γ be a $\mathbf{Z}_2$-invariant Kähler form on S and let η be a nonzero holomorphic 2-form on S. Let $S^\theta = \sum_i C_i$ be the decomposition into the connected components. In [**37**], we introduced the number

$$\tau_M(S,\theta) := \mathrm{vol}(S,\gamma)^{\frac{14-r(M)}{4}} \tau_{\mathbf{Z}_2}(S,\gamma)(\theta) \prod_i \mathrm{Vol}(C_i,\gamma|_{C_i})\tau(C_i,\gamma|_{C_i})$$
$$\times \exp\left[\frac{1}{8}\int_{S^\theta} \log\left(\frac{\eta\wedge\bar\eta}{\gamma^2/2!}\cdot\frac{\mathrm{Vol}(S,\gamma)}{\|\eta\|^2_{L^2}}\right)\Bigg|_{S^\theta} c_1(S^\theta,\gamma|_{S^\theta})\right].$$

By [**37**], $\tau_M(S,\theta)$ is an invariant of the pair (S,θ), so that τ_M descends to a function on $\mathscr{M}^o_{M^\perp}$, the coarse moduli space of 2-elementary $K3$ surfaces of type M.

Theorem 5.14. — *If $m=0$ or $3\le m\le 9$, there exists a constant C_{Λ_m} depending only on Λ_m such that for every 2-elementary $K3$ surface (S,θ) of type $\Lambda_m^\perp$,*

$$\tau_{\Lambda_m^\perp}(S,\theta) = C_{\Lambda_m}\,\|\Phi_m(\varpi_{\Lambda_m^\perp}(S,\theta))\|^{-\frac{1}{2}}.$$

Proof. — Since $\mathscr{M}^o_{\Lambda_m}$ is a point when $m=0$, the result is obvious in this case. When $3\le m\le 9$, the result follows from [**38**, Th. 9.1] and Theorem 4.2 (1). □

Let E be an elliptic curve and let γ be a Kähler form on E. Let ξ be a nonzero holomorphic 1-form on E. We set

$$\tau_{\mathrm{elliptic}}(E) := \mathrm{Vol}(E,\gamma)\,\tau(E,\gamma)\,\exp\left[\frac{1}{12}\int_E \log\left(\frac{\xi\wedge\bar\xi}{\gamma}\right) c_1(E,\gamma)\right].$$

Since $\chi(E)=\int_E c_1(E,\gamma)=0$, $\tau(E)_{\mathrm{elliptic}}$ is independent of the choice of ξ.

Lemma 5.15. — *The following identity holds:*

$$\tau_{\mathrm{elliptic}}(E) = \|\Delta(\Omega(E))\|^{-\frac{1}{6}}.$$

Proof. — The result follows from [**7**, Th. 0.2] and the Kronecker limit formula. □

Theorem 5.16. — *Assume $m=0$ or $4\le m\le 9$. The following identity holds for every Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ of type Λ_m:*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}) = C_m C^4_{\Lambda_m}\,\tau_{\Lambda_m^\perp}(S,\theta)^{-4}\,\tau_{\mathrm{elliptic}}(T)^{-12}.$$

Proof. — The result follows from Theorems 5.7 and 5.14 and Lemma 5.15. □

Conjecture 5.17. — If $\Lambda\subset\mathbb{L}_{K3}$ is a primitive 2-elementary sublattice with $\mathrm{sign}(\Lambda)=(2,r(\Lambda)-2)$, then there exist constants $a(\Lambda)$, $b(\Lambda)$, $C(\Lambda)$ depending only on Λ such that for every Borcea–Voisin threefold $(X_{(S,\theta,T)},\pi_1,\pi_2)$ of type Λ,

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)}) = C(\Lambda)\,\tau_{\Lambda^\perp}(S,\theta)^{a(\Lambda)}\,\tau_{\mathrm{elliptic}}(T)^{b(\Lambda)}.$$

If this conjecture holds, then an explicit formula for the BCOV invariant of the Borcea–Voisin threefolds of type Λ will be obtained from [**38**, Th. 0.1] when $r(\Lambda)\le 11$ or $(r(\Lambda),\delta(\Lambda))=(12,1)$.

Question 5.18. — Let $X_{(S,\theta,T)}$ be a Borcea–Voisin threefold and let $\pi\colon X_{(S,\theta,T)} \to (S \times T)/\mathbf{Z}_2$ be the projection with exceptional divisor $E := \pi^{-1}(\mathrm{Sing}\,(S \times T)/\mathbf{Z}_2)$. Then E has the structure of a $\mathbf{P}^1$-bundle over $\mathrm{Sing}\,(S \times T)/\mathbf{Z}_2$, whose fiber has negative intersection number with E.

