

Astérisque

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Astérisque, tome 328 (2009), p. 223-235

http://www.numdam.org/item?id=AST_2009__328__223_0

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MIRZAKHANI'S RECURSION FORMULA IS EQUIVALENT TO THE WITTEN-KONTSEVICH THEOREM

by

Kefeng Liu & Hao Xu

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper, we give a proof of Mirzakhani's recursion formula of Weil-Petersson volumes of moduli spaces of curves using the Witten-Kontsevich theorem. We also describe properties of intersections numbers involving higher degree κ classes.

Résumé (La formule de récurrence de Mirzakhani est équivalente au théorème de Witten-Kontsevich)

Dans cet article, nous démontrons la formule de récurrence de Mirzakhani sur les volumes de Weil-Petersson des espaces de module de courbes en utilisant le théorème de Witten-Kontsevich. Nous donnons aussi des propriétés des nombres d'intersection associées aux classes κ de degré supérieur.

1. Introduction

Following the notation of Mulase and Safnuk [21], let $\mathcal{M}_{g,n}(\mathbf{L})$ denote the moduli space of bordered Riemann surfaces with n geodesic boundary components of specified lengths $\mathbf{L} = (L_1, \dots, L_n)$ and let $\text{Vol}_{g,n}(\mathbf{L})$ denote its Weil-Petersson volume $\text{Vol}(\mathcal{M}_{g,n}(\mathbf{L}))$. Using her remarkable generalization of the McShane identity, Mirzakhani [19] proved a beautiful recursion formula for these Weil-Petersson volumes

$$\begin{aligned} \text{Vol}_{g,n}(\mathbf{L}) &= \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ n=I \amalg J}} \int_0^{L_1} \int_0^\infty \int_0^\infty xyH(t, x+y) \\ &\quad \times \text{Vol}_{g_1,n_1}(x, \mathbf{L}_I) \text{Vol}_{g_2,n_2}(y, \mathbf{L}_J) dx dy dt \\ &+ \frac{1}{2L_1} \int_0^{L_1} \int_0^\infty \int_0^\infty xyH(t, x+y) \text{Vol}_{g-1,n+1}(x, y, L_2, \dots, L_n) dx dy dt \end{aligned}$$

2000 Mathematics Subject Classification. — 14H10, 14H81.

Key words and phrases. — Weil-Petersson volume, Mirzakhani recursion formula, Witten-Kontsevich theorem.

$$\begin{aligned}
 & + \frac{1}{2L_1} \sum_{j=2}^n \int_0^{L_1} \int_0^\infty x(H(x, L_1 + L_j) + H(x, L_1 - L_j)) \\
 & \qquad \qquad \qquad \times \text{Vol}_{g,n-1}(x, L_2, \dots, \hat{L}_j, \dots, L_n) dx dt,
 \end{aligned}$$

where the kernel function

$$H(x, y) = \frac{1}{1 + e^{(x+y)/2}} + \frac{1}{1 + e^{(x-y)/2}}.$$

Using symplectic reduction, Mirzakhani [20] showed the following relation

$$\begin{aligned}
 \frac{\text{Vol}_{g,n}(2\pi\mathbf{L})}{(2\pi^2)^{3g+n-3}} &= \frac{1}{(3g+n-3)!} \int_{\mathcal{M}_{g,n}} (\kappa_1 + \sum_{i=1}^n L_i^2 \psi_i)^{3g+n-3} \\
 &= \sum_{\substack{d_0+\dots+d_n \\ =3g+n-3}} \prod_{i=0}^n \frac{1}{d_i!} \langle \kappa_1^{d_0} \prod \tau_{d_i} \rangle_{g,n} \prod_{i=1}^\infty L_i^{2d_i}.
 \end{aligned}$$

Combining with her recursion formula of Weil-Petersson volumes, Mirzakhani [20] found a new proof of the celebrated Witten-Kontsevich theorem.

