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GEVREY CLASS OF THE INFINITESIMAL GENERATOR OF A DIFFEOMORPHISM

by

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Abstract. — Let \( F \) be an analytic diffeomorphism in \((\mathbb{C}^m, 0)\) tangent to the identity of order \( n \). The infinitesimal generator of \( F \) is the formal vector field \( X \) such that \( \exp X = F \). In this paper we provide an elementary proof of the fact that \( X \) belongs to the Gevrey class of order \( 1/n \).

Résumé (La classe de Gevrey du générateur infinitésimal d’un difféomorphisme)

Soit \( F \) un difféomorphisme analytique de \( \mathbb{C}^m \) tangent à l’identité à l’ordre \( n \). Le générateur infinitésimal de \( F \) est le champ de vecteurs formel \( X \) tel que \( \exp X = F \). Dans cet article nous donnons une preuve élémentaire du fait que \( X \) appartient à la classe Gevrey d’ordre \( 1/n \).

1. Introduction

For each couple of integers \( m \geq 1 \) and \( n \geq 2 \), let us denote \( \hat{J}_{tn}(\mathbb{C}^m, 0) \) the module of formal vector fields of order \( \geq n \) in \((\mathbb{C}^m, 0)\) and \( \text{Diff}_n(\mathbb{C}^m, 0) \) the group of formal diffeomorphisms in \((\mathbb{C}^m, 0)\) tangent to the identity of order \( \geq n \), i.e, \( F \in \text{Diff}_n(\mathbb{C}^m, 0) \) if and only if \( \nu(F) := \min\{\nu_0(x_i \circ F - x_i) | i = 1, \ldots, m\} - 1 \geq n \). For any \( X \in \hat{J}_n(\mathbb{C}^m, 0) \), the exponential operator of \( X \) is the application \( \exp X : \mathbb{C}[[x]] \to \mathbb{C}[[x]] \) defined by the formula

\[
\exp X(g) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j(g)
\]
where \( X^0(g) = g \) and \( X^{j+1}(g) = X(X^j(g)) \). It is a classical result (for instance, see [5]) that the application

\[
\text{Exp} : \mathfrak{X}_n(C^m, 0) \to \widehat{\text{Diff}}_{n-1}(C^m, 0)
\]

\[
X \mapsto (\exp X(x_1), \ldots, \exp X(x_m))
\]

is a bijection. The formal vector field \( X \) such that \( F = \text{Exp}(X) \) is called the \textit{infinitesimal generator} of \( F \).

Let \( x = (x_1, \ldots, x_m) \) and for any \( s \in \mathbb{R} \) let \( C[[x]]_s \) denote the subset of elements of \( C[[x]] \) that satisfy the \( s \)-Gevrey condition, i.e.

\[
f(x) = \sum_{k=0}^{\infty} f_k(x) \in C[[x]]_s \quad \text{if and only if} \quad \sum_{k=0}^{\infty} \frac{f_k(x)}{k!^s} \in C\{x\},
\]

where \( f_k(x) \) is homogeneous of degree \( k \). Let us observe that \( 0 \)-Gevrey condition means analyticity, and \( C\{x\} \subset C[[x]]_s \subset C[[x]]_t \) if \( 0 < s < t \). Let \( \mathfrak{X}_n(C^m, 0)_s \subseteq \mathfrak{X}_n(C^m, 0) \) be the set of \( s \)-Gevrey vector fields \( X = \sum_{k=1}^{m} X(x_k) \frac{\partial}{\partial x_k} \) with \( X(x_k) \in C[[x]]_s \) and \( \text{Diff}_n(C^m, 0)_s = \widehat{\text{Diff}}_n(C^m, 0) \cap (C[[x]]_s)^m \) the set of \( s \)-Gevrey diffeomorphisms tangent to the identity of order \( \geq n \).

We will prove the following result

\textbf{Theorem 1.1.} — For any \( s \geq \frac{1}{n-1} \) the application \( \text{Exp} \) gives a bijection

\[
\text{Exp} : \mathfrak{X}_n(C^m, 0)_s \to \text{Diff}_{n-1}(C^m, 0)_s.
\]

In particular, the infinitesimal generator of any tangent to the identity analytic diffeomorphism \( F \) is \( \frac{1}{\nu(F)} \)-Gevrey.