Let γ be a Kähler metric on $(S \times T)/\mathbf{Z}_2$ in the sense of orbifolds and let γ_ϵ be a family of Kähler metrics on $X_{(S,\theta,T)}$ converging to γ as $\epsilon \to 0$ such that

$$[\gamma_\epsilon] = \pi^*[\gamma] - \epsilon\, c_1([E]), \qquad 0 < \epsilon \ll 1.$$

It is very likely that $\mathcal{T}_{\mathrm{BCOV}}(X_{(S,\theta,T)}, \gamma_\epsilon)$, $\mathcal{A}(X_{(S,\theta,T)}, \gamma_\epsilon)$, $\mathrm{Vol}_{L^2}(H^2(X_{(S,\theta,T)}, \mathbf{Z}), [\gamma_\epsilon])$ admit the following asymptotic expansions as $\epsilon \to 0$:

$$\begin{aligned}
&\log \mathcal{T}_{\mathrm{BCOV}}(X_{(S,\theta,T)}, \gamma_\epsilon) = \alpha_1 \log \epsilon + \beta_1 + o(1),\\
&\log \mathcal{A}(X_{(S,\theta,T)}, \gamma_\epsilon) = \alpha_2 \log \epsilon + \beta_2 + o(1),\\
&\log \mathrm{Vol}_{L^2}(H^2(X_{(S,\theta,T)}, \mathbf{Z}), [\gamma_\epsilon]) = \alpha_3 \log \epsilon + \beta_3 + o(1).
\end{aligned}$$

It is worth asking explicit formulae for β_1, β_2, β_3, which will lead to direct proofs of Theorems 5.7 and 5.16 and Conjecture 5.13 (and possibly Conjecture 5.17).

Question 5.19. — As an application of the arithmetic Lefschetz formula [**24**], the arithmetic counterpart of the invariant τ_M and hence Φ_m was studied by Maillot–Rössler [**26**]. After [**26**] and Theorem 5.16, it is worth asking the arithmetic counterpart of the BCOV invariant for general Calabi–Yau threefolds.

6. Automorphic forms on the Kähler moduli of a Del Pezzo surface

6.1. Del Pezzo surfaces. — A compact connected smooth complex surface V is a *Del Pezzo surface* if its anti-canonical line bundle K_V^{-1} is ample. The integer $\deg V := c_1(V)^2$ is called the degree of V. Then $1 \leq \deg V \leq 9$. Throughout this section, *V is a Del Pezzo surface.* A Del Pezzo surface of degree $d \neq 8$ is isomorphic to the blow-up of $\mathbf{P}^2$ at $9-d$ points in general position. A Del Pezzo surface of degree 8 is isomorphic to the blow-up of $\mathbf{P}^2$ at one point or to $\mathbf{P}^1 \times \mathbf{P}^1$. If $\deg V = d$, then $H^2(V, \mathbf{Z})$ equipped with the cup-product is isometric to $\mathbb{I}_{1,9-d}$ or to $\mathbb{U}$. Let $\langle \cdot, \cdot \rangle_V$ denote the cup-product pairing on the total integral cohomology lattice of V

$$H(V, \mathbf{Z}) := H^0(V, \mathbf{Z}) \oplus H^2(V, \mathbf{Z}) \oplus H^4(V, \mathbf{Z}).$$

We have an isometry of lattices $(H(V, \mathbf{Z}), \langle \cdot, \cdot \rangle_V) \cong \mathbb{U} \oplus \mathbb{I}_{1,9-\deg V}$ if $V \not\cong \mathbf{P}^1 \times \mathbf{P}^1$ and $(H(V, \mathbf{Z}), \langle \cdot, \cdot \rangle_V) \cong \mathbb{U} \oplus \mathbb{U}$ if $V \cong \mathbf{P}^1 \times \mathbf{P}^1$. The $\mathbf{Z}$-module $H^0(V, \mathbf{Z})$ (resp. $H^4(V, \mathbf{Z})$) has natural generators $[1]$ (resp. $[V]^\vee$) such that $\langle [1], [V]^\vee \rangle_V = 1$.