By taking derivatives with respect to $\mathbf{L} = (L_1, \dots, L_n)$ in Mirzakhani’s recursion, Mulase and Safnuk [21] obtained the following enlightening recursion formula of intersection numbers which is equivalent to Mirzakhani’s recursion.

$$\begin{aligned}
 & (2d_1 + 1)!! \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g \\
 &= \sum_{j=2}^n \sum_{b=0}^a \frac{a!}{(a-b)!} \frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 &+ \frac{1}{2} \sum_{b=0}^a \sum_{r+s=b+d_1-2} \frac{a!}{(a-b)!} (2r+1)!! (2s+1)!! \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
 &+ \frac{1}{2} \sum_{b=0}^a \sum_{\substack{c+c'=a-b \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=b+d_1-2} \frac{a!}{c!c'!} (2r+1)!! (2s+1)!! \beta_b \\
 & \qquad \qquad \qquad \times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},
 \end{aligned}$$

where

$$\beta_b = (2^{2b+1} - 4) \frac{\zeta(2b)}{(2\pi^2)^b} = (-1)^{b-1} 2^b (2^{2b} - 2) \frac{B_{2b}}{(2b)!}.$$

Safnuk [23] gave a proof of the above differential form of Mirzakhani’s recursion formula using localization techniques, but he also used the Mirzakhani-McShane formula. The relationship between Mirzakhani’s recursion and matrix integrals has been studied by Eynard-Orantin [7] and Eynard [6].

Indeed, when $a = 0$, Mulase-Safnuk differential form of Mirzakhani's recursion is just the Witten-Kontsevich theorem [14, 24] in the form of DVV recursion relation [4]. There are several other new proofs of Witten-Kontsevich theorem [3, 12, 13, 22] besides Mirzakhani's proof [20].

More discussions about Weil-Petersson volumes from the point of view of intersection numbers can be found in the papers [5, 10, 18, 26].

In Section 2, we show that Mirzakhani's recursion formula is essentially equivalent to the Witten-Kontsevich theorem via a formula from [11] expressing κ classes in terms of ψ classes. In Section 3, we present certain results of intersection numbers involving higher degree κ classes.

Acknowledgements. — We would like to thank Chiu-Chu Melissa Liu for helpful discussions. We also thank the referees for helpful suggestions.

2. Proof of Mirzakhani's recursion formula

We first give three lemmas. The following lemma can be found in [21].

Lemma 2.1. — *The constants β_b in Mirzakhani's recursion satisfy the following:*

$$\sum_{k=0}^{\infty} \beta_k x^k = \frac{\sqrt{2x}}{\sin \sqrt{2x}}.$$

And its inverse:

$$\left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1} = \frac{\sin \sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{(2k+1)!} x^k.$$

Proof. — Since

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = \frac{x e^{x/2} + e^{-x/2}}{2 e^{x/2} - e^{-x/2}} = \frac{x}{2i} \cot \frac{x}{2i},$$

we have

$$\sum_{k=0}^{\infty} \beta_k x^k = \sqrt{2x} (\cot \sqrt{\frac{x}{2}} - \cot \sqrt{2x}) = \frac{\sqrt{2x}}{\sin \sqrt{2x}}. \quad \square$$

The following elementary result is crucial to our proof.

Lemma 2.2. — *Let $F(m, n)$ and $G(m, n)$ be two functions defined on $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let α_k and β_k be real numbers that satisfy*

$$\sum_{k=0}^{\infty} \alpha_k x^k = \left(\sum_{k=0}^{\infty} \beta_k x^k\right)^{-1}.$$

Then the following two identities are equivalent:

$$G(m, n) = \sum_{k=0}^m \alpha_k F(m - k, n + k), \quad \forall (m, n) \in \mathbb{N} \times \mathbb{N},$$

$$F(m, n) = \sum_{k=0}^m \beta_k G(m - k, n + k), \quad \forall (m, n) \in \mathbb{N} \times \mathbb{N}.$$

Proof. — Assume the first identity holds, then we have

$$\begin{aligned} \sum_{i=0}^m \beta_i G(m - i, n + i) &= \sum_{i=0}^m \beta_i \sum_{j=0}^{m-i} \alpha_j F(m - i - j, n + i + j) \\ &= \sum_{k=0}^m \sum_{i+j=k} (\beta_i \alpha_j) F(m - k, n + k) \\ &= \sum_{k=0}^m \delta_{k0} F(m - k, n + k) \\ &= F(m, n). \end{aligned}$$

So we proved the second identity. The proof of the other direction is the same. \square

The fact that intersection numbers involving both κ classes and ψ classes can be reduced to intersection numbers involving only ψ classes was already known to Witten [9], and has been developed by Arbarello-Cornalba [2], Faber [8] and Kaufmann-Manin-Zagier [11] into a nice combinatorial formalism.