In general, \( X \) may be divergent for a convergent \( F \), for instance, Szekeres [7] and Baker [2] proved that every entire holomorphic function tangent to the identity of order \( k \) in dimension 1 has a non-convergent infinitesimal generator, Ahern and Rosay [1] proved that this kind of diffeomorphisms cannot be the time-1 map of a \( C^{3k+3} \)-vector field, and finally J. Rey [6] showed that they cannot be the time-1 map of a \( C^{k+1} \)-vector field, which is the best possible bound. Thus, the map \( \text{Exp} : \mathfrak{X}_n(C^m, 0)_0 \to \text{Diff}_{n-1}(C^m, 0)_0 \) is not surjective for any couple of positive integers \( m, n \). In addition, in dimension 1, using resummation arguments, it is proved that if an analytic diffeomorphism \( f(x) = x + a_{k+1}x^{k+1} + \cdots \) with \( a_{k+1} \neq 0 \) has a divergent infinitesimal generator \( X \), then \( X \) is \( k \)-summable, so \( X \) is Gevrey of order \( \frac{1}{k} \), but not smaller (see [4], [3] and [5]). Therefore, the condition \( s \geq \frac{1}{n-1} \) is necessary.

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2. Technical estimations

In this paper, we take the following notations:

- $h_k(x)$ will denote the homogeneous polynomial $\sum_{\alpha \in \mathbb{N}^m, |\alpha| = k} x^\alpha$.
- $H_{s,n}(x)$ the series $\sum_{q=n}^{\infty} (q + m - n)!^s h_q(x)$.
- $\frac{\partial}{\partial x}$ the differential operator $\sum_{k=1}^{m} \frac{\partial}{\partial x_k}$.

For formal series $f(x) = \sum_{\alpha} f_\alpha x^\alpha$ and $g(x) = \sum_{\alpha} g_\alpha x^\alpha$, we say that $f \preceq g$ if $|f_\alpha| \leq |g_\alpha|$ for any $\alpha \in \mathbb{N}^m$. We get in this way a partial order in $\mathbb{C}[[x]]$, and also in $\mathcal{K}_n(\mathbb{C}^m, 0)$ and $\text{Diff}_n(\mathbb{C}^m, 0)$, working on the component function. From the definition of Gevrey condition, it can be seen that $X \in \mathcal{K}_n(\mathbb{C}^m, 0)_s$ if and only if there exists $a \in \mathbb{R}^+$ such that, for all $q \geq n$,

$$\text{Coeff}_q(X) \preceq (q + m - n)!^s a^q h_q(x) \frac{\partial}{\partial x},$$

where $\text{Coeff}_q(X)$ denotes the homogeneous term of $X$ of degree $q$. Thus $X \in \mathcal{K}_n(\mathbb{C}^m, 0)_s$ if and only if there exists $a \in \mathbb{R}^+$ such that $X \preceq H_{s,n}(ax)\frac{\partial}{\partial x}$.

We need the following technical lemmas:

**Lemma 2.1.** — For every $k, l \in \mathbb{N}^*$

$$h_k \frac{\partial}{\partial x} h_l \preceq (l + m - 1) \min \left\{ \left( \frac{k + m - 1}{m - 1} \right), \left( \frac{l + m - 2}{m - 1} \right) \right\} h_{k+l-1}.$$

**Proof.** — Observe that

$$\frac{\partial}{\partial x} h_l = \sum_{k=1}^{m} \frac{\partial}{\partial x_k} \sum_{\alpha \in \mathbb{N}^m, |\alpha|=l} x^\alpha = \sum_{k=1}^{m} \sum_{\alpha \in \mathbb{N}^m, |\alpha|=l} \alpha_k \frac{x^\alpha}{x_k}$$

$$= \sum_{|\beta|=l-1} \sum_{k=1}^{m} (\beta_k + 1)x^\beta = (l + m - 1)h_{l-1}$$

Now, the coefficient of $x^\alpha$ in the product $h_k(x)h_{l-1}(x)$ is less than or equal to the minimum between the number of monomials of $h_k$ and the number of monomials of $h_{l-1}$, and the number of monomials of $h_j$ is $(j+m-1)$, that corresponds to the number of ordered partitions of $j$ in $m$ parts; therefore,

$$h_k \frac{\partial}{\partial x} h_l = (l + m - 1)h_k h_{l-1} \preceq (l + m - 1) \left( \min\{k, l-1\} + m - 1 \right) h_{k+l-1}. \quad \square$$

**Lemma 2.2.** — Let $\Theta(y) = \sum_{j=n}^{\infty} \binom{m-1+j}{m-1} y^{j-n}$. Then $\Theta(y)$ converges for any $|y| < 1$. 