Let $1 \leq m \leq 9$ and let $P_1, \dots, P_{m-1}$ be $m-1$ points of $\mathbf{P}^2$ in general position. Let $\pi\colon V \to \mathbf{P}^2$ be the blow-up of $\mathbf{P}^2$ at $P_1, \dots, P_{m-1}$. Then V is a Del Pezzo surface of degree $10-m$. Set $E_i = \pi^{-1}(P_i)$. Then $E_1, \dots, E_{m-1}$ are (-1)-curves of V. Set

$H := \pi^* c_1(\mathcal{O}_{\mathbf{P}^2}(1)) \in H^2(V, \mathbf{Z})$ and $D_i := c_1([E_i])$, where $[E_i]$ is the line bundle on V defined by the divisor E_i. Then $\{H, D_1, \dots, D_{m-1}\}$ is a basis of $H^2(V, \mathbf{Z})$ over $\mathbf{Z}$ with Gram matrix $\mathbb{I}_{1,m-1}$. By the adjunction formula, we have

$$c_1(V) = c_1(K_V^{-1}) = 3H - (D_1 + \cdots + D_{m-1}).$$

Recall that the basis $\{h, d_1, \dots, d_{m-1}\}$ of $\mathbb{I}_{1,m-1}(2)$ and the Weyl vector $\varrho_m \in \mathbb{I}_{1,m-1}(2)^\vee$ were defined in Sect. 4.3. Let $\mathfrak{i}\colon H^2(V, \mathbf{Z}) \to \mathbb{I}_{1,m-1}(2)$ be the isomorphism of $\mathbf{Z}$-modules defined by

$$\mathfrak{i}(H) = h, \qquad \mathfrak{i}(D_i) = d_i \qquad (1 \le i \le m-1).$$

The following identities hold:

$$\langle \mathfrak{i}(v), \mathfrak{i}(w) \rangle_{\mathbb{I}_{1,m-1}(2)} = 2\langle v, w \rangle_V, \qquad \forall\, v, w \in H^2(V, \mathbf{Z}), \tag{6.1}$$

$$\mathfrak{i}(c_1(V)) = 2\varrho_m. \tag{6.2}$$

Set

$$\mathrm{Exc}(V) := \{c_1([C]) \in H^2(V, \mathbf{Z});\ C \text{ is a } (-1)\text{-curve on } V\}.$$

By [**27**, Th. 26.2 (i)],

$$\mathfrak{i}(\mathrm{Exc}(V)) = \Pi_m. \tag{6.3}$$

The set of effective classes on V is the subset of $H^2(V, \mathbf{Z})$ defined by

$$\mathrm{Eff}(V) := \{c_1(L) \in H^2(V, \mathbf{Z});\ L \in H^1(V, \mathcal{O}_V^*),\ h^0(L) > 0\}.$$

We set $\mathrm{Eff}(V)_{\ge m} := \{\alpha \in \mathrm{Eff}(V);\ \alpha^2 \ge m\}$ for $m \in \mathbf{Z}$. Let $\mathcal{K}_V \subset H^2(V, \mathbf{R})$ be the set of Kähler classes on V. By Nakai's criterion [**1**, Chap. IV Cor. 5.4], $\mathcal{K}_V$ is the cone of $H^2(V, \mathbf{R})$ given by $\mathcal{K}_V = \{x \in H^2(V, \mathbf{R});\ x^2 > 0,\ \langle x\ , \alpha \rangle_V > 0,\ \forall\, \alpha \in \mathrm{Eff}(V)\}$. If D is an irreducible projective curve on V with arithmetic genus $a(D)$, we get $c_1([D])^2 = 2a(D) - 2 + \deg(K_V^{-1}|_D) \ge 2a(D) - 1 \ge -1$ by the adjunction formula and the ampleness of K_V^{-1}. If $c_1([D])^2 = -1$ for an irreducible curve $D \subset V$, then $a(D) = 0$ and D must be a (-1)-curve by [**27**, Th. 26.2 (i)]. Hence $c_1([D])^2 \ge 0$ if $c_1([D]) \notin \mathrm{Exc}(V)$. Since $H^2(V, \mathbf{R})$ is a Lorentzian vector space, this implies that

$$\mathcal{K}_V = \{x \in H^2(V, \mathbf{R});\ x^2 > 0,\ \langle x\ , \delta \rangle_V > 0,\ \forall\, \delta \in \mathrm{Exc}(V)\}. \tag{6.4}$$

By Proposition 4.1 (3) and (6.3), (6.4), we get

$$\mathcal{W}_m = \mathfrak{i}(\mathcal{K}_V). \tag{6.5}$$

Lemma 6.1. — *Let L be a holomorphic line bundle on V with $c_1(L)^2 \ge -1$. Then $c_1(L) \cdot \mathcal{K}_V > 0$ if and only if L is effective, i.e., $h^0(L) > 0$.*