Lemma 2.3 ([11]). — For $m > 0$,

$$\langle \prod_{j=1}^n \tau_{d_j} \kappa_1^m \rangle_g = \sum_{k=1}^m \frac{(-1)^{m-k}}{k!} \sum_{\substack{m_1 + \dots + m_k = m \\ m_i > 0}} \binom{m}{m_1, \dots, m_k} \langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{m_j+1} \rangle_g.$$

Proof. — (sketch) Let $\pi_{n+p,n} : \overline{\mathcal{M}}_{g,n+p} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism which forgets the last p marked points and denote $\pi_{n+p,n*}(\psi_{n+1}^{a_1+1} \dots \psi_{n+p}^{a_p+1})$ by $R(a_1, \dots, a_p)$, then we have the formula from [2]

$$R(a_1, \dots, a_p) = \sum_{\sigma \in \mathbb{S}_p} \prod_{\substack{\text{each cycle } c \\ \text{of } \sigma}} \kappa_{\sum_{j \in c} a_j},$$

where we write any permutation σ in the symmetric group \mathbb{S}_p as a product of disjoint cycles.

A formal combinatorial argument [11] leads to the following inversion equation

$$\kappa_{a_1} \dots \kappa_{a_p} = \sum_{k=1}^p \frac{(-1)^{p-k}}{k!} \sum_{\substack{\{1, \dots, p\} = S_1 \amalg \dots \amalg S_k \\ S_k \neq \emptyset}} R\left(\sum_{j \in S_1} a_j, \dots, \sum_{j \in S_k} a_j\right),$$

from which the result follows easily. \square

Proposition 2.4. — *We have*

$$\begin{aligned} & \sum_{b=0}^a (-1)^b \binom{a}{b} \frac{(2(d_1 + b) + 1)!!}{(2b + 1)!!} \langle \tau_{d_1+b} \prod_{i=2}^n \tau_{d_i} \kappa_1^{a-b} \rangle_g \\ &= \sum_{j=2}^n \frac{(2d_1 + 2d_j - 1)!!}{(2d_j - 1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\ &+ \frac{1}{2} \sum_{r+s=d_1-2} (2r + 1)!!(2s + 1)!! \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\ &+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I \amalg J = \{2, \dots, n\}}} \binom{a}{c} \sum_{r+s=d_1-2} (2r + 1)!!(2s + 1)!! \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}. \end{aligned}$$

Proof. — Let LHS and RHS denote the left and right hand side of the equation respectively. By Lemma 2.3 and the Witten-Kontsevich theorem, we have

$$\begin{aligned} & (2d_1 + 1)!! \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g \\ &= (2d_1 + 1)!! \sum_{k=0}^a \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1 + \dots + m_k = a \\ m_i > 0}} \binom{a}{m_1, \dots, m_k} \langle \prod_{j=1}^n \tau_{d_j} \prod_{j=1}^k \tau_{m_j+1} \rangle_g \\ &= \sum_{k=0}^a \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1 + \dots + m_k = a \\ m_i > 0}} \binom{a}{m_1, \dots, m_k} \\ &\times \left(\sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \prod_{i=1}^k \tau_{m_i+1} \rangle_g \right. \\ &+ \sum_{j=1}^k \frac{(2(d_1 + m_j) + 1)!!}{(2m_j + 1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^n \tau_{d_i} \prod_{i \neq j} \tau_{m_i+1} \rangle_g \\ &+ \frac{1}{2} \sum_{r+s=d_1-2} (2r + 1)!!(2s + 1)!! \langle \tau_r \tau_s \prod_{i=2}^n \tau_{d_i} \prod_{i=1}^k \tau_{m_i+1} \rangle_{g-1} \\ &+ \frac{1}{2} \sum_{\substack{I \amalg J = \{2, \dots, n\} \\ I' \amalg J' = \{1, \dots, k\}}} \sum_{r+s=d_1-2} (2r + 1)!!(2s + 1)!! \\ &\left. \times \langle \tau_r \prod_{i \in I} \tau_{d_i} \prod_{i \in I'} \tau_{m_i+1} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \prod_{i \in J'} \tau_{m_i+1} \rangle_{g-g'} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j=2}^n \frac{(2d_1 + 2d_j - 1)!!}{(2d_j - 1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 &+ \frac{1}{2} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
 &+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I \amalg J = \{2, \dots, n\}}} \binom{a}{c} \sum_{r+s=d_1-2} (2r+1)!!(2s+1)!! \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \\
 &+ \sum_{k=0}^a \frac{(-1)^{a-k}}{k!} \sum_{\substack{m_1+\dots+m_k=a \\ m_i > 0}} \binom{a}{m_1, \dots, m_k} \\
 &\times \sum_{j=1}^k \frac{(2(d_1 + m_j) + 1)!!}{(2m_j + 1)!!} \langle \tau_{d_1+m_j} \prod_{i=2}^n \tau_{d_i} \prod_{i \neq j} \tau_{m_i+1} \rangle_g \\
 &= RHS + \sum_{k \geq 0} \frac{(-1)^{a-k-1}}{(k+1)!} \sum_{b=1}^a \sum_{\substack{m_1+\dots+m_k=a-b \\ m_i > 0}} \binom{a}{b} \binom{a-b}{m_1, \dots, m_k} \\
 &\times (k+1) \frac{(2(d_1 + b) + 1)!!}{(2b + 1)!!} \langle \tau_{d_1+b} \prod_{i=2}^n \tau_{d_i} \prod_{i=1}^k \tau_{m_i+1} \rangle_g \\
 &= RHS - \sum_{b=1}^a (-1)^b \binom{a}{b} \frac{(2(d_1 + b) + 1)!!}{(2b + 1)!!} \langle \tau_{d_1+b} \prod_{i=2}^n \tau_{d_i} \kappa_1^{a-b} \rangle_g \\
 &= RHS - LHS + (2d_1 + 1)!! \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g.
 \end{aligned}$$