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Proof. — Since \[ \sum_{j=n}^{\infty} y^{m-1+j} = \frac{y^{m+n-1}}{1-y} \] converges for any \( |y| < 1 \) then
\[
\Theta(y) = \frac{1}{(m-1)!} \frac{1}{y^n} d^{m-1} \left( \frac{y^{m+n-1}}{1-y} \right)
\]
converges for any \( |y| < 1 \). \( \square \)

Lemma 2.3. — For any \( s > 0 \) and integers \( m \geq 1 \) and \( n \geq 2 \), the sequence \( \{b_q\}_{q \geq 2n-1} \)
given by
\[
b_q = \sum_{j=n}^{\lfloor q+1 \rfloor} \frac{(j + m - n)! (q - j + 1 + m - n)!}{m! (q + m - n)!} \left( \frac{q - j + m}{m} \right)^{n-1} \left( j - m - 1 \right),
\]
is bounded.

Proof. — Observe that
\[
\frac{(q - j + m)^{n-1}}{(q - j + 2 + m - n) \cdots (q - j + m)} < \left( \frac{q - j + m}{q - j + 2 + m - n} \right)^{n-1} \leq \left( \frac{\frac{q-1}{2} + m}{\frac{q-1}{2} + 2 + m - n} \right)^{n-1} \leq \left( \frac{m + n - 1}{m + 1} \right)^{n-1}
\]
then
\[
b_n \leq \left( \frac{m + n - 1}{m + 1} \right)^{s(n-1)} \sum_{j=n}^{\lfloor q+1 \rfloor} \frac{(j + m - n)! (q - j + m)!}{m! (q + m - n)!} \left( j - m - 1 \right).
\]
In addition
\[
\frac{m + 1}{q + m - j + 1} < \frac{m + 2}{q + m - j + 2} < \cdots < \frac{j + m - n}{q + m - n}
\]
and
\[
\frac{j + m - n}{q + m - n} \leq \frac{\frac{q+1}{2} + m - n}{q + m - n} \leq \max \left\{ \frac{1}{2}, \frac{m}{m + n - 1} \right\} = C_{m,n} < 1;
\]
from lemma 2.2,
\[
b_q \leq \left( \frac{m + n - 1}{m + 1} \right)^{s(n-1)} \Theta(C^s_{m,n}). \quad \square
\]

Proposition 2.4. — Let \( s \geq \frac{1}{n-1} \), \( X \in \tilde{\mathcal{X}}(C^m, 0) \) and \( a \in \mathbb{R}^+ \) such that
\[
\text{Coef}_q(X) \leq (q + m - n)! a^q h_q(x) \frac{\partial}{\partial x}
\]
for all \( n \leq q \leq N \), and let us denote \( A = 2m!^s \left( \frac{m+n-1}{m+1} \right)^{s(n-1)} \Theta(C^s_{m,n}) \). For every \( q, k \) with \( n \leq q \leq N + k - 1 \),
\[
\text{Coef}_q(X^k) \leq (aA)^{k-1} (q + m - n)! a^q h_q(x) \frac{\partial}{\partial x},
\]
Proof. — Since 
\[ X^k = \sum_{i=1}^{m} X^k(x_i) \frac{\partial}{\partial x_i}, \]

it is enough to prove the affirmation for 
\[ X^k(x_i), \]

where \( i \in \{1, 2, \ldots, m\} \). Let us write 
\[ X = \sum_{j=n}^{\infty} X_j, \]

where \( X_j \) is homogeneous of degree \( j \). We will proceed by induction on \( k \); if \( k = 1 \), by hypothesis

\[ X_q(x_i) \leq (q + m - n)! s a^q h_q(x) \quad \text{for every } n \leq q \leq N. \]