Proof. — Assume that $c_1(L) \cdot \mathcal{K}_V > 0$ and $c_1(L)^2 \geq -1$. By the Riemann-Roch theorem, $h^0(L) - h^1(L) + h^0(K_V \otimes L^{-1}) = 1 + \{\langle c_1(V), c_1(L)\rangle_V + c_1(L)^2\}/2$. Since $c_1(V) \in \mathcal{K}_V$ and $c_1(L)^2 \geq -1$, we get $h^0(L) + h^0(K_V \otimes L^{-1}) \geq 1$. Since K_V^{-1} is ample, we get $\langle c_1(K_V^{-1}), c_1(K_V \otimes L^{-1})\rangle_V = -c_1(K_V^{-1})^2 - \langle c_1(K_V^{-1}), c_1(L)\rangle_V < 0$ by the condition $c_1(L) \cdot \mathcal{K}_V > 0$. It follows from Nakai's criterion [**1**, Chap. 4 Cor. 5.4] that $K_V \otimes L^{-1}$ is not effective, i.e., $h^0(K_V \otimes L^{-1}) = 0$. Thus we get $h^0(L) > 0$.

If $h^0(L) > 0$ and $c_1(L)^2 \geq -1$, then L is effective and hence $\langle c_1(L), \kappa\rangle_V > 0$ for every Kähler class $\kappa \in H^2(V, \mathbf{R})$ on V. This proves the converse. □

Recall that the subset $\Pi_m^{+\,(\delta)}$ was defined in Theorem 4.2 (2).

Lemma 6.2. — *The following identities hold:*

(1) $\mathfrak{i}^{-1}(\Pi_m^{+(0)}) = \mathrm{Eff}(V)_{\geq -1}$.

(2) $\mathfrak{i}^{-1}(\Pi_m^{+(1)}) = \{\alpha \in H^2(V, \mathbf{Q});\ 2\alpha \in \mathrm{Eff}(V)_{\geq 0},\ \alpha \equiv c_1(V)/2 \mod H^2(V, \mathbf{Z})\}$.

Proof. — By (6.1), (6.5), the result is a consequence of Lemma 6.1 and the definition of $\Pi_m^{+\,(\delta)}$. □

6.2. An automorphic form on the Kähler moduli of V. — The *complexified Kähler cone* of V is the tube domain of $H^2(V, \mathbf{C})$ defined as $H^2(V, \mathbf{R}) + i\,\mathcal{K}_V$. Recall that $\mathcal{C}_{H^2(V,\mathbf{Z})}$ is the positive cone of the Lorentzian vector space $H^2(V, \mathbf{Z})$. Let $\mathcal{C}_V^+$ be the component of $\mathcal{C}_{H^2(V,\mathbf{Z})}$ containing $\mathcal{K}_V$. The complexified Kähler cone of V is regarded as an open subset of $\Omega^+_{H(V,\mathbf{Z})}$ via (2.2):

$$H^2(V, \mathbf{R}) + i\,\mathcal{K}_V \ni \eta \to \left[[1] + \eta - \frac{\eta^2}{2}[V]^\vee\right] \in \Omega^+_{H(V,\mathbf{Z})}.$$

Definition 6.3. — Define a formal infinite product $\Phi_V(w)$ on $H^2(V, \mathbf{R}) + i\,\mathcal{K}_V$ by

$$\Phi_V(w) := e^{\pi i \langle c_1(V), w\rangle_V} \prod_{\alpha \in \mathrm{Eff}(V)} \left(1 - e^{2\pi i \langle \alpha, w\rangle_V}\right)^{c^{(0)}_{\deg V}(\alpha^2)}$$

$$\times \prod_{\beta \in \mathrm{Eff}(V),\ \beta/2 \equiv c_1(V)/2 \mod H^2(V,\mathbf{Z})} \left(1 - e^{\pi i \langle \beta, w\rangle_V}\right)^{c^{(1)}_{\deg V}(\beta^2/4)}.$$

This is an analogue of similar infinite products for algebraic $K3$ surfaces [**16**].