So we have proved $RHS = LHS$. □

Proposition 2.4 is also implicitly contained in the arguments of Mulase and Safnuk [21].

Theorem 2.5. — *We have*

$$\begin{aligned}
 &\frac{(2d_1 + 1)!!}{a!} \langle \prod_{j=1}^n \tau_{d_j} \kappa_1^a \rangle_g \\
 &= \sum_{b=0}^a \sum_{j=2}^n \frac{(2(b + d_1 + d_j) - 1)!!}{(a - b)!(2d_j - 1)!!} \beta_b \langle \kappa_1^{a-b} \tau_{b+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 &+ \frac{1}{2} \sum_{b=0}^a \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{(a-b)!} \beta_b \langle \kappa_1^{a-b} \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \sum_{b=0}^a \sum_{\substack{c+c'=a-b \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=b+d_1-2} \frac{(2r+1)!!(2s+1)!!}{c!c'} \beta_b \\
 & \qquad \qquad \qquad \times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},
 \end{aligned}$$

where the constants β_k are given by

$$\left(\sum_{k=0}^{\infty} \beta_k x^k \right)^{-1} = \frac{\sin \sqrt{2x}}{\sqrt{2x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)!!} x^k.$$

Proof. — Denote the LHS by $F(a, d_1)$. Let

$$\begin{aligned}
 G(a, d_1) &= \sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{a!(2d_j - 1)!!} \langle \kappa_1^a \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 &+ \frac{1}{2} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{a!} \langle \kappa_1^a \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
 &+ \frac{1}{2} \sum_{\substack{c+c'=a \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=d_1-2} \frac{(2r+1)!!(2s+1)!!}{c!c'} \times \langle \kappa_1^c \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa_1^{c'} \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},
 \end{aligned}$$

Note that Proposition 2.4 is just

$$\sum_{b=0}^a \frac{(-1)^b}{b!(2b+1)!!} F(a-b, d_1+b) = G(a, d_1).$$

By Lemmas 2.1 and 2.2, we have

$$F(a, d_1) = \sum_{b=0}^a \beta_b G(a-b, d_1+b) = RHS.$$

So we conclude the proof. □

3. Higher Weil-Petersson volumes

Mirzakhani's formula provides a recursive way of computing the following Weil-Petersson volumes of moduli spaces of curves

$$WP(g) := \int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^{3g-3+n}.$$

Mirzakhani's formula resorts to intersection numbers of mixed ψ and κ classes.

A natural question is whether there exist an explicit formula expressing $WP(g)$ in terms of those $WP(g')$ with $g' < g$. Recall the following beautiful formula due to Itzykson-Zuber [9].