Suppose that the lemma is true for every \( k \leq p \), then, since the order of \( X^j \) is greater than or equal to \((n-1)j+1\), Coef\( _q(X^{p+1}) = 0 \) for \( n \leq q \leq (n-1)p + n - 1 \) and for \((n-1)p + n \leq q \leq N + p \) we have

\[
\text{Coef}_q(X^{p+1}(x_i)) = \text{Coef}_q(X(X^p(x_i))) = \text{Coef}_q \left( \sum_{j=n}^{\infty} X_j(X^p(x_i)) \right)
\]

\[
= \sum_{j=n}^{q-(n-1)p} X_j \text{Coef}_{q+1-j}(X^p(x_i))
\]

\[
\leq \sum_{j=n}^{q-(n-1)p} (j + m - n)! s a^q h_j(x) \frac{\partial}{\partial x} \left((aA)^{p-1}(q - j + 1 + m - n)! s a^{q+1-j} h_{q+1-j}(x)\right)
\]

\[
\leq \sum_{j=n}^{q-(n-1)p} (j + m - n)! s (q - j + 1 + m - n)! s (q - j + m) \left(\min_{m-1}^{(j+1)} \right) A^{p-1} a^{q+p} h_q,
\]

\[
\leq 2 \sum_{j=n}^{q-(n-1)p} ((j + m - n)! (q - j + 1 + m - n)! (q - j + m) \min_{m-1}^{(j+1)} A^{p-1} a^{q+p} h_q.
\]

Now, observe that

\[
b_q m! s (q + m - n)! s = \sum_{j=n}^{\left\lfloor \frac{q+s+1}{2} \right\rfloor} ((j + m - n)! (q - j + 1 + m - n)! (q - j + m) \min_{m-1}^{(j+1)}) s \left(\frac{j + m - 1}{m - 1}\right),
\]

where \( \{b_q\} \) is the sequence defined in lemma 2.3; it follows that

\[
\text{Coef}_q(X^{p+1}(x_i)) \leq 2 b_q m! s (q + m - n)! s A^{p-1} a^{q+p} h_q
\]

\[
\leq (q + m - n)! s (aA)^p a^q h_q. \quad \square
\]


To prove that the application \( \text{Exp} : \mathcal{X}_n(C^m, 0)_s \to \text{Diff}_{n-1}(C^m, 0)_s \) is well defined for \( s \geq \frac{1}{n-1} \), let \( X \in \mathcal{X}_n(C^m, 0)_s, a > 0 \) be such that \( X \sim H_s, n(ax) \), and \( A \) as in proposition 2.4.

Then by proposition 2.4 we have

\[
\text{Coef}_q(\text{exp} X(x_j)) = \sum_{k=1}^{\infty} \frac{1}{k!} \text{Coef}_q(X^k(x_j))
\]

\[
\leq \sum_{k=1}^{\infty} \frac{1}{k!} (aA)^{k-1} (q + m - n)! s a^q h_q(x)
\]

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therefore $\text{Exp}(X) \lesssim \sum_{k=1}^{\infty} \frac{(aA)^{k-1}}{k!} H_{s,n}(ax)$. Now, to prove that $\text{Exp}$ is surjective, let us consider a diffeomorphism $F(x) = (x_1 + f_1(x), \ldots, x_m + f_m(x)) \in \text{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ where $f_j(x) = \sum_{q=n}^{\infty} f_{j,q}(x) \in \mathbb{C}[[x]]_s$ and $f_{j,q}(x)$ is a homogeneous polynomial of degree $q$. Then there exists $a > 0$ such that $f_{j,q}(x) \preceq (q + m - n)!^s a^q h_q(x)$. Observe that, making a linear change of coordinates, we can suppose that $a$ is small enough such that $\sum_{k=2}^{\infty} \frac{1}{k!} (2aA)^{k-1} \leq \frac{1}{2}$. If $X = \sum_{q=n}^{\infty} X_q$ is the infinitesimal generator of $F(x)$, we will show by induction on $q$ that

$$X_q \preceq (q + m - n)!^s (2a)^q h_q(x) \frac{\partial}{\partial x}.$$ 

For $q = n$

$$X_n(x_j) = f_{j,n}(x) \preceq m!^s a^n h_n(x) \preceq m!^s (2a)^n h_n(x).$$

Suppose that the claim is true for any integer between $n$ and $q$, it follows that

$$f_{j,q+1}(x) = \text{Coef}_{q+1} \left( \sum_{k=1}^{\infty} \frac{1}{k!} X^k(x_j) \right) = X_{q+1}(x_j) + \sum_{k=2}^{q} \frac{1}{k!} \text{Coef}_{q+1} (X^k(x_j)),$$