Theorem 6.4. — *The following identity holds:*

$$\Phi_V(w) = \Phi_{10-\deg V}(\mathfrak{i}(w)/2).$$

In particular, $\Phi_V(w)$ converges absolutely for $w \in H^2(V, \mathbf{R}) + i\,\mathcal{K}_V$ with $(\mathrm{Im}\, w)^2 \gg 0$. Under the identification $H^2(V, \mathbf{R}) + i\,\mathcal{C}^+_{H^2(V,\mathbf{Z})} \cong \Omega^+_{H(V,\mathbf{Z})}$ given by (2.2), Φ_V extends to an automorphic form on $\Omega_{H(V,\mathbf{Z})}$ for $O^+(H(V, \mathbf{Z}))$ of weight $\deg V + 4$ with zero divisor $\sum_{\delta \in H(V,\mathbf{Z}),\, \delta^2 = -1} H_\delta$.

Proof. — Set $m = 10 - \deg V$. By Theorem 4.2 (2), we get

$$\begin{aligned}
\Phi_m(\mathrm{i}(w)/2) &= e^{2\pi i\langle \varrho_m, \mathrm{i}(w)/2\rangle} \prod_{\delta\in\mathbf{Z}_2} \prod_{\lambda\in\Pi_m^{+(\delta)}} \left(1 - e^{2\pi i\langle\lambda,\mathrm{i}(w)/2\rangle}\right)^{c_{10-m}^{(\delta)}(\lambda^2/2)} \\
&= e^{\pi i\langle c_1(V), w\rangle_V} \prod_{\delta\in\mathbf{Z}_2} \prod_{\alpha\in\mathrm{i}^{-1}(\Pi_m^{+(\delta)})} \left(1 - e^{2\pi i\langle\alpha,w\rangle_V}\right)^{c_{10-m}^{(\delta)}(\alpha^2)} \\
&= e^{\pi i\langle c_1(V), w\rangle_V} \prod_{\alpha\in\mathrm{Eff}(V)} \left(1 - e^{2\pi i\langle\alpha,w\rangle_V}\right)^{c_{\deg V}^{(0)}(\alpha^2)} \\
&\quad\times \prod_{2\alpha\in\mathrm{Eff}(V),\ \alpha\equiv c_1(V)/2 \mod H^2(V,\mathbf{Z})} \left(1 - e^{2\pi i\langle\alpha,w\rangle_V}\right)^{c_{\deg V}^{(1)}(\alpha^2)} \\
&= \Phi_V(w),
\end{aligned}$$

where the second equality follows from (6.1), (6.2) and the third equality follows from Lemma 6.2 and the vanishings $c_m^{(0)}(\ell) = 0$ for $\ell < -1$ and $c_m^{(1)}(\ell) = 0$ for $\ell < 0$. The rest of the theorem follows from Theorems 4.2 (1) and 4.5. □

Remark 6.5. — Let Λ be the total cohomology lattice of a $K3$ surface. In [**12**, Example 15.2], Borcherds constructed an $O^+(\Lambda)$-invariant real analytic function on the Grassmannian $G^+(\Lambda)$ with singularities along the subgrassmannians orthogonal to vectors of Λ of norm -2. The automorphic form Φ_V may be regarded as an analogue of this Borcherds' function for Del Pezzo surfaces.

Let (S,θ) be a 2-elementary $K3$ surface of type M with $M^\perp \cong H(V,\mathbf{Z})(2)$. By definition, there is an isometry $\mathrm{j}\colon H^2_-(S,\mathbf{Z}) \to H(V,\mathbf{Z})(2)$. By (2.2), there is a vector $\widehat{\varpi}_M(S,\theta,\mathrm{j}) \in H^2(V,\mathbf{R}) + i\,\mathscr{C}_V^+$ with

$$\mathrm{j}(H^{2,0}(S,\mathbf{C})) = \left[[1] + \widehat{\varpi}_M(S,\theta,\mathrm{j}) - \frac{1}{2}\widehat{\varpi}_M(S,\theta,\mathrm{j})^2\,[V]^\vee\right] \in \Omega^+_{H(V,\mathbf{Z})}.$$

Theorem 6.6. — *If* $\deg V \leq 7$*, there is a constant* $C_{\deg V}$ *depending only on* $\deg V$ *such that for every* 2*-elementary* $K3$ *surface* (S,θ) *of type* M *with* $M^\perp \cong \Lambda_{10-\deg V}$*,*

$$\tau_M(S,\theta) = C_{\deg V}\,\|\Phi_V(\widehat{\varpi}_M(S,\theta,\mathrm{j}))\|^{-\frac{1}{2}}.$$

Notice that the left hand side is a function on the moduli space of 2-elementary $K3$ surfaces of type M, while the right hand side is a function on the Kähler moduli of the Del Pezzo surface V.