Proposition 3.1 (Itzykson-Zuber). — *Let $g \geq 0$. Then*

$$\phi_{g+1} = \frac{25g^2 - 1}{24} \phi_g + \frac{1}{2} \sum_{m=1}^g \phi_{g+1-m} \phi_m,$$

where $\phi_0 = -1, \phi_1 = \frac{1}{24}$ and

$$\phi_g = \frac{(5g - 5)(5g - 3)}{2^g(3g - 3)!} \langle \tau_2^{3g-3} \rangle_g, \quad g \geq 2.$$

By projection formula, we have

$$\langle \tau_2^{3g-3} \rangle_g = \langle \kappa_1^{3g-3} \rangle_g + \dots,$$

where \dots denote terms involving higher degree kappa classes. Also note that $\langle \kappa_1^{3g-3} \rangle_g$ is conjecturally [16] the largest term in the right hand side.

To our disappointment, so far, all recursion formulae for $WP(g)$ stemming from the Witten-Kontsevich theorem involve either ψ class or higher degree κ classes inevitably.

Mirzakhani, Mulase and Safnuk’s arguments use Wolpert’s formula [25]

$$\kappa_1 = \frac{1}{2\pi^2} \omega_{WP},$$

where ω_{WP} is the Weil-Petersson Kähler form. We have no similar formulae for higher degree κ classes. So a priori κ_1 may be rather special in the intersection theory. However, as we will see, this is not the case.

First we fix notations as in [11]. Consider the semigroup N^∞ of sequences $\mathbf{m} = (m(1), m(2), \dots)$ where $m(i)$ are nonnegative integers and $m(i) = 0$ for sufficiently large i .

Let $\mathbf{m}, \mathbf{t}, \mathbf{a}_1, \dots, \mathbf{a}_n \in N^\infty$, $\mathbf{m} = \sum_{i=1}^n \mathbf{a}_i$, $\mathbf{m} \geq \mathbf{t}$ and $\mathbf{s} := (s_1, s_2, \dots)$ be a family of independent formal variables.

$$|\mathbf{m}| := \sum_{i \geq 1} im(i), \quad \|\mathbf{m}\| := \sum_{i \geq 1} m(i), \quad \mathbf{s}^{\mathbf{m}} := \prod_{i \geq 1} s_i^{m(i)}, \quad \mathbf{m}! := \prod_{i \geq 1} m(i)!,$$

$$\binom{\mathbf{m}}{\mathbf{t}} := \prod_{i \geq 1} \binom{m(i)}{t(i)}, \quad \binom{\mathbf{m}}{\mathbf{a}_1, \dots, \mathbf{a}_n} := \prod_{i \geq 1} \binom{m(i)}{a_1(i), \dots, a_n(i)}.$$

Let $\mathbf{b} \in N^\infty$, we denote a formal monomial of κ classes by

$$\kappa(\mathbf{b}) := \prod_{i \geq 1} \kappa_i^{b(i)}.$$

We are interested in the following intersection numbers

$$\langle \kappa(\mathbf{b}) \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \kappa(\mathbf{b}) \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

When $d_1 = \dots = d_n = 0$, these intersection numbers are called higher Weil-Petersson volumes of moduli spaces of curves. The details of the following discussions are contained in [17].

The following lemma is a direct generalization of Lemma 2.2.

Lemma 3.2. — Let $F(\mathbf{L}, n)$ and $G(\mathbf{L}, n)$ be two functions defined on $N^\infty \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Let $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ be real numbers depending only on $\mathbf{L} \in N^\infty$ that satisfy $\alpha_0 \beta_0 = 1$ and

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}}\beta_{\mathbf{L}'} = 0, \quad \mathbf{b} \neq 0.$$

Then the following two identities are equivalent:

$$G(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}}F(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N},$$

$$F(\mathbf{b}, n) = \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \beta_{\mathbf{L}}G(\mathbf{L}', n + |\mathbf{L}|), \quad \forall (\mathbf{b}, n) \in N^\infty \times \mathbb{N}.$$

We may generalize Mirzakhani's recursion formula to include higher degree κ classes.