using proposition 2.4

$$X_{q+1}(x_j) \preceq (q + 1 + m - n)!^s a^{q+1} h_{q+1}(x)$$

$$+ \sum_{k=2}^{\infty} \frac{1}{k!} (2aA)^{k-1} (q + 1 + m - n)!^s (2a)^q h_{q+1}(x)$$

$$\preceq \left( \frac{1}{2a} + \sum_{k=2}^{\infty} \frac{1}{k!} (2aA)^{k-1} \right) (q + 1 + m - n)!^s (2a)^q h_{q+1}(x)$$

$$\preceq (q + 1 + m - n)!^s (2a)^{q+1} h_{q+1}(x),$$

in other words $X \preceq H_{s,n}(2a) \frac{\partial}{\partial x}$. \hfill \Box

4. Case $0 \leq s < \frac{1}{n-1}$

As we indicated in the introduction, in this case, there exists $F \in \text{Diff}_{n-1}(\mathbb{C}^m, 0)_s$ such that its infinitesimal generator is not $s$-Gevrey, but the reciprocal is true, i.e.

**Proposition 4.1.** — Let $0 \leq s \leq \frac{1}{n-1}$, and $X \in \mathcal{X}_n(\mathbb{C}^m, 0)_s$. Then $\text{Exp}(X) \in \text{Diff}_{n-1}(\mathbb{C}^m, 0)_s$.

Observe that the case $s = 0$ is a classical result about the existence of solution of an analytic differential equation. To prove this proposition in the case $s > 0$ we need the following lemma
Lemma 4.2. — Let \( t, r \in \mathbb{R} \) such that \( 0 < t < 1 \) and \( 1 - t < r < 1 \). Let \( \{a_k\} \) be the sequence defined by \( a_1 = a > 0 \) and for \( k \geq 1 \), \( a_{k+1} = \sup_{q \in \mathbb{N}^*} \sqrt[k]{\frac{(q+m)^{1-t}}{(k+1)^{r}}} a_k \). Then \( \{a_k\} \) is increasing and convergent.

Proof. — Taking \( q \gg k \) it is clear that \( \sqrt[k]{\frac{(q+m)^{1-t}}{(k+1)^{r}}} > 1 \), and then \( a_{k+1} > a_k \). Now, we know by Bernoulli inequality that

\[
q + m \left( \frac{1}{k+1} \right)^{1-t} < 1 + \frac{1}{q+k} \left( \frac{(q+m)^{1-t}}{(k+1)^{r}} - 1 \right) < 1 + \frac{1}{(k+1)^{1-t}}
\]

for \( k > m \), so

\[
a_{k+1} < \left( 1 + \frac{1}{(k+1)^{1-t}} \right)^{1-t} a_k < \left( \prod_{j=m+1}^{k+1} \left( 1 + \frac{1}{j^{1-t}} \right) \right)^{1-t} a_m,
\]

and since \( \frac{r}{1-t} > 1 \) it follows that \( \{a_k\} \) is bounded, thereby it is convergent.

Proof of proposition 4.1. — If \( s \in (0, \frac{1}{n-1}) \), \( X \in \mathcal{X}_n(C^m, 0) \) and \( a \in \mathbb{R}^+ \) such that \( X \leq H_{s,n}(ax) \frac{\partial}{\partial x} \), then for \( t = s(n-1) \), \( r = (1-t, 1) \) and \( \{a_k\} \) as in lemma 4.2, using the arguments of proposition 2.4 and the fact that \( k^r a_k^{k+q-1} \geq (q+m)^{1-t} a_{k-1}^{k+q-1} \) for every \( q \geq 2 \), we can prove that

\[
X^k \leq (a_k A) k! H_{s,n}(a_k x) \frac{\partial}{\partial x},
\]

where \( A = 2m! s \left( \frac{m+n-1}{m+1} \right)^s (n-1) \Theta(C^m_{s,n}) \). Let \( c = \lim_{k \to \infty} a_k \). Therefore we have

\[
\text{Coef}_q(\exp(X)(x_j)) = \sum_{k=1}^{\infty} \frac{1}{k!} \text{Coef}_q(X^k(x_j)) \leq \sum_{k=1}^{\infty} \frac{(cA)^{k-1}}{k!^{1-r}} (m+q-n)! c^q h_q(x)
\]

Thus \( \text{Exp}(X) \leq \sum_{k=1}^{\infty} \frac{(cA)^{k-1}}{k!^{1-r}} H_{s,n}(cx) \frac{\partial}{\partial x} \).

References


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