Proof. — Since $\mathrm{i}\circ\mathrm{j}\colon H^2_-(S,\mathbf{Z}) \to \mathbb{I}_{1,m-1}(2)$ is an isometry of lattices, the point $(-\frac{1}{4}\widehat{\varpi}_M(S,\theta,\mathrm{j})^2, 1, \frac{1}{2}\mathrm{i}(\widehat{\varpi}_M(S,\theta,\mathrm{j}))) \in \Omega^+_{\Lambda_m}$ is the period of (S,θ). By Theorems 4.2, 5.13 and 6.4, we get

$$\tau_M(S,\theta) = C_M \left\| \Phi_m\left(\frac{1}{2}\mathrm{i}(\widehat{\varpi}_M(S,\theta,\mathrm{j}))\right)\right\|^{-\frac{1}{2}} = C_M \left\|\Phi_V(\widehat{\varpi}_M(S,\theta,\mathrm{j}))\right\|^{-\frac{1}{2}}.$$

Since the isometry class of M is determined by $\deg V$, we get the result. □

6.3. The functional equations of Φ_V. — Let $H^2(V,\mathbf{Z})_0$ be the maximal *even* sublattice of $H^2(V,\mathbf{Z})$:

$$H^2(V,\mathbf{Z})_0 := \{\alpha \in H^2(V,\mathbf{Z});\ \langle \alpha, c_1(V)\rangle_V \equiv 0 \mod 2\}.$$

Set $W(V) := \{g \in O^+(H^2(V,\mathbf{Z}));\ g(c_1(V)) = c_1(V)\}$. By [**27**, Th. 23.9], $W(V)$ is the Weyl group of the root system with root lattice $c_1(V)^\perp \subset H^2(V,\mathbf{Z})_0$. Set

$$\Gamma_V := H^2(V,\mathbf{Z})_0 \rtimes O^+(H^2(V,\mathbf{Z})), \qquad \widetilde{W}(V) := c_1(V)^\perp \rtimes W(V) \subset \Gamma_V.$$

Then $\widetilde{W}(V)$ is the affine Weyl group of the root system with root lattice $c_1(V)^\perp$. The group Γ_V preserves both of $H^2(V,\mathbf{R}) + i\,\mathcal{K}_V$ and $H^2(V,\mathbf{R}) + i\,\mathcal{C}_V^+$ and is regarded as a subgroup of $O^+(H(V,\mathbf{Z}))$ by the following injective homomorphism $\varphi\colon \Gamma_V \to O^+(H(V,\mathbf{Z}))$: For $(a,x,b) = a[1] + x + b[V]^\vee$,

$$\varphi_\lambda(a,x,b) := \begin{cases} a[1] + (x + a\lambda) + \left(b - \frac{\lambda^2}{2}a - \langle\lambda, x\rangle_V\right)[V]^\vee & (\lambda \in H^2(V,\mathbf{Z})_0), \\ a[1] + \lambda(x) + b[V]^\vee & (\lambda \in O^+(H^2(V,\mathbf{Z}))). \end{cases}$$

Then $\varphi(\Gamma_V)$ is the stabilizer of the isotropic vector $[1] \in H^0(V,\mathbf{Z})$ in $O^+(H(V,\mathbf{Z}))$.

Let G_V be the subgroup of $O^+(H(V,\mathbf{Z}))$ generated by the set

$$\varphi(\Gamma_V), \qquad \{s_{[1]+\delta}\}_{\delta\in\mathrm{Exc}(V)}, \qquad s_{[1]-[V]^\vee}, \qquad -1.$$

Following [**11**, Sect. 2], one can verify that G_V is a cofinite subgroup of $O^+(H(V,\mathbf{Z}))$ when $1 \leq \deg V \leq 7$. We give explicit functional equations of Φ_V for the above system of generators of G_V. We set $\Lambda = H(V,\mathbf{Z})$ and $l_{H(V,\mathbf{Z})} = [V]^\vee$ in Sect. 4.1.

Let $\mathfrak{W}^{(1)}(V)$ be the subgroup of $O^+(H^2(V,\mathbf{Z}))$ generated by the reflections $\{s_\delta\}_{\delta\in\mathrm{Exc}(V)}$. Since $\mathcal{K}_V$ is a fundamental domain for the $\mathfrak{W}^{(1)}(V)$-action on $\mathcal{C}_V^+$ and since $W(V)$ is the stabilizer of $\mathcal{K}_V$ in $O^+(H^2(V,\mathbf{Z}))$, $O^+(H^2(V,\mathbf{Z}))$ is generated by $\mathfrak{W}^{(1)}(V)$ and $W(V)$. Let $\epsilon\colon O^+(H^2(V,\mathbf{Z})) \to \{\pm 1\}$ be the character such that $\epsilon(g) = 1$ for $g \in W(V)$ and $\epsilon(g) = \det(g)$ for $g \in \mathfrak{W}^{(1)}(V)$.