Theorem 3.3. — There exist (uniquely determined) rational numbers $\alpha_{\mathbf{L}}$ depending only on $\mathbf{L} \in N^\infty$, such that for any $\mathbf{b} \in N^\infty$ and $d_j \geq 0$, the following recursion relation of mixed ψ and κ intersection numbers holds.

$$(2d_1 + 1)!! \langle \kappa(\mathbf{b}) \prod_{j=1}^n \tau_{d_j} \rangle_g$$

$$= \sum_{j=2}^n \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2(|\mathbf{L}| + d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{L}') \tau_{|\mathbf{L}|+d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g$$

$$+ \frac{1}{2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}} (2r + 1)!! (2s + 1)!! \langle \kappa(\mathbf{L}') \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \coprod J = \{2, \dots, n\}}} \sum_{r+s=|\mathbf{L}|+d_1-2} \alpha_{\mathbf{L}} \binom{\mathbf{b}}{\mathbf{L}, \mathbf{e}, \mathbf{f}} (2r + 1)!! (2s + 1)!!$$

$$\times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.$$

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{|\mathbf{L}|} \alpha_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! (2|\mathbf{L}'| + 1)!!} = 0, \quad \mathbf{b} \neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}' \neq 0}} \frac{(-1)^{|\mathbf{L}'|} \alpha_{\mathbf{L}}}{\mathbf{L}! \mathbf{L}'! (2|\mathbf{L}'| + 1)!!}, \quad \mathbf{b} \neq 0,$$

with the initial value $\alpha_0 = 1$.

Theorem 3.4. — *We have*

$$\begin{aligned}
 \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \frac{(2d_1 + 2|\mathbf{L}| + 1)!!}{(2|\mathbf{L}| + 1)!!} \langle \kappa(\mathbf{L}') \tau_{d_1+|\mathbf{L}|} \prod_{j=2}^n \tau_{d_j} \rangle_g \\
 = \sum_{j=2}^n \frac{(2(d_1 + d_j) - 1)!!}{(2d_j - 1)!!} \langle \kappa(\mathbf{b}) \tau_{d_1+d_j-1} \prod_{i \neq 1, j} \tau_{d_i} \rangle_g \\
 + \frac{1}{2} \sum_{r+s=|\mathbf{L}|-2} (2r + 1)!! (2s + 1)!! \langle \kappa(\mathbf{b}) \tau_r \tau_s \prod_{i \neq 1} \tau_{d_i} \rangle_{g-1} \\
 + \frac{1}{2} \sum_{\substack{\mathbf{e}+\mathbf{f}=\mathbf{b} \\ I \amalg J = \{2, \dots, n\}}} \sum_{r+s=|\mathbf{L}|-2} \binom{\mathbf{b}}{\mathbf{e}} (2r + 1)!! (2s + 1)!! \\
 \times \langle \kappa(\mathbf{e}) \tau_r \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \kappa(\mathbf{f}) \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.
 \end{aligned}$$

Theorem 3.3 and Theorem 3.4 implies each other through Lemma 3.2.

Both Theorems 3.3 and 3.4 are effective recursion formulae for computing higher Weil-Petersson volumes with the three initial values

$$\langle \tau_0 \kappa_1 \rangle_1 = \frac{1}{24}, \quad \langle \tau_0^3 \rangle_0 = 1, \quad \langle \tau_1 \rangle_1 = \frac{1}{24}.$$

From the following Proposition 3.4, we have

$$\langle \kappa(\mathbf{b}) \rangle_g = \frac{1}{2g - 2} \sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \kappa(\mathbf{L}') \rangle_g.$$

We have computed a table of $\alpha_{\mathbf{L}}$ for all $|\mathbf{L}| \leq 15$ and have written a Maple program [1] implementing Theorems 3.3 and 3.4.

In fact, we find that ψ and κ classes are compatible in the sense that recursions of pure ψ classes can be neatly generalized to recursions including both ψ and κ classes by the same proof as Proposition 2.4. In view of Theorem 3.8 below, this can be rephrased as differential equations governing generating functions of ψ classes also govern generating functions of mixed ψ and κ classes.

We present some examples below.

Proposition 3.5. — *Let $\mathbf{b} \in N^\infty$ and $d_j \geq 0$. Then*

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{|\mathbf{L}|} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = (2g - 2 + n) \langle \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g.$$

The above proposition is a generalization of the dilaton equation. In the special case $\mathbf{b} = (m, 0, 0, \dots)$, it has been proved by Norman Do and Norbury [5].

Proposition 3.6. — *Let $\mathbf{b} \in N^\infty$. Then*

$$\begin{aligned} \langle \tau_0 \tau_1 \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g &= \frac{1}{12} \langle \tau_0^4 \prod_{j=1}^n \tau_{d_j} \kappa(\mathbf{b}) \rangle_g \\ &+ \frac{1}{2} \sum_{\substack{\mathbf{L} + \mathbf{L}' = \mathbf{b} \\ n=I \coprod J}} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_0^2 \prod_{i \in I} \tau_{d_i} \kappa(\mathbf{L}) \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \kappa(\mathbf{L}') \rangle_{g-g'}. \end{aligned}$$

The above proposition, together with the projection formula, can be used to derive an effective recursion formula for higher Weil-Petersson volumes [17] (without ψ classes).

Let $\mathbf{s} := (s_1, s_2, \dots)$ and $\mathbf{t} := (t_0, t_1, t_2, \dots)$, we introduce the following generating function

$$G(\mathbf{s}, \mathbf{t}) := \sum_g \sum_{\mathbf{m}, \mathbf{n}} \langle \kappa_1^{m_1} \kappa_2^{m_2} \dots \tau_0^{n_0} \tau_1^{n_1} \dots \rangle_g \frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},$$

where $\mathbf{s}^{\mathbf{m}} = \prod_{i \geq 1} s_i^{m_i}$.

Following Mulase and Safnuk [21], we introduce the following family of differential operators for $k \geq -1$,

$$\begin{aligned} V_k &= -\frac{1}{2} \sum_{\mathbf{L}} (2(|\mathbf{L}| + k) + 3)!! \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}!(2|\mathbf{L}| + 1)!!} \mathbf{s}^{\mathbf{L}} \frac{\partial}{\partial t_{|\mathbf{L}|+k+1}} \\ &+ \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k) + 1)!!}{(2j - 1)!!} t_j \frac{\partial}{\partial t_{j+k}} + \frac{1}{4} \sum_{d_1+d_2=k-1} (2d_1 + 1)!!(2d_2 + 1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} \\ &+ \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{48}. \end{aligned}$$

Theorem 3.7 ([17, 21]). — *The recursion of Theorem 3.4 implies*

$$V_k \exp(G) = 0.$$

Moreover, we can check directly that the operators V_k , $k \geq -1$ satisfy the Virasoro relations

$$[V_n, V_m] = (n - m)V_{n+m}.$$

The Witten-Kontsevich theorem states that the generating function for ψ class intersections

$$F(t_0, t_1, \dots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ -function for the KdV hierarchy.

Theorem 3.8 ([17, 21]). — *We have*

$$G(\mathbf{s}, t_0, t_1, \dots) = F(t_0, t_1, t_2 + p_2, t_3 + p_3, \dots),$$

where p_k are polynomials in \mathbf{s} given by

$$p_k = \sum_{|\mathbf{L}|=k-1} \frac{(-1)^{|\mathbf{L}|-1}}{\mathbf{L}!} \mathbf{s}^{\mathbf{L}}.$$

In particular, for any fixed values of \mathbf{s} , $G(\mathbf{s}, \mathbf{t})$ is a τ -function for the KdV hierarchy.

At a final remark, it would be interesting to prove that $\alpha_{\mathbf{L}}$ in Theorem 3.3 are positive for all $\mathbf{L} \in N^\infty$. This problem is kindly pointed out to us by a referee.

More generally the question can be formulated as following: two sequences $\alpha_{\mathbf{L}}$ and $\beta_{\mathbf{L}}$ with $\alpha_0 = \beta_0 = 1$ are said to be inverse to each other if they satisfy

$$\left(\sum_{\mathbf{L}} \alpha_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \right) \cdot \left(\sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}} \right) = 1.$$

Find sufficient conditions on $\beta_{\mathbf{L}}$ such that $\alpha_{\mathbf{L}} > 0$ for all \mathbf{L} .

We conjecture that $\alpha_{\mathbf{L}}$ are positive when $\sum_{\mathbf{L}} \beta_{\mathbf{L}} \mathbf{s}^{\mathbf{L}}$ equals any of the following.

$$\sum_{\mathbf{L}} \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}!(2|\mathbf{L}|+1)!!} \mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}!(2|\mathbf{L}|-1)!!} \mathbf{s}^{\mathbf{L}}, \quad \sum_{\mathbf{L}} \frac{(-1)^{|\mathbf{L}|}}{\mathbf{L}!|\mathbf{L}|!} \mathbf{s}^{\mathbf{L}}.$$

The latter two arise when we consider Hodge integrals involving λ classes [17].

For works on the positivity criteria of coefficients of reciprocal power series of a single variable, see for example [15]. However it seems there is no literature dealing with the coefficients of reciprocal series of several variables.

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