By Proposition 4.3 (1), (2), (3), we get the following equations for $\varphi(\Gamma_V)$:

$$\text{(a)}\quad \Phi_V(w + l) = \Phi_V(w), \qquad \forall\, l \in H^2(V,\mathbf{Z})_0,$$

$$\text{(b)}\quad \Phi_V(g(w)) = \epsilon(g)\,\Phi_V(w), \quad \forall\, g \in O^+(H^2(V,\mathbf{Z})).$$

In particular, $\Phi_V(w)$ is invariant under the action of the affine Weyl group $\widetilde{W}(V)$.

Let $\delta \in \mathrm{Exc}(V)$. Since

$$s_{[1]+\delta}\left([1]+(w+\delta)-\frac{(w+\delta)^2}{2}[V]^\vee\right)$$
$$=-\langle w,w\rangle_V\left\{[1]+\left(-\frac{w}{\langle w,w\rangle_V}+\delta\right)-\frac{1}{2}\left(-\frac{w}{\langle w,w\rangle_V}+\delta\right)^2[V]^\vee\right\}$$

and since Φ_V vanishes of order 1 on $H_{[1]+\delta}$, the automorphic property of Φ_V with respect to $s_{[1]+\delta}$ (cf. Sect. 4.1) implies that

$$\text{(c)}\qquad \Phi_V\left(-\frac{w}{\langle w,w\rangle_V}+\delta\right)=-(-\langle w,w\rangle_V)^{\deg V+4}\,\Phi_V(w+\delta),\qquad \forall\,\delta\in\mathrm{Exc}(V).$$

Since $s_{[1]-[V]^\vee}([1]+w-\frac{w^2}{2}[V]^\vee)=-\frac{w^2}{2}[1]+w+[V]^\vee$, the automorphic property of Φ_V with respect to $s_{[1]-[V]^\vee}$ implies that $\Phi_V(-\frac{2w}{\langle w,w\rangle_V})=\epsilon(-\frac{\langle w,w\rangle_V}{2})^{\deg V+4}\,\Phi_V(w)$, $\epsilon\in\{\pm 1\}$. Since $[1]-[V]^\vee\in H(V,\mathbf{Z})$ is a vector of norm -2 and since Φ_V does not vanish on the divisor $H_{[1]-[V]^\vee}\subset\Omega^+_{H(V,\mathbf{Z})}$, we get $\epsilon=1$, i.e.,

$$\text{(d)}\qquad \Phi_V\left(-\frac{2w}{\langle w,w\rangle_V}\right)=\left(-\frac{\langle w,w\rangle_V}{2}\right)^{\deg V+4}\Phi_V(w).$$

Remark 6.7. — When $1\le\deg V\le 7$, the conditions $\mathrm{div}(\Phi_V)=\sum_{\delta\in H(V,\mathbf{Z}),\,\delta^2=-1}H_\delta$ and (a), (b), (c), (d) are sufficient to characterize Φ_V up to a constant, since $|O^+(H(V,\mathbf{Z}))/G_V|<\infty$.

6.4. Borcherds Φ-function as an analogue of Φ_V for Enriques surfaces. — Consider the case $N=1$ and $L=\mathbb{U}(2)\oplus\mathbb{E}_8(2)$ in (4.1). Then $L^\vee=\frac{1}{2}L$, $\mathbf{1}_L=0$, $\Delta_L=\varnothing$. By [**12**, Th. 10.4], we get $\varrho(L,F_L,\mathcal{W})=0$. Substituting these into (4.1), we get another expression of the Borcherds Φ-function [**12**, Example 13.7]

$$\text{(6.6)}\qquad \Psi_{\mathbb{U}\oplus L}(z,F_{\mathbb{U}\oplus L})=\prod_{\lambda\in L\cap\overline{\mathcal{C}}^+_L}\left(\frac{1-e^{\pi i\langle\lambda,z\rangle_L}}{1+e^{\pi i\langle\lambda,z\rangle_L}}\right)^{c_0^{(0)}(\lambda^2/2)},$$

which is the Fourier expansion of the Borcherds Φ-function at the *level* 1 *cusp* and is the denominator function of the fake monster superalgebra [**34**]. We see that (6.6) is regarded as an analogue of Theorem 6.4 in the case of Enriques surfaces.

Let S be an Enriques surface [**1**, Chap. VIII] and let $p\colon\widetilde{S}\to S$ be the universal covering. Let $\theta\in\pi_1(\widetilde{S})$ be the generator. Hence $S=\widetilde{S}/\theta$. Assume that S contains no rational curves. Let $\mathcal{K}_S\subset H^2(S,\mathbf{R})$ be the Kähler cone of S. We define the infinite product Φ_S on the complexified Kähler cone $H^2(S,\mathbf{R})+i\,\mathcal{K}_S$ by

$$\text{(6.7)}\qquad \Phi_S(w):=\prod_{\alpha\in H^2(S,\mathbf{Z})\cap\overline{\mathcal{K}}_S}\left(\frac{1-e^{2\pi i\langle\alpha,w\rangle_S}}{1+e^{2\pi i\langle\alpha,w\rangle_S}}\right)^{c_0^{(0)}(\alpha^2)}.$$

We set $H_+(\widetilde{S}, \mathbf{Z}) := \{v \in H(\widetilde{S}, \mathbf{Z}) = H^0(\widetilde{S}, \mathbf{Z}) \oplus H^2(\widetilde{S}, \mathbf{Z}) \oplus H^4(\widetilde{S}, \mathbf{Z});\ \theta^* v = v\}$. Then $H_+(\widetilde{S}, \mathbf{Z}) \cong \mathbb{U} \oplus H^2(\widetilde{S}, \mathbf{Z}) \cong \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2) = \mathbb{U} \oplus L$. The pull-back $p^* \colon H^2(S, \mathbf{Z}) \to H^2(\widetilde{S}, \mathbf{Z})$ induces the following embedding:

$$p^* \colon H^2(S, \mathbf{R}) + i\,\mathscr{K}_S \hookrightarrow H^2_+(\widetilde{S}, \mathbf{R}) + i\,\mathscr{C}^+_{H^2_+(\widetilde{S}, \mathbf{Z})} \cong \Omega^+_{H_+(\widetilde{S}, \mathbf{Z})},$$

where $H^2_+(\widetilde{S}, \mathbf{Z}) = H^2(\widetilde{S}, \mathbf{Z}) \cap H_+(\widetilde{S}, \mathbf{Z})$ and the last isomorphism is given by (2.2). By (6.6), Φ_S is an automorphic form on $\Omega^+_{H(\widetilde{S}, \mathbf{Z})}$ for $O^+(H_+(\widetilde{S}, \mathbf{Z}))$ of weight 4.

There is a formula for the analytic torsion of a Ricci-flat Enriques surface [**37**, Th. 8.3] analogous to Theorem 6.6: For every Ricci-flat Enriques surface (S, ω),

$$\mathrm{Vol}(S, \omega)^{\frac{1}{2}} \tau(S, \omega) = \mathrm{Const.}\, \|\Phi_S(\widetilde{\varpi}(\widetilde{S}, \theta))\|^{-\frac{1}{2}}.$$

Question 6.8. — After Theorem 4.7, it is worth asking the limiting situation in Theorem 6.4. Let W be the blow-up of $\mathbf{P}^2$ at 9 points. Is Φ_{10} regarded as an automorphic form on $H^2(W, \mathbf{R}) + i\,\mathscr{C}^+_W$? If this is the case, the Fourier expansion of the Borcherds Φ-function at the *level* 2 *cusp* would be regarded as an automorphic form on the complexified Kähler cone of W by Theorem 4.7. The case when these 9 points are given by the intersection of two generic cubics in $\mathbf{P}^2$ will be the most interesting, in which case W is a rational elliptic surface.

Question 6.9. — Let X be a smooth projective surface with $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$. As before, the tube domain $H^2(X, \mathbf{R}) + i\,\mathscr{C}^+_X$ is isomorphic to a bounded symmetric domain of type IV of dimension $b_2(X)$. As we have seen, there is a nice automorphic form on $H^2(X, \mathbf{R}) + i\,\mathscr{C}^+_X$ when X is a Del Pezzo surface or an Enriques surface. Is there a canonical way of constructing a nice Borcherds product on $H^2(X, \mathbf{R}) + i\,\mathscr{C}^+_X$? For example, when X is of general type with $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$ or when X is rational, is there an analogue of the Borcherds Φ-function for X?

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